1.10. Discrete systems.

We first present some abstract results for discrete dynamical systems and then make some remarks on applications to nonautonomous differential equations which are periodic in time, periodic orbits of autonomous equations, flows on a cylinder or torus and discrete approximations to differential equations. We give more details on these applications in later chapters.

We denote by a *discrete dynamical system* in $\mathbb{R}^d$ any continuous map $T : \mathbb{R}^d \to \mathbb{R}^d$. For any $\xi \in \mathbb{R}^d$, we let $T^0\xi = \xi$, $T^{n+1}\xi = T(T^n\xi), n = 0, 1, \ldots$. The *positive orbit* $\gamma^+(\xi)$ through $\xi \in \mathbb{R}^d$ is defined by $\gamma^+(\xi) = \{ T^n\xi : n \geq 0 \}$ and a *negative orbit* $\gamma^-(\xi)$ through $\xi \in \mathbb{R}^d$ is defined by $\gamma^-(\xi) = \{ x_{-n}, n \geq 0 : x_0 = \xi, Tx_{-(j+1)} = x_{-j}, j \geq 0 \}$.. When we are given a negative orbit, we will denote the elements sometimes by $T^n\xi$, $n \leq 0$. A *complete orbit* $\gamma(\xi)$ through $\xi \in \mathbb{R}^d$ is the union of a positive and a negative orbit. We cannot say *the* negative orbit and complete orbit because we have not assumed that the map T is one-to-one. If we do not want to distinguish a particular point on the orbit, we will write γ, γ^+, γ^- for the orbit, positive orbit, negative orbit, respectively.

We remark that we often are interested in maps T which are defined on a domain $D \subset \mathbb{R}^d$ and not on all of $\mathbb{R}^d$. The treatment of this situation is no more difficult but simply requires a little more notation.

The *ω-limit set* $\omega(\xi)$ of a point ξ (or, equivalently, the ω-limit set of the positive orbit containing ξ) is defined by the following relation:

$$\omega(\xi) = \cap_{m \geq 0} \overline{\cup_{n \geq m} T^n\xi}$$

If $\gamma^-(\xi)$ is a negative orbit through ξ, then we can define the α-limit set $\alpha(\gamma^-(\xi))$ of the negative orbit $\gamma^-(\xi)$ through ξ in a similar way as

$$\alpha(\gamma^-(\xi)) = \cap_{m \leq 0} \overline{\cup_{n \leq m} T^n\xi}$$

If the mapping T is one-to-one, then there is at most one negative orbit through ξ and then $\alpha(\gamma^-(\xi))$ depends only upon ξ. If the mapping T is not assumed to be one-to-one, there may be several negative orbits through a given point ξ. If we want to define the α-limit set of the point ξ (in contrast to the α-limit set of a particular orbit through ξ, then we must consider the set $H(\xi)$ consisting of all negative orbits through ξ. We can represent the set $H(\xi)$ as $H(\xi) = \{A_{-n}(\xi) : n \geq 0\}$, where $TA_{-(n+1)}(\xi) = A_{-n}(\xi)$ for $n \geq 0$. With this notation, we can define the α-limit set $\alpha(\xi)$ of the point ξ as

$$\alpha(\xi) = \cap_{m \geq 0} \overline{\cup_{n \geq m} A_{-n}(\xi)}$$

It is easy to see that a point $y \in \omega(\xi)$ (resp. $\alpha(\gamma^-(\xi))$) if and only if there is a sequence of real numbers $\{ n_k \}$, $n_k \to \infty$ as $k \to \infty$ such that $T^{n_k}(\xi) \to y$ (resp. $x_{-n} \to y$) as $k \to \infty$. Can you describe the elements in $\alpha(\xi)$ in terms of sequences?

If B is a bounded set in $\mathbb{R}^d$, then we define $TB = \cup_{\xi\in B} T\xi$ and the ω-limit set of $B, \omega(B)$, by the relation

$$\omega(B) = \cap_{m\geq 0}\overline{\cup_{n\geq m}T^n B}.$$

If $\gamma^-(B)$ is a negative orbit through B, then we can define the α-limit set $\alpha(\gamma^-(B))$ of the orbit $\gamma^-(B)$ in a similar way. It also is possible to define the α-limit set of B by using all possible orbits through B. We do not give this latter definition precisely since it will not be of interest to us in the following.

A set $A \subset \mathbb{R}^d$ is an *invariant set* of T if $TA = A$. A set $A \subset \mathbb{R}^d$ is a *positively invariant set* of T if $TA \subset A$. A set $A \subset \mathbb{R}^d$ is a *negatively invariant set* of T if $A \subset TA$.

Exercise 10.1. If B is a closed, bounded, positively invariant set, show that

$$\omega(B) = \cap_{n\geq 0}T^n B\,.$$

As in the proof of Theorem 6.2, we obtain the following result.

Theorem 10.1. *The sets $\omega(B)$, $\alpha(B)$ are closed and invariant. If $B \subset \mathbb{R}^d$ is nonempty and bounded and $\gamma^+(B)$ (respectively, $\gamma^-(B)$) is bounded, then $\omega(B)$ (respectively, $\alpha(B)$) is nonempty, compact and dist $(T^n(B), \omega(B))$ (respectively, dist $(T^n(B), \alpha(B))$) $\to 0$ as $n \to \infty$ (respectively, $n \to -\infty$).*

Notice that we said nothing about the connectedness of the ω-limit set in Theorem 10.1. For maps, this set may not be connected. In fact, if $d = 1$ and $T(x) = -x(2-x^2)$, then $T(1) = -1$, $T(-1) = 1$. Therefore, $\omega(1) = \{1, -1\}$ and $\omega(-1) = \{1, -1\}$, which are disconnected sets.

As in Section 1.7, we define the concepts of stability of invariant sets. Suppose that J is an invariant set of T. We say that J is *stable* if, for any $\epsilon > 0$, there is a $\delta > 0$ such that, if $\xi \in B(J, \delta)$, then $T^n\xi \in B(J, \epsilon)$ for $n \geq 0$. We say that J is *unstable* if it is not stable. We say that J *attracts points locally* if there exists a constant $b > 0$ such that, if $\xi \in B(J, b)$, then $\mathrm{dist}(T^n\xi, J) \to 0$ as $n \to \infty$; that is, for any $\eta > 0$ and any $\xi \in B(J, b)$, there is an $n_0(\xi, \eta)$ with the property that $T^n\xi \in B(J, \eta)$ for $n \geq n_0(\xi, \eta)$. We say that J is a *local attractor* if there exists a constant $c > 0$ such that dist $(T^nB(J, c), J) \to 0$ as $n \to \infty$; that is, for any $\eta > 0$, there is an $n_0(\eta)$ with the property that, if $\xi \in B(J, c)$, then $T^n\xi \in B(J, \eta)$ for $n \geq n_0(\eta)$. If, in addition, for any bounded set $B \subset \mathbb{R}^d$, we have dist $(T^nB, A) \to 0$ as $n \to \infty$, then we say that A is a *global attractor* of T.

The following statement is proved in the same way as Theorem 7.2.

Theorem 10.2. *If J is a compact invariant set of T, then J is stable and attracts points locally if and only if it is a local attractor.*

We say that T is *point dissipative* if there exists a bounded set $B \in \mathbb{R}^d$ such that, for any $\xi \in \mathbb{R}^d$, there is an $n_0(\xi) \geq 0$ such that $T^n\xi \in B$ for $n \geq n_0(\xi)$.

Theorem 10.3. *If T is point dissipative, then there exists a global attractor for T. In addition, there is a fixed point of T.*

Proof. The proof of the first part is the same as the proof of Theorem 7.3. The existence of the fixed point is a consequence of Theorem A.1.6.

1.10.1. Periodic Systems.

If the vector field in the differential equation (1.1) $\dot{x} = f(t, x)$ is such that it has period p in t, then we say that (1.1) is a *p-periodic system*. Suppose that $f \in C^1(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$, (1.1) is p-periodic and $x(t, \tau, \xi)$ is the solution of the initial value problem (1.2) through ξ at time τ. Then uniqueness of the solution of the initial value problem implies that

$$x(t, \tau, \xi) = x(t + p, \tau + p, \xi). \tag{10.1}$$

The *Poincaré map* π at time 0 for the p-periodic system (1.1) is defined by $\pi\xi = x(p, 0, \xi)$. The map π is a discrete dynamical system and $\pi^n\xi = x(np, 0, \xi)$ for all n. Here, we are implicitly assuming that each solution of the differential equation exists on an interval of length at least p. If this is not the case, then the discussion below should be restricted to the set of initial data for which this is true.

A point ξ_0 is a *fixed point* of π if $\pi\xi_0 = \xi_0$. The fixed points of π are in one-to-one correspondence with the initial values of the p-periodic solutions of (1.1). In fact, if $\psi(t)$ is a p-periodic solution of (1.1), then $\psi(p) = \psi(0)$ and $\psi(0)$ is a fixed point of π. Conversely, suppose that ξ is a fixed point of π. If $\varphi(t, \xi)$ is the solution of (1.1) with $\varphi(0, \xi) = \xi$, then $\varphi(t + p, \xi)$ is also a solution of (1.1) since $f(t, x)$ is p-periodic in t. Since ξ is a fixed point of π, uniqueness of solutions of the initial value problem imply that $\varphi(t, \xi)$ is p-periodic.

In general, we define a *k-periodic point* of π to be a point ξ_0 such that $\pi^k\xi_0 = \xi_0$ and $\pi^j\xi_0 \neq \xi_0$ for $0 \leq j < k$. The k-periodic points of π are in one-to-one correspondence with the initial values of the kp-periodic solutions of (1.1).

Example 10.1. If $a(t)$ is a continuous periodic function of minimal period p, we consider the equation $\dot{x} = a(t)x$. The solution $x(t, \tau, \xi) = e^{\int_\tau^t a(s)\, ds}\xi$ and so the Poincaré map is given by $\pi\xi = e^\alpha\xi$, where $\alpha = \int_0^p a(s)\, ds$. Notice that $e^\alpha > 0$ so that the Poincaré map is monotone. If $\alpha < 0$ (resp. > 0), then $\pi^n\xi \to 0$ (resp., $\to \infty$) as $n \to \infty$. This implies that the solutions of the differential equation approach 0 (resp., become unbounded) as $t \to \infty$ is $\alpha < 0$ (resp., > 0). If $\alpha = 0$, then each point is fixed under the Poincaré map and corresponds to a p-periodic solution of the differential equation.

Exercise 10.2. Consider the scalar p-periodic equation $\dot{x} = f(t, x)$, where f is a C^1-function. Suppose that each solution is defined for all t and let π be the Poincaré map. Show that $\pi\xi$ is monotone in ξ and that $\omega(\xi)$ and $\alpha(\xi)$ are fixed points of π

if they are nonempty. Show that each solution of the equation that is bounded for $t \geq 0$ (resp. $t \leq 0$) approaches a p-periodic solution as $t \to \infty$ (resp. $t \to -\infty$).

Exercise 10.3. Give an example of a linear equation $\dot{x} = g(t)x + h(t)$, g, h of period 1 with g of minimal period 1, such that there is a periodic solution of period $\frac{1}{2}$.

Exercise 10.4. Consider the linear scalar equation $\dot{x} = -x + h(t)$, where $h(t)$ is a continuous p-periodic function. Show that the global attractor for the Poincaré map is a single point. Interpret this result in terms of periodic solutions of the differential equation. Do not integrate the equation but use Theorem 10.3 to obtain the existence of the global attractor and the existence of a fixed point. Then show the fixed point is unique.

Exercise 10.5. Consider the linear scalar equation $\dot{x} = -x^3 + h(t)$, where $h(t)$ is a continuous p-periodic function. Show that the global attractor for the Poincaré map is a single point.

Exercise 10.6. Suppose $r(t)$, $k(t)$ are 1-periodic, positive continuous functions and consider the equation

$$\dot{x} = r(t)x[1 - \frac{x}{k(t)}].$$

Show that the set $D = [0, \infty)$ is positively invariant under the flow and show that the Poincaré map π has a global attractor in D. Show there exists at least two fixed points of π in D and thus two 1-periodic solutions of the differential equation in D.

Exercise 10.7. Can you construct a system of linear periodic differential equations in $\mathbb{R}^2$ of minimal period p such that the Poincaré map has an eigenvalue $= -1$? If so, then there will be periodic solutions of the differential equation which has minimal period $2p$.

If we suppose that A is an invariant set for π, then the cylindrical set $C_A = \{(t, \zeta) : \zeta = x(t, 0, \xi), \xi \in A\}$ has the property that it is invariant for (1.1); that is, for any $(\tau, \xi) \in C_A$, we have $(t, x(t, \tau, \xi)) \in C_A$ for $t \in \mathbb{R}$. If A is a global attractor for π, then, for any bounded set $B \subset \mathbb{R}^d$, $\tau \in \mathbb{R}$, we have dist $((t, x(t, \tau, B)), C_A) \to 0$ as $t \to \infty$. In this sense, the set C_A is a global attractor for (1.1). The structure of the solutions of (1.1) on C_A is completely determined by the discrete dynamical system π on A. If π is point dissipative, then there is a fixed point of π and therefore a p-periodic solution of (1.1).

Exercise 10.8. An autonomous differential system $\dot{x} = f(x)$ is p-periodic for any $p > 0$. For $p = 1$, consider the equation in Exercise 1.6.3 and find the global attractor for the Poincaré map π. Is there an invariant circle for π? What is the nature of the flow defined by π on this circle?

Remark 10.1. For an autonomous equation, $\dot{x} = f(x)$, in $\mathbb{R}^d$, $d = 1$, we have seen that the existence of a bounded solution implies the existence of an equilibrium point.

We will show in the next section that the same is true for $d = 2$. If $d = 3$, it is possible to construct an example (but it is not trivial) for which *every* solution is bounded and yet there is no equilibrium point. On the other hand, if the equation is point dissipative, then there is an equilibrium point.

Exercise 10.9. Consider the equation

$$\dot{x}_1 = x_2 + x_1(1 - r^2), \quad \dot{x}_2 = -x_1 + x_2(1 - r^2) + \epsilon \cos 2\pi t$$

where $r^2 = x_1^2 + x_2^2$ and ϵ is a real parameter. Let π_ϵ be the Poincaré map for this system. Show that there is an $\epsilon_0 > 0$ such that, for $0 \le \epsilon < \epsilon_0$, the map π_ϵ is point dissipative and that there is a fixed point of π_ϵ. Speculate about the nature of the global attractor and the nature of the flow.
Hint. Compute the derivative of r^2 along the solutions of the equation.

1.10.2. Poincaré Map of a Periodic Orbit.

Suppose that γ is a periodic orbit of an autonomous equation (6.1). Our objective is associate with this periodic orbit a mapping, called the *Poincaré mapping*, on a $(d-1)$ surface transversal to a point on γ. To do this, we need precise definitions of these terms.

If $U \subset \mathbb{R}^d$ is an open set, we say that $\Sigma \subset U$ is a *(d-1)-dimensional C^k-surface* if there is a function $F \in C^k(U, \mathbb{R})$ such that, for each $x \in U$, rank $\nabla F(x) = 1$ and $\Sigma = \{\, x \in U : F(x) = 0 \,\}$.

Suppose that $f \in C^k(U, \mathbb{R}^d)$ and $\Sigma \subset U$ is a $(d-1)$-dimensional C^k-surface. We say that Σ is *transverse to f at $\xi \in \Sigma$* if $\nabla F(\xi) \cdot f(\xi) \neq 0$. We also refer to Σ as a *transversal to f at ξ* or a *cross section at ξ*. If we let $T_\xi(\Sigma)$ denote the set of vectors tangent to Σ at ξ; that is, the set of vectors orthogonal to $\nabla F(\xi)$, then Σ is transverse to f at ξ if and only if

$$T_\xi(\Sigma) \oplus f(\xi) = \mathbb{R}^d \,.$$

Let γ be a periodic orbit of minimal period $p > 0$ of (6.1) and suppose that Σ is a transversal to f at ξ. Let

$$\begin{aligned} D(\pi) = \{\, x \in \Sigma : {} & \text{there exists a } \tau(x) > 0 \text{ such that} \\ & \varphi^{\tau(x)}(x) \in \Sigma,\ \varphi^t(x) \notin \Sigma \text{ for } 0 < t < \tau(x) \,\} \,. \end{aligned}$$

The *Poincaré map* $\pi : D(\pi) \to \Sigma$ is defined by $\pi(x) = \varphi^{\tau(x)}(x)$. From Theorem 3.1, we know that $D(\pi)$ is not empty and we know also that $\pi \in C^k(D(\pi), \Sigma)$.

It is intuitively clear that, if we understand the discrete dynamical system defined by π, then we will understand the dynamical system defined by (6.1) in a neighborhood of the periodic orbit γ. We will make this more precise at a later time.

Exercise 10.10. For the equation in Exercise 1.6.3, discuss the behavior of iterates of the Poincaré map for the transversal $\Sigma = \{\, (x_1, 0) : x_1 > 0 \,\}$.

Exercise 10.11. Suppose that γ is a periodic orbit of an autonomous differential equation in $\mathbb{R}^2$ and let π be the Poincaré map associated with a transversal to this orbit at some point. The transversal is the image of a line segment and therefore has the natural order of the real numbers. Show that π is a monotone function. If $q \in \gamma$ and $0 < D\pi(q) < 1$, show that γ is a local attractor. If $D\pi(q) > 1$, show that γ is unstable.
Hint. For the first part, you must use the *Jordan Curve Theorem*: Every closed curve $\mathcal{C}$ in the plane separates the plane into two open disconnected sets called the inside and outside of $\mathcal{C}$. For the other parts, you need a simple estimate using Taylor's Theorem.

1.10.3. General Poincaré Maps.

It is clear that we can define Poincaré maps in a more general setting than in the previous section. In fact, if $f \in C^k(U, \mathbb{R}^d)$ and $\Sigma \subset U$ is a transversal to f at $\xi \in \Sigma$, then we can define the Poincaré map in the same way as the previous section.

Remark 10.2. In this general setting, it may be that $D(\pi)$ is the empty set. In fact, for the equation $\dot{x}_1 = -x_1$, $\dot{x}_2 = -x_2$, if Σ is the vertical line $\Sigma = \{ (1, x_2) \}$, then $D(\pi)$ is empty.

Example 10.12. If (1.1) is a periodic system as defined in Section 1.6.2 and if we identify $\{ 0 \} \times \mathbb{R}^d$ with $\{ p \} \times \mathbb{R}^d$, then (1.1) can be regarded as a differential equation on a cylinder with the new differential equation being $\dot{s} = 1 \pmod p$, $\dot{x} = f(s, x)$. The set $\Sigma = \{ 0 \} \times \mathbb{R}^d$ is obviously transverse to the vector field $(1, f(x))$ at every $x \in \mathbb{R}^d$. The Poincaré map obtained in this way is the same as the one obtained in Section 1.6.2.

Example 10.13. Let $T^2 = \{ \theta = (\theta_1, \theta_2) : 0 \le \theta_j \le 1, j = 1, 2 \}$ be a torus and consider the differential equation $\dot{\theta}_1 = 1$, $\dot{\theta}_2 = \alpha \pmod 1$ on T^2. It is clear that $\Sigma = \{ (0, \theta_2) : 0 \le \theta_2 \le 1 \}$ is a transversal to the vector field at each point of Σ. The Poincaré map π on Σ is defined by $\pi(0, \xi) = (0, \alpha + \xi) \pmod 1$. Recall that you have discussed in Exercise 1.6.5 the properties of this flow as a function of α. Interpret your result for the map π.

1.10.4. Discretizations.

In the applications, it is necessary to use numerical methods to obtain specific quantitative information about the solutions of a differential equation. If this is done on a digital computer, then we become involved with maps. In this section, we illustrate this for a scalar equation and a particularly simple discretization.

For a fixed constant a, let us consider the scalar logistic equation

$$\dot{x} = ax(1 - x)$$

on the finite interval $[0, \alpha]$. Let us use Euler's method for approximating the solutions of this equation. If we subdivide $[0, \alpha]$ into N equal parts of length h, let $X^{(n)} = x(nh)$, replace $\dot{x}(nh)$ by $[x(nh+h) - x(nh)]/h$, then $X^{(n+1)} = X^{(n)}$ for $n \geq 0$, where the map $T : \mathbb{R} \to \mathbb{R}$ is defined by

$$Ty = by(\frac{1+b}{b} - y)$$

where $b = ha$. Thus, $X^{(n)} = T^n X^{(0)}$ is the n^{th} iterate of $x^{(0)}$ by the map T.

If we make the change of variables $y = ((1+b)/b)z$, then we obtain the equivalent map œ$\tilde{T} : \mathbb{R} \to \mathbb{R}$ given by the quadratic map (the *logistic map*)

(10.2) $$\tilde{T}z = \lambda z(1 - z)\,,$$

where $\lambda = 1 + b$.

For h small, we expect that the n^{th} iterate of the map T and therefore $\tilde{T}$ will behave in much the same way as the solution x of the differential equation evaluated at nh. It is not too difficult to show that this is the case and, in fact, we did this when we were explaining a proof of the Peano theorem of existence of solutions using ϵ-approximate solutions. Of course, the degree of smallness of the step size h depends upon the vector field and, in this example, upon the size of the constant a. If the quantity b is too large, then the iterates of T and therefore of $\tilde{T}$ may have nothing whatsoever to do with the solution x. To see why this will be the case, let us mention only that the map must at least be monotone if the iterates are to have properties similar to the differential equation. There are two fixed points of $\tilde{T}$ given by 0, $\xi_\lambda = 1 - \frac{1}{\lambda}$. If $2 < \lambda < 3$, then $D\tilde{T}(\xi_\lambda) < 0$ and so the map will not be monotone in a neighborhood of 0. Phenomena can occor in the map which does not occur in the differential equation! This makes it clear that much effort is needed to determine if the results that are obtained on the computer are realistic interpretations of the solutions of differential equations.

In a later section, we will discover that the logistic map exhibits many important phenomena encountered in dynamical systems.

Exercise 10.12. Use Taylor's Theorem to show that the fixed point ξ_λ for the logistic map is a local attractor for $1 < \lambda < 3$ and discuss the manner in which the iterates of $\tilde{T}$ approach ξ_λ.

æ