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Exotic Smoothings of open 4-manifolds

X^4 any 4-manifold

genus function $G: H_2(X) \to \mathbb{Z}^{\geq 0}$
$G(\alpha)$ = min genus of a smoothly embedded surface representing α

Adjunction inequality: can bound G using Seiberg-Witten ...

Open 4-manifolds: $\mathbb{R}^4$ many smooth strs.

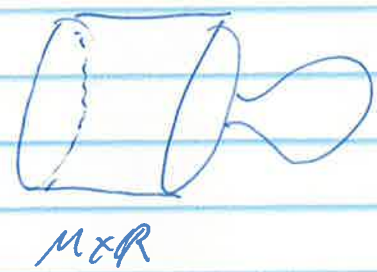
[G-Stipsicz] book 9.4 - survey upto mid '90's

Sample Thm: $\forall$ open X^4 admits uncountably many smoothings provided X embeds topologically in $\#_n \mathbb{CP}^2$ or there is a finite cover $\tilde{X}$ - smoothly... in $\#_n \mathbb{CP}^n$ or $\tilde{X} - K$ embeds ...

Example: M^3 closed oriented
$M \times \mathbb{R}$ frequently admits uncountably smoothings

Every $M^3 \times \mathbb{R}$ admits infinitely many smoothings (Bizaca-Etnyre)

$\Rightarrow$ Taylor invariant



Problems: 1 lack of invariants
2 Constructions are indirect
3 Don't affect internal structure (G)

Thm: let X be interior of a 2-handle body
(0, 1, 2-handles, possibly
infinitely many)

1 X admits an exotic smooth str.
2 X admits infinitely many if $H_2(X) \neq 0$
or X is not a $K(\pi, 1)$
3 X admits uncountably many if $H_2(X)$
is not finitely generated

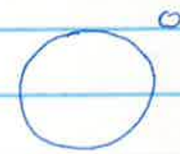
(Known from Taylor inv't if $H_2(X) \neq 0$
is finitely gen.)

Examples: 1 $\mathbb{R}^2$-bundles over surfaces
(if surface $\mathbb{RP}^2$, nothing
previously
was known)
now uncountably many smoothings

(2) Oriented $\mathbb{R}^2$-bundle over oriented surface F

any $g \geq g(F)$ is $G(\text{gen})$ for uncountably many smoothings

(3) $S^2 \times \mathbb{R}^2 = \text{int}(S^2 \times D^2)$

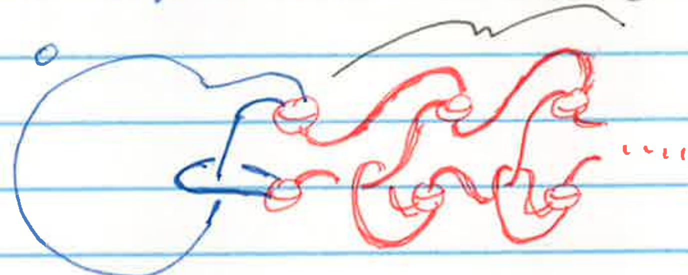


need proof (see below)

example with $G(\text{gen}) = 1$

topologically 2-handle "s"

Casson handle



get Seif surface of genus 1

Thm ([Amels] and 90's)

every 2-handle body interior is homeo to a Stein surface realizing any preassigned c_1 with $c_1|_2 = w_2$

c_1 given by a cocycle $r(h) = $ rot # of attaching S^1

Proof of Th^{m}: ① Suffices to assume $X \approx K(\pi_1,1)$

Apply Taylor (to univ. cover)

② $H_2(X) \neq 0$, X oriented

$\forall k \in \mathbb{Z}_+$ find a smoothing so that $\forall \alpha \in H_2 - \{0\}$ $|\alpha \cdot \alpha| \leq k$ we have $G(\alpha) > k$

then take std smoothing and α_0 choose $k > |\alpha_0 \cdot \alpha_0|, G(\alpha_0)$

let $\{h_i\}$ be 2-handles of X

let S_0 be any Stein structure as in Th^{m} above

for each i and sign let $S_i^{\pm}$ be obtained from S_0 by "refining" only h_i so that $r(h_i) \begin{cases} > 3k & + \\ < -3k & - \end{cases}$

as in example above (add Casson handle) →

S = common refinement of $S_i^{\pm}$ (embeds in each $S_i^{\pm}$ Casson handles ⊂ regular handles)

Given $\alpha \neq 0$ in $H_2(X)$

α represented by cycle $mh_i + z$ some $m \neq 0$

suppose $\alpha = [F]$ $\quad |\alpha \cdot \alpha| \leq k$

$$F \subset S \subset S_1^{\pm}$$

so $2g(F) - 2 \geq F \cdot F - |\langle c_1, [F] \rangle|$

adjunction $\leq$

$\geq -k + |m\, r(h_2) + \langle r, z \rangle|$

$\geq -k \pm |r(h_2)|$

right choice of sign

$\geq -k + 3k = 2k$

Idea for 3

For any smooth 4-manifold X and $g \geq 0$ define $\Gamma_g \subset H_2(X)$ to be span of surfaces with genus $\leq g$ and $|\alpha \cdot \alpha| \leq g$

filtration $\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \ldots \subset H_2(X)$

(not inv of diffeo)

define $\gamma : \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ $\quad \gamma(g) = \operatorname{rank} \Gamma_g$

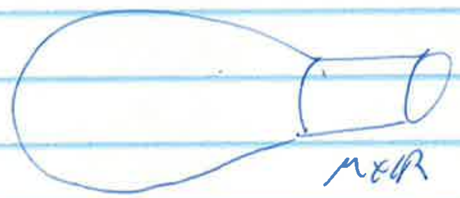
use this to distinguish

$M^3 \times \mathbb{R}$ $\qquad$ $M = \partial X^4$ $\quad$ $X^4 = $ 0-4 $\cup$ 2-4's

incl $M \hookrightarrow X$ π_1-injective

apply above to int X.
recover min genus info from X.

for manifolds like



can talk about
genus at infinity.