1.6. Autonomous systems.

A differential equation is said to be *autonomous* if the vector field does not depend upon the independent variable t. In this section, we give a few general but very important properties of autonomous systems.

If $U \subset \mathbb{R}^d$ is an open set, $f \in C^r(U, \mathbb{R}^d), r \geq 1$, we consider the autonomous differential equation

$$\dot{x} = f(x)\,. \tag{6.1}$$

Since we are assuming that f is at least C^1, Theorem 3.1 implies that, for any $\tau \in \mathbb{R}$, $\xi \in U$, there is a unique solution $x(t,\, \tau,\, \xi)$ of (6.1) satisfying $x(\tau,\, \tau,\, \xi) = \xi$.

Since the differential equation (6.1) remains invariant under translation in the independent variable, it follows that, for any $\tau \in \mathbb{R}$, $x(t-\tau, 0, \xi)$ is a solution. Thus, $x(t-\tau, 0, \xi)$ and $x(t, \tau, \xi)$ are solutions of the equation (6.1) with the same values at the initial time $t = \tau$. It follows from the uniqueness of the solution of the initial value problem that $x(t,\, \tau,\, \xi) = x(t-\tau,\, 0,\, \xi)$ for all t for which the solution is defined. Therefore, there is no loss in generality in assuming that the initial time is 0. We denote by $\varphi^t(\xi)$ the maximal solution of (6.1) satisfying $\varphi^0(\xi) = \xi$ on the interval I_ξ.

If x is a solution of (6.1) on a maximal interval I, the *trajectory* of x is the set in $\mathbb{R} \times \mathbb{R}^d$ defined by $\{\, (t,\, x(t)) : t \in I \,\}$. The *orbit* $\gamma(x)$ of x is the set in $\mathbb{R}^d$ defined by $\gamma(x) = \{\, x(t) : t \in I \,\}$. We also designate the orbit of x sometimes as $\gamma(\xi)$ where ξ is a point in $\gamma(x)$. The *phase space* for (6.1) is the subset U of $\mathbb{R}^d$ and thus the orbit $\gamma(x)$ is the projection along $\mathbb{R}$ in $\mathbb{R} \times \mathbb{R}^d$ of the trajectory of x into the phase space. The orbit of x is uniquely defined. However, there are many natural parametrizations of $\gamma(x)$. In fact, $\gamma(x(\cdot)) = \gamma(x(\cdot + c))$ for any real number c.

Theorem 6.1. *For any $\xi \in U$, the function $\varphi^t(\xi)$ satisfies the following properties for $t,\, s,\, t+s \in I_\xi$:*
(i) $\varphi^0(\xi) = \xi$
(ii) $\varphi^t(\varphi^s(\xi)) = \varphi^{t+s}(\xi)$
(iii) *$\varphi^t(\xi)$ is a C^r-function in x and a C^{r+1}-function in t if $f \in C^r(U, \mathbb{R}^d)$.*

Proof. *(i)* is by definition, *(ii)* is a consequence of the uniqueness of solutions and *(iii)* follows from Theorem 3.1.

We refer to the collection of orbits of (6.1) together with the sense of direction in time along these orbits as the *flow defined by (6.1).* If (6.1) has the property that there is an interval I such that, for each $(0,\, \xi) \in U$, we have $\varphi^t(\xi)$ defined for all $t \in I$, then we say that (6.1) defines a *C^r-dynamical system on I*. The most interesting dynamical systems are those for which the interval I is either $[0,\, \infty)$ or $(-\infty,\, \infty)$. The collection $\{\, \varphi^t,\, t \in I \,\}$ of maps is called a *group of transformations* if $I = (-\infty,\, \infty)$ and and a *semigroup of transformations* if $I = [0,\, \infty)$. In the literature, a dynamical system on $[0,\, \infty)$ is sometimes referred to as a semidynamical system.

In Section 1.5, we have discussed in some detail the special case of (6.1) corresponding to case where $f(x)$ is the linear vector field Ax, where A is a $d \times d$ constant matrix. Each solution of the equation is defined for all $t \in (-\infty, \infty)$. The mappings $\{ \varphi^t, t \in I \}$ are given by the family of matrices $\{ e^{At}, t \in I \}$ and represent a group of transformations on $\mathbb{R}^d$.

If $\gamma(\xi) = \{ \xi \}$, then ξ is called a *critical point* or *equilibrium point*. It is clear that ξ is a critical point if and only if $f(\xi) = 0$. A point in U is said to be *regular* if it is not a critical point.

Exercise 6.1. Prove the following fact: *If $f \in C^r(U, \mathbb{R}^d)$, $r \geq 1$, x_0 is a critical point and $\varphi^t(\xi)$ $\xi \neq x_0$ tends to x_0 with increasing (or decreasing) t, then it must be that $t \to \infty$ (or $t \to -\infty$).*

A *closed curve* (or *Jordan curve*) in $\mathbb{R}^d$ is the homeomorphic image of a circle in $\mathbb{R}^d$. If $\gamma(\xi)$ is a closed curve, then it is called a *periodic orbit*. It is easy to see that $\gamma(\xi)$ is a periodic orbit if and only if $\varphi^t(\xi)$ is defined for all $t \in \mathbb{R}$ and there is a constant $p > 0$ such that $\varphi^{t+p}(\xi) = \varphi^t(\xi)$ for $t \in \mathbb{R}$ and $\varphi^t(\xi) \neq \xi$ for $t \in (0, p)$. We call such a solution *periodic of minimal period p*.

We want to reemphasize the difference between an orbit and a trajectory by using a periodic orbit as an example. If γ is a periodic orbit, then we can define a cylinder $C_\gamma \equiv \mathbb{R} \times \gamma$ in $\mathbb{R} \times \mathbb{R}^d$. For any point ξ on γ, if $\varphi^t(\xi)$ is the solution through ξ at initial time 0, then $\varphi^t(\xi)$ is periodic of minimal period p and the trajectory is a curve on the cylinder C_γ and it goes around the cylinder. If we move to another point on γ, say a point $\zeta = \varphi^c(\xi)$ for some real $c \in (0, p)$, then the trajectory corresponding to the solution $\varphi^{t+c}(\xi)$ through ζ at initial time 0 is a different curve on C_γ. These two curves on the cylinder project onto the same orbit γ in $\mathbb{R}^d$. See Figure 6.1.

Figure 6.1. Periodic orbit vs periodic trajectory.

Since the sense of direction in time is lost when we look at orbits rather than trajectories, we use a pictorial convention by placing an arrow on the orbit to designate the direction that a particular solution is moving along the orbit as time increases.

Example 6.1. Consider the equation $\dot{x} = -x$. The solution $x = 0$ is an equilibrium point and therefore an orbit. If $\xi > 0$, then $\gamma(\xi) = \{\, \varphi^t(\xi), t \in \mathbb{R} \,\} = (0, \infty)$ and, if $\xi < 0$, then $\gamma(\xi) = (-\infty, 0)$. The solution is decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$.

Exercise 6.2. For each of the following equations, list all of the orbits and sketch them in the phase space:

$$
\begin{aligned}
&(i) \dot{x} = x(1-x)\,, \\
&(ii) \dot{x} = x(1-x^2)\,, \\
&(iii) \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1\,, \\
&(iv) \dot{x}_1 = x_2(x_2^2 - x_1^2), \quad \dot{x}_2 = -x_1(x_2^2 - x_1^2)\,.
\end{aligned}
$$

Exercise 6.3. Discuss the orbits and phase portrait for the equation

$$\dot{x}_1 = -x_2 + x_1(1-r^2), \quad \dot{x}_2 = x_1 + x_2(1-r^2)\,,$$

where $r^2 = x_1^2 + x_2^2$. *Hint*: Let $x_1 = r\,\cos\theta$, $x_2 = r\,\sin\theta$.

Exercise 6.4. On the circle, consider the equation $\dot{\theta} = a + \sin\theta \pmod{2\pi}$, where a is a positive constant. Show that every solution is periodic if $|a| > 1$. If $|a| < 1$, show that there are two eqiulibrium points. Draw the flows.

A torus is the homeomorphic image of the cross product of two circles. If we suppose that φ, ψ, $0 \le \varphi, \psi \le 2\pi$, are the angles describing the circles, then a point on the torus is represented by a coordinate pair (φ, ψ). A differential equation on a torus is given by a relation $\dot{\varphi} = A(\varphi, \psi)$, $\dot{\psi} = B(\varphi, \psi)$, where A, B are 2π-periodic functions of both variables. The flow on the torus also can be depicted in the (φ, ψ)-plane if we identify each of the lines $\{\,(\varphi, 2\pi k)\,\}$, k an integer, with the line $\{\,(\varphi, 0)\,\}$ and each of the lines $\{\,(2\pi k, \psi)\,\}$, k an integer, with the line $\{\,(0, \psi)\,\}$. In this coordinate system, it is sometimes easier to see the manner in which orbits rotate around a torus. The following exercise is concerned with the simplest flow possible on the torus and yet it illustrates that the flow may be complicated.

Exercise 6.5. Suppose that φ, ψ are the angles describing a torus and suppose that we have a differential equation on the torus for which the angles satisfy the differential equation $\dot{\varphi} = 1$, $\dot{\psi} = \omega$, where ω is a fixed constant. An orbit goes around the torus traversing the angle φ with period 2π and the angle ψ with period $\frac{2\pi}{\omega}$. If ω is rational, prove that every orbit is periodic. If ω is irrational, prove that every orbit is dense in the torus.

We now turn to the important topic which is concerned with the limits of orbits. In addition to the notation for an orbit $\gamma(\xi)$, we need the *positive orbit* $\gamma^+(\xi) = \{\, \varphi^t(\xi) : t \ge 0 \,\}$ and the *negative orbit* $\gamma^-(\xi) = \{\, \varphi^t(\xi) : t \le 0 \,\}$. If we do not want to distinguish a particular point on the orbit, we will write γ, γ^+, γ^- for the orbit, positive orbit, negative orbit, respectively.

The *ω-limit set* $\omega(\xi)$ and *α-limit set* $\alpha(\xi)$ of an orbit γ containing ξ are defined by the following relations:

$$\omega(\xi) = \cap_{\tau\geq 0}\overline{\cup_{t\geq\tau}\varphi^t(\xi)}$$

$$\alpha(\xi) = \cap_{\tau\leq 0}\overline{\cup_{t\leq\tau}\varphi^t(\xi)}\,.$$

These sets also can be defined by using sequences. For example, if we assume that the solution through ξ is defined for all $t \in \mathbb{R}$, then it is easy to see that a point $y \in \omega(\xi)$ (or $\alpha(\xi)$) if and only if there is a sequence of real numbers $\{\, t_k \,\}$, $t_k \to \infty$ (or $-\infty$) as $k \to \infty$ such that $\varphi^{t_k}(\xi) \to y$ as $k \to \infty$.

If B is a bounded set in $\mathbb{R}^d$, then we define $\varphi^t(B) = \cup_{\xi\in B}\varphi^t(\xi)$ and the ω-limit set of $B, \omega(B)$, and the α-limit set of $B, \alpha(B)$, by the relations

$$\omega(B) = \cap_{\tau\geq 0}\overline{\cup_{t\geq\tau}\varphi^t(B)}$$

$$\alpha(B) = \cap_{\tau\leq 0}\overline{\cup_{t\leq\tau}\varphi^t(B)}$$

An equivalent formulation of these definitions will involve double sequences. In fact, if we assume that all solutions are defined for all $t \geq 0$, then it is possible to show that $y \in \omega(B)$ if and only if there are sequences of real numbers $t_k \to \infty$ as $k \to \infty$ and points $x_k \in B$ such that $\varphi^{t_k}(x_k) \to y$ as $k \to \infty$. A similar definition holds for $\alpha(B)$.

Exercise 6.6. Give a complete proof that the definition of $\omega(B)$, $\alpha(B)$ in terms of double sequences is equivalent to the original one.

It might have seemed reasonable to take the definition of $\omega(B)$ to be the set $\cup_{\xi\in B}\omega(\xi)$. It is clear that $\omega(B) \supset \cup_{\xi\in B}\omega(\xi)$. On the other hand, it is possible to give an example so that

$$\omega(B) \neq \cup_{\xi\in B}\omega(\xi)\,. \tag{6.2}$$

In fact, this latter situation is to expected except in the most trivial types of problems.

We give an example of a scalar equation for which (6.2) is satisfied. Consider the equation $\dot{x} = x - x^3$. There are three equilibrium points 0, ± 1 and, for any $\xi \in (-\infty, \infty)$, we have that $\omega(\xi)$ is one of these three points. On the other hand, if B is the interval $(-2, 2)$, then $\omega(B) = [-1, 1]$ and (6.2) is not satisfied. By taking the union of the ω-limit sets of points, we discover all of the equilibrium points, but this set does not reveal the fact that the equilibrium points are connected by orbits. This information is contained in the above definition of $\omega(B)$.

Exercise 6.7. For the equation $\dot{x}_1 = x_1 - x_1^3$, $\dot{x}_2 = -x_2$, determine $\cup_{\xi\in\mathbb{R}^2}\omega(\xi)$. If B is the interval $\{\,(x_1, x_2) : x_2 = 1, -0.5 < x_1 < 0.5\,\}$, determine $\omega(B)$. If B is the interval $\{ (x_1, x_2) : x_2 = 1, -2 < x_1 < 2 \}$, determine $\omega(B)$. If B is the interval $\{ (x_1, x_2) : x_2 = 0, -0.5 < x_1 < 0.5 \}$, determine $\omega(B)$.

A set $A \subset \mathbb{R}^d$ is an *invariant set* of (6.1) if $\varphi^t(A) = A$ for all $t \geq 0$. A set $A \subset \mathbb{R}^d$ is a *positively invariant set* of (6.1) if $\varphi^t(A) \subset A$ for all $t \geq 0$. A set $A \subset \mathbb{R}^d$ is a *negatively invariant set* of (6.1) if $\varphi^t(A) \subset A$ for all $t \leq 0$.

Exercise 6.8. If B is a closed positively invariant set, show that

$$\omega(B) = \cap_{t \geq 0} \varphi^t(B)\,.$$

Exercise 6.9. For each of the examples in Exercises 6.2, 6.3 and 6.5, discuss the ω-limit sets and α-limit sets of all orbits. What is the largest bounded invariant set in each of these examples? Is this largest bounded invariant set the ω-limit set of some bounded set B? What are some positively invariant sets which are bounded?

To state the next result, we need some additional notation. If B is a set in $\mathbb{R}^d$ and x is a point in $\mathbb{R}^d$, define the *distance* $\mathrm{dist}(x, B)$ *from* x *to* B as $\mathrm{dist}\,(x, B) = \inf_{b \in B} |x - b|$. If A, B are sets in $\mathbb{R}^d$, we define the *distance* $\mathrm{dist}(A, B)$ *from* A *to* B as

$$\mathrm{dist}(A, B) = \sup_{a \in A} \mathrm{dist}(a, B) = \sup_{a \in A} \inf_{b \in B} |a - b|.$$

Notice that $\mathrm{dist}(A, B) \neq \mathrm{dist}(B, A)$ in general.

Theorem 6.2. *Suppose that $B \subset \mathbb{R}^d$ is nonempty, bounded and $\varphi^t(B)$ is defined for $t \geq 0$(resp., $t \leq 0$). If $\gamma^+(B)$ (resp., $\gamma^-(B)$) is bounded, then $\omega(B)$ (resp., $\alpha(B)$) is nonempty, compact, invariant and dist$(\varphi^t(B), \omega(B)) \to 0$ as $t \to \infty$ (resp., dist$(\varphi^t(B), \alpha(B)) \to 0$ as $t \to -\infty$). If, in addition, B is connected, then $\omega(B)$ (resp., $\alpha(B)$) is connected.*

Proof. We prove the theorem only for $\omega(B)$ since the other case is similar. From the definition, it is clear that $\omega(B)$ is nonempty and compact. We now prove that $\mathrm{dist}(\varphi^t(B), \omega(B)) \to 0$ as $t \to \infty$. If this is not the case, then there is an $\epsilon > 0$, a sequence $t_j \to \infty$ as $j \to \infty$ and a sequence $y_j \in B$ such that $\mathrm{dist}(\varphi^{t_j}(y_j), \omega(B)) > \epsilon$ for $j = 1, 2 \ldots$. Since the set $\{ \varphi^{t_j}(B), j \geq 1 \}$ belongs to a compact set, there is a convergent subsequence of the $\{ \varphi^{t_j} y_j \}$ which we label the same. Since the limit of this sequence must belong to $\omega(B)$, we obtain a contradiction.

Using an argument very similar to the one in the previous paragraph, it is easy to show that $\mathrm{dist}(\varphi^t(B), \omega(B)) \to 0$ as $t \to \infty$ implies that $\omega(B)$ is connected if B is connected.

It remains to show that $\omega(B)$ is invariant. From the fact that φ^t is continuous in t, we can deduce that $\varphi^t(\omega(B)) \subset \omega(B)$. In fact, suppose that $y \in \omega(B)$. Then there is a sequence $t_j \to \infty$ as $j \to \infty$ and a sequence $y_j \in B$ such that $\varphi^{t_j}(y_j) \to y$ as $j \to \infty$. For any fixed $t \in (-\infty, \infty)$, there is an integer n_0 such that $t + t_j \geq 0$ for

$j \geq n_0$. Therefore, $\varphi^{t+t_j}(B)$ is defined for $j \geq n_0$. Also, $\varphi^{t+t_j}(y_j) \to \varphi^t(y)$ as $j \to \infty$. This shows that the orbit through y belongs to $\omega(B)$. From the definition, it follows immediately that $\omega(B) \subset \varphi^t(\omega(B))$ and thus $\omega(B)$ is invariant. This completes the proof of the theorem.

Remark 6.2. We want to emphasize that $\omega(B)$ is invariant and so $\varphi^t(\omega(B))$ is defined for all $t \in (-\infty, \infty)$ even though we may not have $\varphi^t(B)$ defined for all $t \in (-\infty, \infty)$. Therefore, (6.1) may define a dynamical system only on the interval $[0, \infty)$ and yet the sets $\omega(B)$ are invariant.

An important special case of Theorem 6.2 is the following statement.

Corollary 6.1. *If $B = \{\xi\}$ for a fixed $\xi \in \mathbb{R}^d$ and $\gamma^+(\xi)$ (respectively, $\gamma^-(\xi)$) is bounded, then $\omega(\xi)$ (respectively, $\alpha(\xi)$) is nonempty, compact and connected.*

Exercise 6.10. Give an example of a two dimensional system which has an orbit whose ω-limit set is not empty and disconnected.

In later sections, we need the following important concept. A set J is said to be a *minimal set* of (6.1) if J is closed, invariant and, if $J_1 \subset J$ is a closed invariant set, then $J_1 = J$.

Theorem 6.3. *If E is a nonempty compact invariant of (6.1), then there exists a minimal set in E.*

Proof. Let $\mathcal{E} = \{S \subset E : S \text{ is a nonempty compact invariant set}\}$. Define a partial ordering on $\mathcal{E}$ as follows: $S_1 \leq S_2$ if and only if $S_1 \subset S_2$. If $\mathcal{E}_0 \subset \mathcal{E}$ is totally ordered and $J = \cap_{s \in \mathcal{E}_0} S$, then J is a compact invariant set. The family $\mathcal{E}_0$ has the finite intersection property. Indeed, if S_1, S_2 are in $\mathcal{E}_0$, then either $S_1 < S_2$ or $S_2 < S_1$ and, in either case, $S_1 \cap S_2$ is nonempty compact invariant and belongs to $\mathcal{E}_0$. The same holds true for any finite collection of sets of $\mathcal{E}_0$. Thus, J is a nonempty compact invariant set and J is a least element. Since each totally ordered subfamily of $\mathcal{E}$ has a least element, it follows from Zorn's lemma that there is a minimal element of $\mathcal{E}$. This completes the proof.

The properties of the flow of a differential equation can be used sometimes to assert the existence of a zero of a vector valued function.

Theorem 6.4. *If K is a positively invariant compact set of (6.1) and K is homeomorphic to the unit ball in $\mathbb{R}^d$, then there is at least one equilibrium point of (6.1) in K.*

Proof. For any $\tau_1 > 0$, consider the mapping $\xi \mapsto \varphi^{\tau_1}(\xi)$. From The Brouwer Fixed Point Theorem, there is a $\xi_1 \in K$ such that $\varphi^{\tau_1}(\xi_1) = \xi_1$ and, thus, a periodic orbit of (6.1) of period τ_1. We choose a sequence $\tau_m > 0$, $\tau_m \to 0$ as $m \to \infty$ and the corresponding points ξ_m so that $\varphi^{\tau_m}(\xi_m) = \xi_m$. Since K is compact, we may assume that this sequence converges to a point $\xi^* \in K$ as $m \to \infty$. For any t and any m, there is an integer $k_m(t)$ such that $k_m(t)\tau_m \le t < k_m(t)\tau_m + \tau_m$ and $\varphi^{k_m(t)\tau_m}(\xi_m) = \xi_m$ for all t since $\varphi^t(\xi_m)$ is periodic of period τ_m in t. Furthermore,

$$\begin{aligned}|\varphi^t(\xi^*) - \xi^*| &\le |\varphi^t(\xi^*) - \varphi^t(\xi_m)| + |\varphi^t(\xi_m) - \xi_m| + |\xi_m - \xi^*| \\ &= |\varphi^t(\xi^*) - \varphi^t(\xi_m)| + |\varphi^{t-k_m(t)\tau_m}(\xi_m) - \xi_m| + |\xi_m - \xi^*|\,.\end{aligned}$$

Since the right hand side of this inequality approaches zero as $m \to \infty$, we have that ξ^* is an equilibrium point and the theorem is proved.

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