1.9. The principle of Wazewski.

To motivate the discussion of the results of this section, let us first introduce the following concepts for the differential equation

$$\dot{x} = f(x) \tag{9.1}$$

in $\mathbb{R}^d$.

Defiinition 9.1. Suppose that U is a given set in $\mathbb{R}^d$. A set A is said to be the *maximal invariant set* in U if $A \subset U$, A is an invariant set for (9.1) and, if $K \subset U$ is invariant for (9.1), then $K \subset A$.

Defiinition 9.2. An invariant set A for (9.1) is said to be an *isolated invariant set* if there is an open neighborhood U of A such that A is the maximal invariant set in $\bar{U}$. The set U is said to be an *isolating neighborhood of* A.

We remark that, if U is an open set in $\mathbb{R}^d$ and if the maximal invariant set A in U is closed, then A is an isolated invariant set and U is an isolating neighborhood of A.

An important problem in differential equations is to know if the maximal invariant set in an open set U is not empty. In our study of Liapunov functions in the previous section, we have given conditions under which this is true. In fact, suppose that $V \in C^1(\mathbb{R}^d l; \mathbb{R})$ and $U = \{x \in \mathbb{R}^d : V(x) < k\}$. If U is nonempty and bounded, then every orbit with initial data in U remains in U, is bounded and has a nonempty ω-limit set in $\bar{U}$. If we further assume that $\dot{V} \leq 0$ in U, then the ω-limit set of any orbit with initial data in U must be in U and the maximal invariant set A in U is not empty. In fact, Corollary 8.1 implies that A is compact, nonempty and is a local attractor for U. Therefore, A is an isolated invariant set and U is an isolating neighborhood.

From this point of view, the principle of Wazewski which we present here is an important extension of the method of Liapunov functions for determining the asymptotic behavior of the solutions of differential equations. Since the principle requires rather sophisticated concepts from analysis and topology, we give several examples to bring out the ideas.

Example 9.1. Consider a scalar differential equation; that is, (9.1) with $d = 1$, and suppose that there is an interval $U = (a, b)$ such that $f(a) < 0$ and $f(b) > 0$. Then there must be a zero of f in U and thus the maximal invariant set in U is not empty.

In the previous example, the hypotheses on the vector field imply that every solution beginning on ∂U must leave U. Of course, if we replace t by $-t$, then every solution beginning on ∂U must enter U and we have the same situation as in Corollary 8.1. A more interesting situation is when some solutions on the boundary of U enter U and some solutons leave U. The next example illustrates this point.

Example 9.2. Consider the equation

$$\dot{x}_1 = -x_1^3, \quad \dot{x}_2 = x_2^3 \tag{9.2}$$

and let U be the square $U = (-1,1) \times (-1,1)$. If we define $U_+ = \{(x_1, 1) : x_1 \in [-1,1]\} \cup \{(x_1, -1) : x_1 \in [-1,1]\}$, then U_+ is the *egress set* from U and is that part of ∂U which has the property that the solution with initial data in U_+ leaves U_+ for positive time. The corresponding *ingress set* $U_- = \{(-1, x_2) : x_2 \in (-1,1)\} \cup \{1, x_2) : x_2 \in (-1,1)\}$ of U has the property that the solution with initial data in U_- enters U_- for positive time.

Let us now make a perturbation of (9.2) to the equation

$$\dot{x}_1 = -x_1^3 + g_1(x_1, x_2), \quad \dot{x}_2 = x_2^3 + g_2(x_1, x_2), \tag{9.3}$$

where the function $g(x) = (g_1(x), g_2(x))$ is C^1 and so small that the sets U_+ and U_- are respectively the egress and ingress sets for the rectangle U.

We now prove the intuitive obvious fact that there must be some solution of (9.3) that enters U and remains in U for all positive time. To do this, we define $\varphi(t, x_0) = (\varphi_1(t, x_0), \varphi_2(t, x_0))$ be the solution of (9.3) through x_0 and define

$$\begin{aligned}
S^* &= S_+^* \cup S_-^*, \\
S_+^* &= \{(-1,1)\} \cup \{x_0 \in U_- : \exists \tau \text{ with } \varphi_2(\tau, x_0) = 1\}, \\
S_-^* &= \{(-1,-1)\} \cup \{x_0 \in U_- : \exists \tau \text{ with } \varphi_2(\tau, x_0) = -1\}.
\end{aligned}$$

The set S^* represents the set of initial values in the closure $\overline{U_-^*}$ of U_- of those solutions of (9.3) which leave the region U at some time, S_+^* (resp. S_-^*) the ones which leave the top (resp. bottom) of the rectangle.

If we know that S_+^* and S_-^* are open sets in the relative topology of S^*, then the fact that $S_+^* \cap S_-^* = \emptyset$ implies that $\overline{U_-^*} \setminus S^*$ is not empty; that is, there is an $x_0 \in U_-$ such that the solution $x(t, x_0) \in U$ for $t > 0$. In particular, this solution is bounded, has ω-limit set in U and thus there is an invariant set in U. There also is a maximal invariant set in U which is compact (why?).

It remains to show that S_+^* and S_-^* are open sets in the relative topology of S^*. Let $x_0 \in S_+^*$, $x_0 \neq (-1,1)$ and let τ be such that $\varphi_2(t, x_0) = 1$. Then there exists an $\epsilon > 0$ such that $\varphi_2(t, x_0) > 1$ for $\tau < t \leq \tau + \epsilon$. Choose a neighborhood V of $\varphi(\tau + \epsilon, x_0)$ such that $y = (y_1, y_2) \in V$ implies that $y_2 > 1$. By continuous dependence on initial data, there is a neighborhood W of x_0 such that, for every $z \in W$, there is a $\bar{\tau}(z)$ such that the solution $\varphi(t, z)$ of (9.3) through z satisfies $\varphi(\bar{\tau}(z), z) \in V$. In particular, $\varphi_2(\bar{\tau}(z), z) > 1$. As a consequence, there is a $\tau(z)$ such that $\varphi(\tau(z), z) = 1$. This shows that S_+^* is open. A similar argument shows that S_-^* is open.

What is the nature of the compact invariant set in the rectangle U? Without more information about the perturbation U, we can say very little since the equilibrium point $(0,0)$ of (9.2) is not hyperbolic. For an arbitrary perturbation g satisfying the above general limitations, there must be at least one equilibrium point (why?). If we choose the perturbation g so that the system remains a gradient system, then the invariant set in U consists only of equilibrium points and connections between them. It is possible to choose the gradient system so that the perturbed equation has exactly 9 equilibrium points (show this). If we allow the perturbation g to vary in the class of non-gradient systems, then it is possible to obtain a periodic orbit (construct such a perturbation).

Exercise 9.1. Consider the three dimensional system

$$\begin{aligned}
\dot{x}_1 &= -x_1(x_1^2+x_2^2)^3 + x_2 \\
\dot{x}_1 &= -x_2(x_1^2+x_2^2)^3 - x_1 \\
\dot{x}_3 &= x_3^3.
\end{aligned}$$

In polar coordinates, we have the system

$$\dot{\theta} = 1, \quad \dot{r} = -r^3, \quad \dot{x}_3 = x_3^3.$$

Let $U = \{(\theta, r, x_3) : \theta \in [0, 2\pi], 0 \le r < 1, -1 \le x_3 \le 1\}$, $U_+ = \{(\theta, r, 1) : \theta \in [0, 2\pi], 0 \le r < 1\} \cup \{(\theta, r, -1) : \theta \in [0, 2\pi], 0 \le r < 1\}$, $U_- = \{(\theta, 1, x_3) : \theta \in [0, 2\pi], x_3 \in (-1, 1)\}$. Choose a perturbation g of the vector field so small that U_+ (resp. U_-) is the egress (resp. ingress) set for U. Show that there must be a nonempty invariant set in U for the perturbed system. Must there be a periodic orbit? Can the perturbation be made in such a way that there is no equilibrium point in U?

Let us now generalize the ideas in the previous examples to d-dimensions. We consider the differential equation (9.1) and assume that f is a C^1-function. We let $\varphi(t, x_0$ denote the solution of (9.1) through x_0, let $(\alpha(x_0), \beta(x_0))$ denote the maximal interval of existence of $\varphi(t, x_0)$, and, for any interval $I \in (\alpha(x_0), \beta(x_0))$, let $\varphi(I, x_0) = \{\varphi(t, x_0) : t \in I\}$.

Definition 9.3. Let U be an open set in $\mathbb{R}^d$. A point $x_0 \in \partial U$ is a point of *egress* from U with respect to (9.1) if there is a $\delta > 0$ such that $\varphi([-\delta, 0), x_0) \subset U$. An egress point x_0 is a point of *strict egress* from U if there a $\delta > 0$ such that $\varphi((0, \delta], x_0) \subset \mathbb{R}^d \setminus \bar{U}$. The set of points of egress from U is denoted by S and the set of points of strict egress from U is denoted by S^*.

Definition 9.4. If $A \subset B$ are any two sets of a topological space and $K : B \to A$ is a continuous map with $K(P) = P$ for all points P in A, then K is said to be a *retraction* from B to A and A is said to be a *retract* of B.

With these definitions, we are in a position to prove the following result which is known as the *principle of Wazewski*.

Theorem 9.1. *If $S = S^*$ and there is a set $Z \subset U \cup S$ such that $Z \cap S$ is a retract of S but not a retract of Z, then there is at least one point x_0 in $Z \setminus S$ such that $\varphi([0, \beta(x_0)), x_0) \subset U$. In particular, if U is a bounded set and the above hypotheses are satisfied, then there is a positive orbit which remains in U and thus an invariant set in U.*

Proof. For any point $x_0 \in U$ for which $\varphi([0, \beta(x_0)), x_0) \cap (\mathbb{R}^d \setminus U) \neq \emptyset$, there is a time t_{x_0} for which $\varphi(t_{x_0}, x_0) \in S$ and $\varphi(t, x_0) \in U$ for $t \in [0, t_{x_0})$. The point $\varphi(t_{x_0}, x_0)$ is called the *consequent* of x_0 and denoted by $C(x_0)$. The set of points in U for which a consequent exists is designated by G and is called the *left shadow* of S.

Suppose now that $S = S^*$ and define the map $K : G \cup S \to S$, $K(x) = C(x)$ for $x \in G$, $K(x) = x$ for $x \in S$. We first prove that K is continuous. If $x \in U$ and $C(x) = \varphi(t_x, x)$, then $S = S^*$ implies that there is a $\delta > 0$ such that $\varphi((-\delta, 0), x) \subset U$, $\varphi((0, \delta), x) \subset \mathbb{R}^d \setminus U$. Since $\varphi(s, x)$ is continuous in (s, x), for any $\epsilon > 0$, there is an $\eta > 0$ such that $|\varphi(s, y) - \varphi(s, x)| < \epsilon$ for $s \in (-\delta, \delta)$ if $|y - x| < \eta$. This clearly implies that $C(y) \to C(x)$ if $y \to x$. If $x \in S = S^*$, then one repeats the same type of argument to obtain that K is continuous.

Since K is continuous, K is a retract of $G \cup S$ into S.

If the conclusion of the theorem is not true, then $Z \setminus S \subset G$, the left shadow of S. Thus, $Z \subset G \cup S$. Since $Z \cap S$ is a retract of S, there is a mapping $H : S \to Z \cap S$ such that $H(x) = x$ for $x \in Z \cap S$. The map $HK : G \cup S \to Z \cap S$ is continuous and $HK(x) = x$ for $x \in Z \cap S$. Thus, $G \cup S$ is a retract of $Z \cap S$. Since $z \subset G \cup S$, the map $HK : Z \to Z \cap S$ is a retraction of z onto $Z \cap S$. This contradiction proves the theorem.

Let us see how to apply this result to the Example 9.2.

Example 9.2 (*Revisited*). For Equation (9.3) with the imposed smallness conditions on g, it is clear that $S = S^*$. For the set Z, choose $Z = \{(-1, x_2) : x_2 \in [-1, 1]\}$. Then $Z \cap S$ is a retract of S but not a retract of Z. Therefore, the conclusion of Theorem 9.1 holds true. What we actually did in Example 9.2 was to prove Theorem 9.1 for the special case of Equation 9.3.

Exercise 9.2. Redo Exercise 9.1 using Theorem 9.1.

Exercise 9.3. Consider the scalar equation $\dot{x} = f(t, x)$, $t \geq 0$, and suppose that there is a $\delta > 0$ such that $xf(t, x) \geq \delta$ for $|x| = 1$. Show that there is a solution $x(t)$ such that $|x(t)| < 1$ for $t \geq 0$.
Hint. Rewrite the equation as $\dot{x}_1 = 1, \dot{x}_2 = f(x_1, x_2)$ and use the principle of Wazewski.

Exercise 9.4. Consider the system

$$\begin{aligned}
\dot{x}_1 &= -x_1^3 + f_1(t, x_1, x_2) \\
\dot{x}_2 &= x_2^3 + f_2(t, x_1, x_2).
\end{aligned}$$

Give conditions on f_1, f_2 on the sets $U_+ = \{(x_1, 1) : x_1 \in [-1, 1]\} \cup \{(x_1, -1) : x_1 \in [-1, 1]\}$, $U_- = \{(-1, x_2) : x_2 \in (-1, 1)\} \cup \{1, x_2) : x_2 \in (-1, 1)\}$, to ensure that there is a solution satisfying $|x_1(t)| < 1, |x_2(t)| < 1$.

æ