Random knots: a preliminary report

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with

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others.

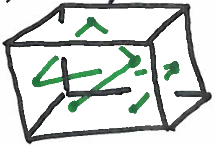
Random Knots: Pick one

$$\left\{ K : S^1 \underset{\text{smooth}}{\hookrightarrow} S^3 \Big/ \text{isotopy} \right\}$$

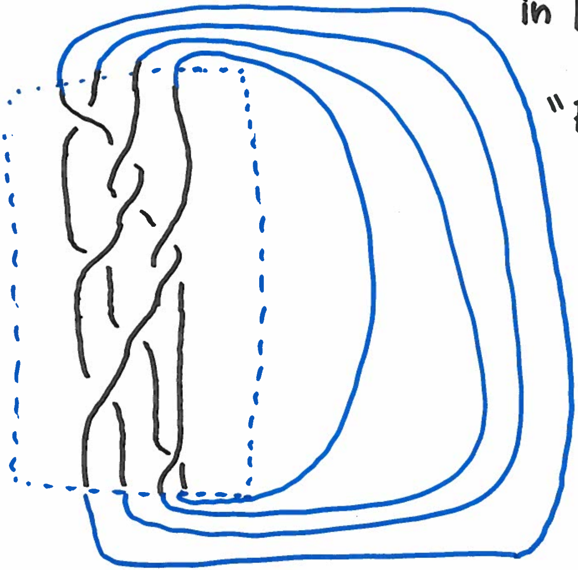
...

Some Models:

- Random walks in $\mathbb{R}^3$.
- Random triples of periodic fn's.
- Physically motivated

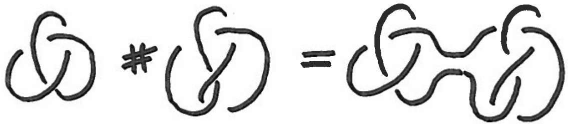


- Random Braids: Close off random walk in B_n w.r.t $\{\sigma_i^{\pm 1}\}$.



"Fixed # of gens with few long relators."

Connect Sum:



Random Prime Knots:

$$\mathcal{K}\mathcal{T}_n = \left\{ \begin{array}{l} \text{isotopy classes of prime} \\ \text{knots w/ projections of} \leq n \\ \text{crossings.} \end{array} \right\}$$

$$|\mathcal{K}\mathcal{T}_{16}| \approx 1.6 \text{ million}$$

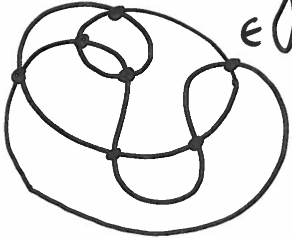
Conj: $\mathbb{E}(\operatorname{rank} \pi_1(S^3 \setminus K)) \longrightarrow \infty$ as $n \longrightarrow \infty$.

$$\mathcal{P}_n = \left\{ \begin{array}{l} \text{4-valent planar graphs} \\ \text{with } n \text{ vertices} \end{array} \right\}$$

[Tutte 60's]

$$\#\mathcal{P}_n = \frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}$$

$$\sim \frac{c}{n^{5/2}} 12^n$$



$\in \mathcal{P}_7$

[Schaeffer 2000] Can sample from $\mathcal{P}_n$ uniformly in $O(n)$ time.

Model: $\mathcal{P}_n' = \{G \in \mathcal{P}_n \mid G$ can't be disconn by removing 2 edges$\}$

- Select $G \in \mathcal{P}_n'$ uniformly at random
- At each vertex

- Make twist regims uniform

Issue: Gives a link (typically)

To get a knot, either

ⓐ Throw back until get a knot.
(Practical to at least $n = 500$)

ⓑ Take longest component.

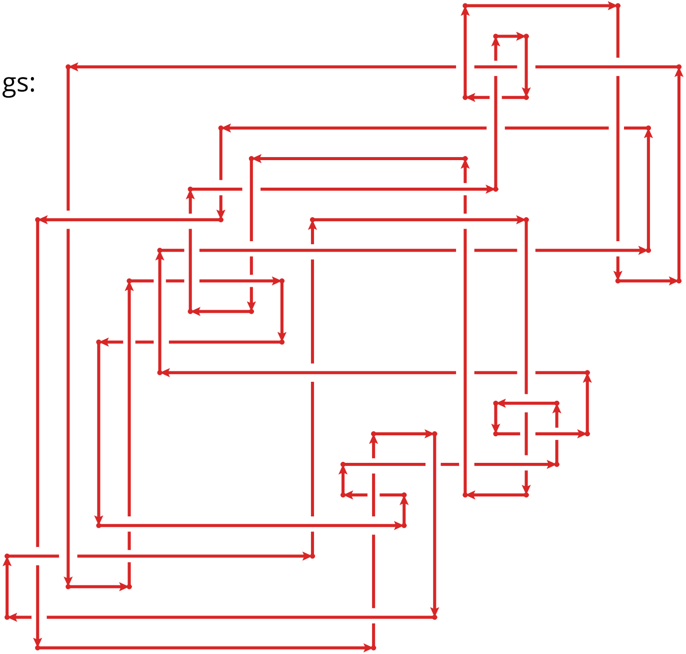
ⓐ Get a prime knot w/ prob > 90%

ⓑ Not usually prime, but:

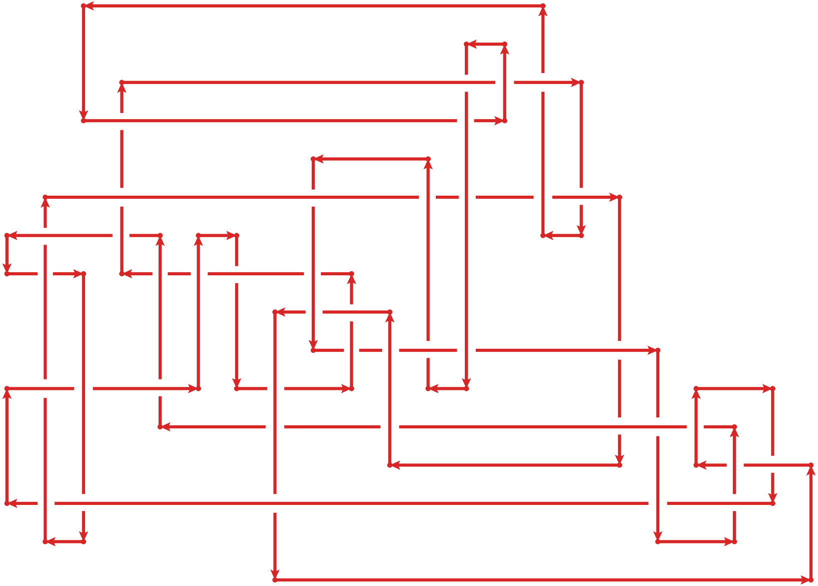
Init Knot → simplify ⇄ Deconnect sum → Prime Knot.

0

50 crossings:

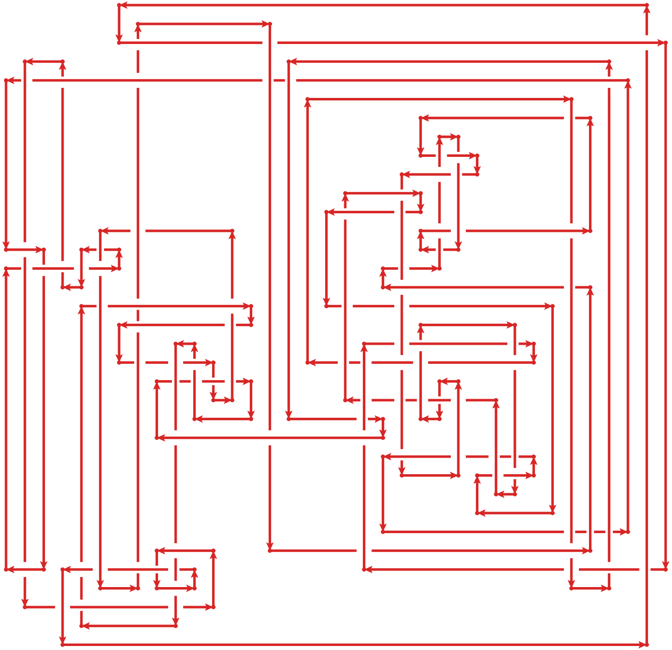


0 Simplified: 41



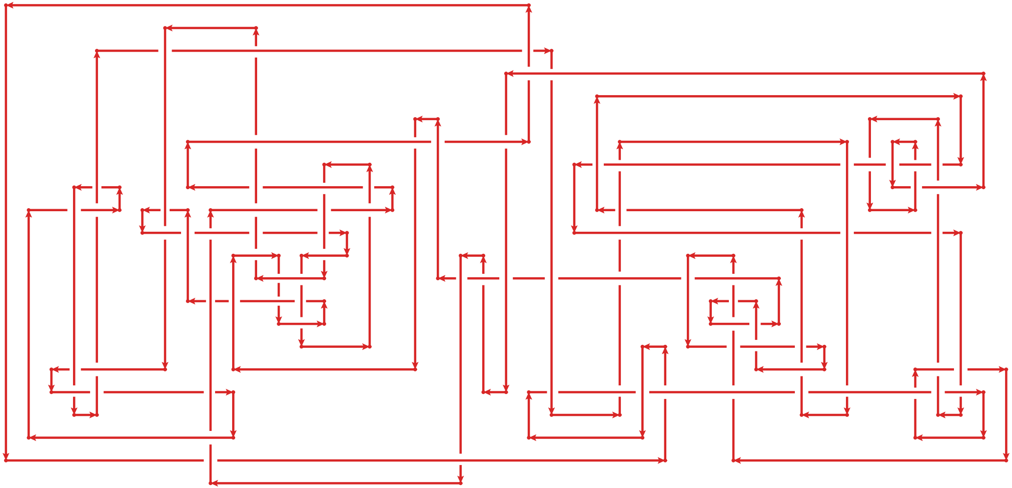
0

100 crossings:



0

Simplified: 81



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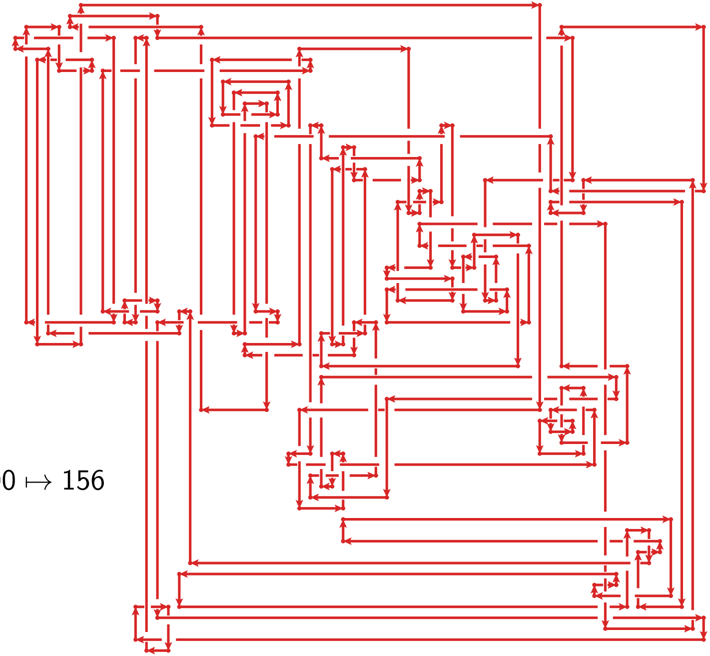
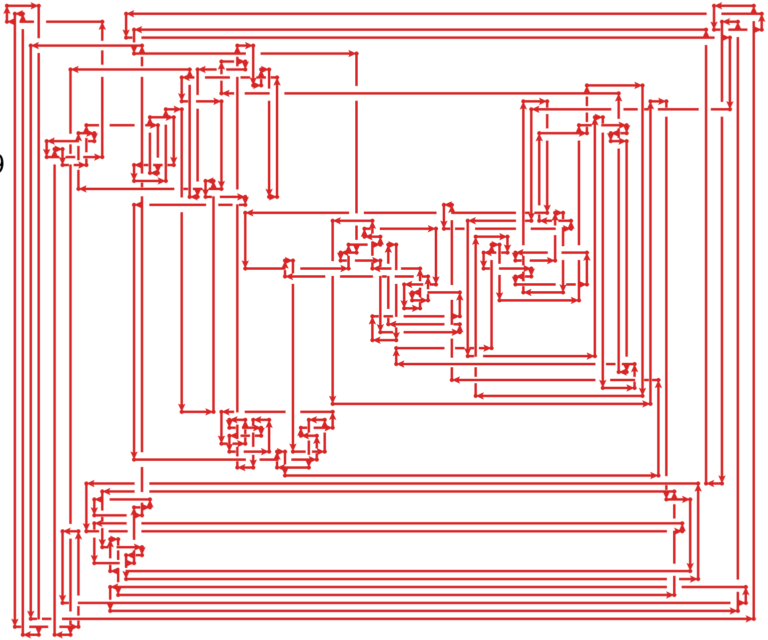
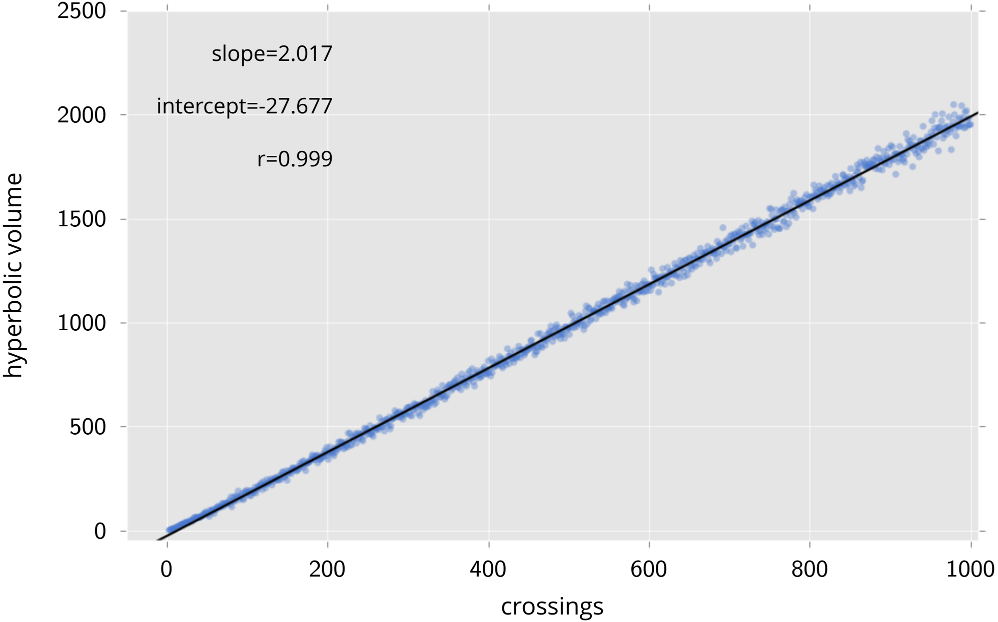


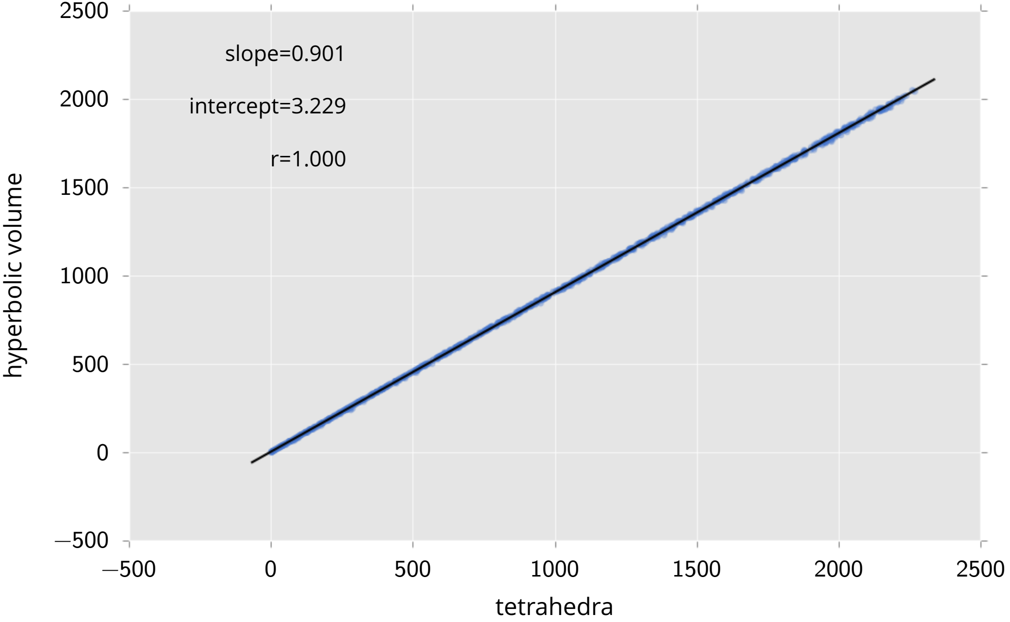
Simplified: 200 $\mapsto$ 156

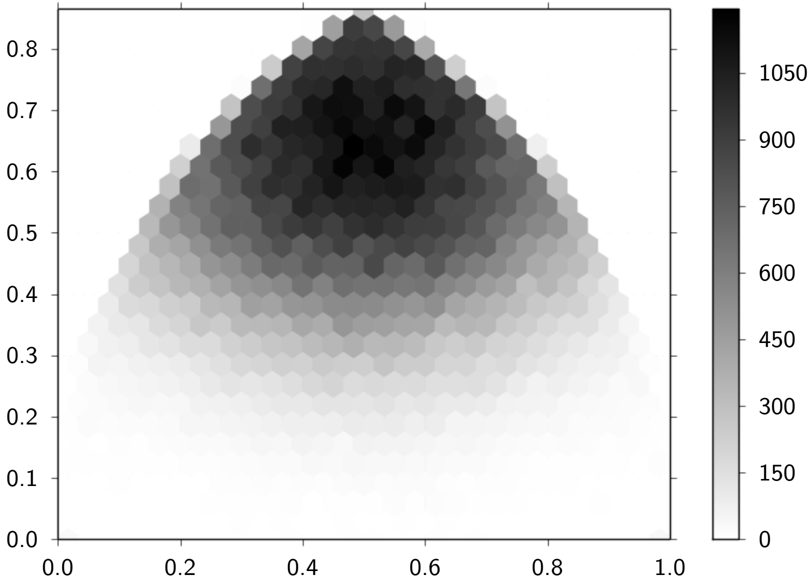
0

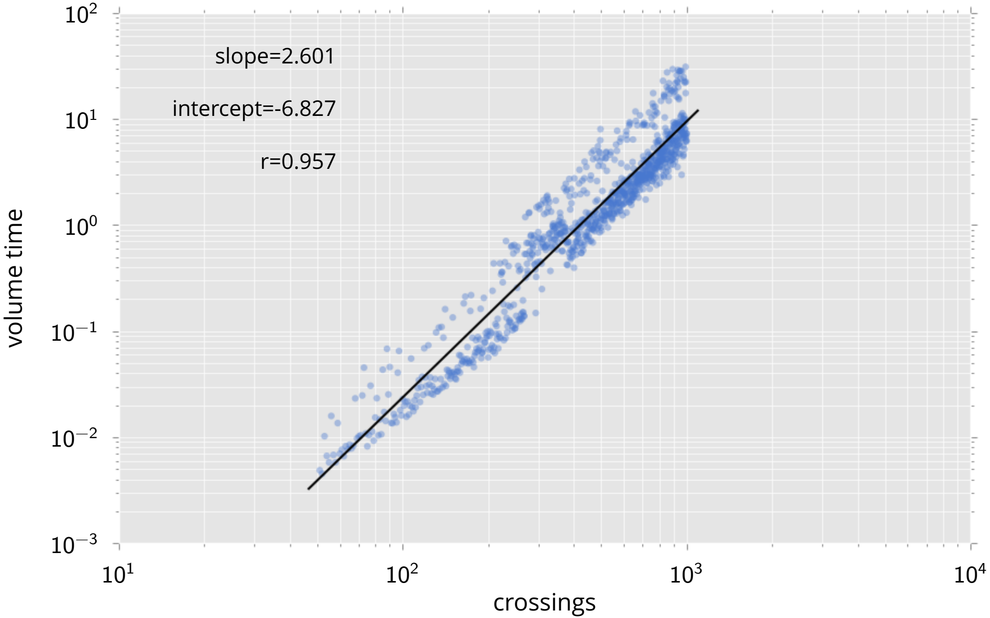
Simplified: $300 \mapsto 229$



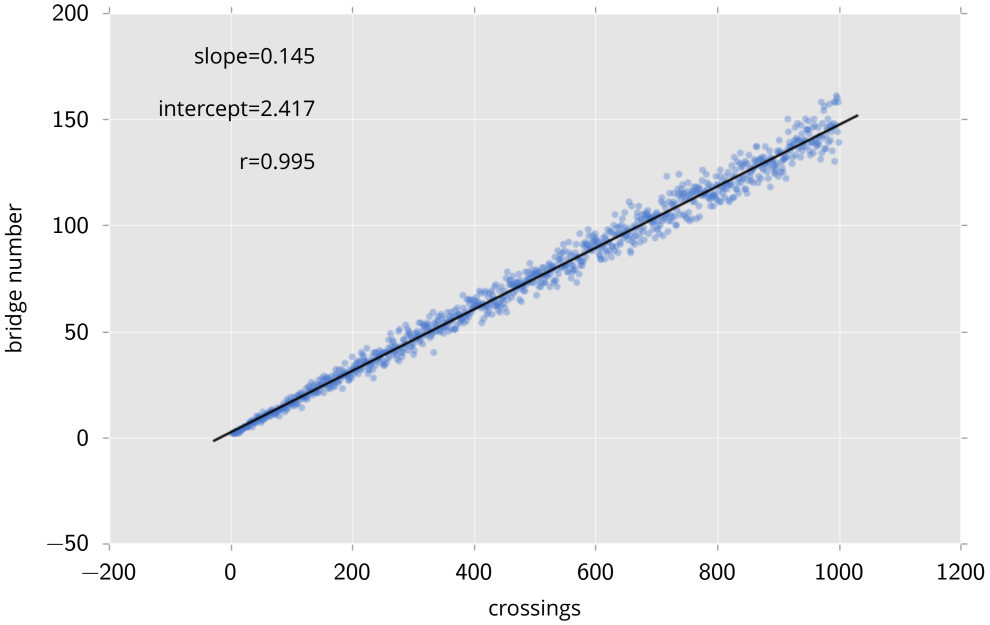

slope=2.017
intercept=-27.677
r=0.999
hyperbolic volume
crossings
0
200
400
600
800
1000
500
1000
1500
2000
2500


slope=0.901
intercept=3.229
r=1.000
hyperbolic volume
tetrahedra
2500
2000
1500
1000
500
0
−500
−500
0
500
1000
1500
2000
2500

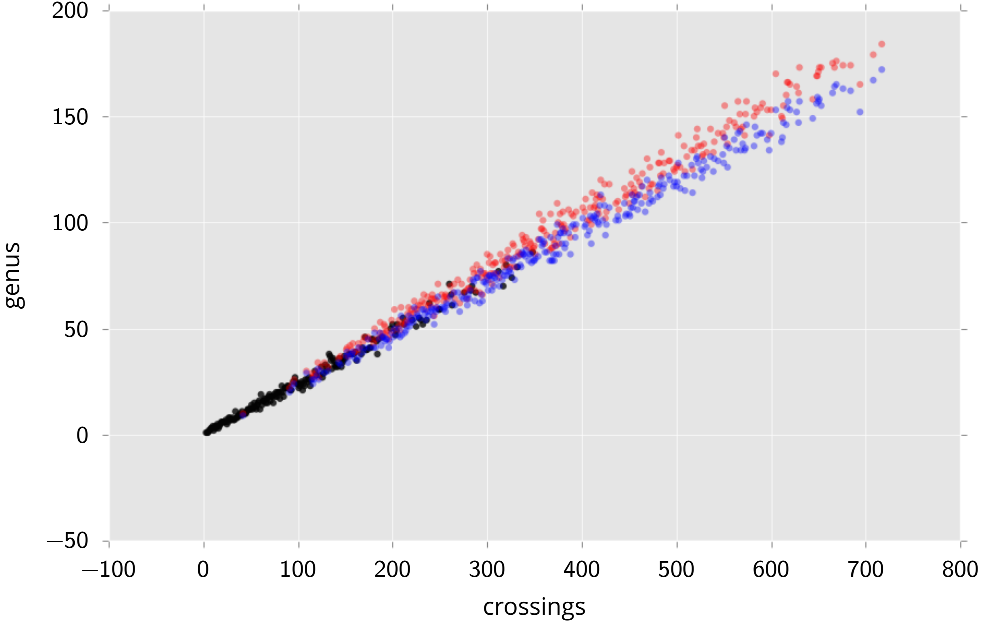

0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1050
900
750
600
450
300
150
0



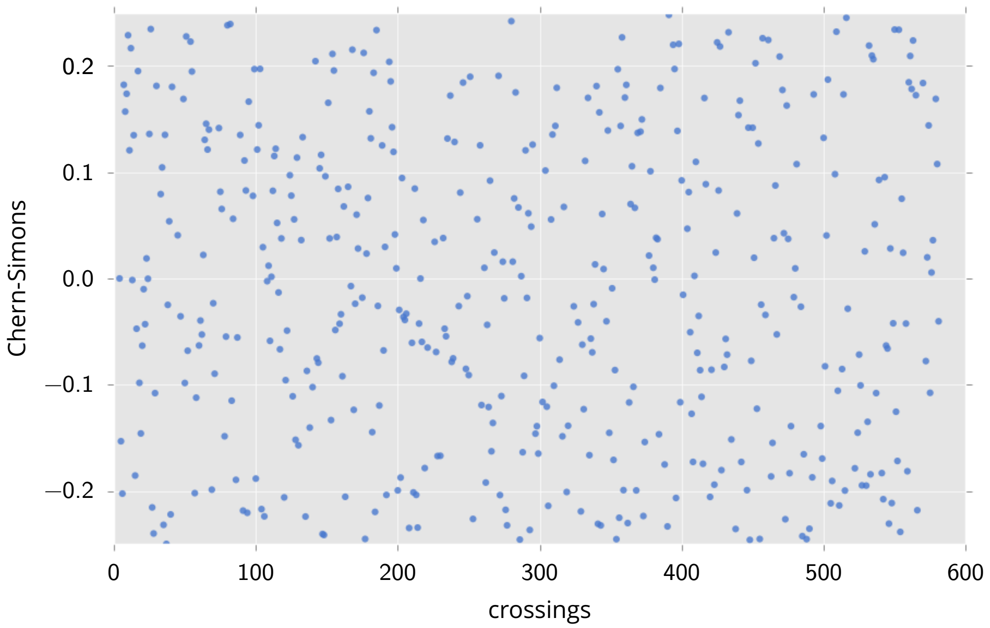
slope=2.601
intercept=-6.827
r=0.957
volume time
crossings
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{1}
10^{2}
10^{3}
10^{4}



slope=0.145
intercept=2.417
r=0.995
bridge number
crossings
200
150
100
50
0
−50
−200
0
200
400
600
800
1000
1200



200
150
100
50
0
−50
genus
−100
0
100
200
300
400
500
600
700
800
crossings


Chern-Simons
0.2
0.1
0.0
−0.1
−0.2
0
100
200
300
400
500
600
crossings

Knot Genus: $\mathcal{T}$ triangulation of M^3, K knot $\subseteq \mathcal{T}^{(1)}$, and $g \in \mathbb{Z}_{\geq 0}$. } INPUT

Is K the bdry of an embedded orient. surface of genus $\leq g$?

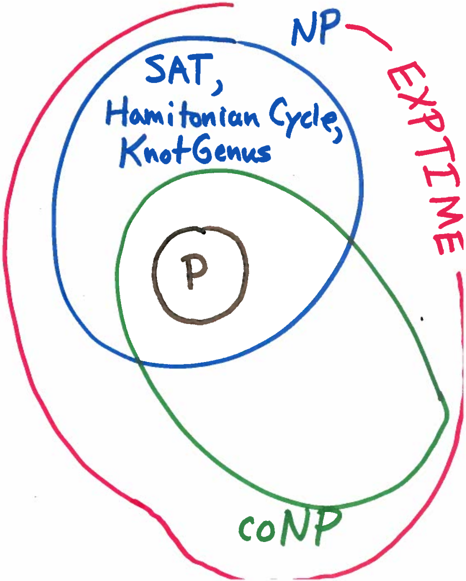
[Agol-Hass-W. Thurston 2006] KnotGenus is NP-complete.

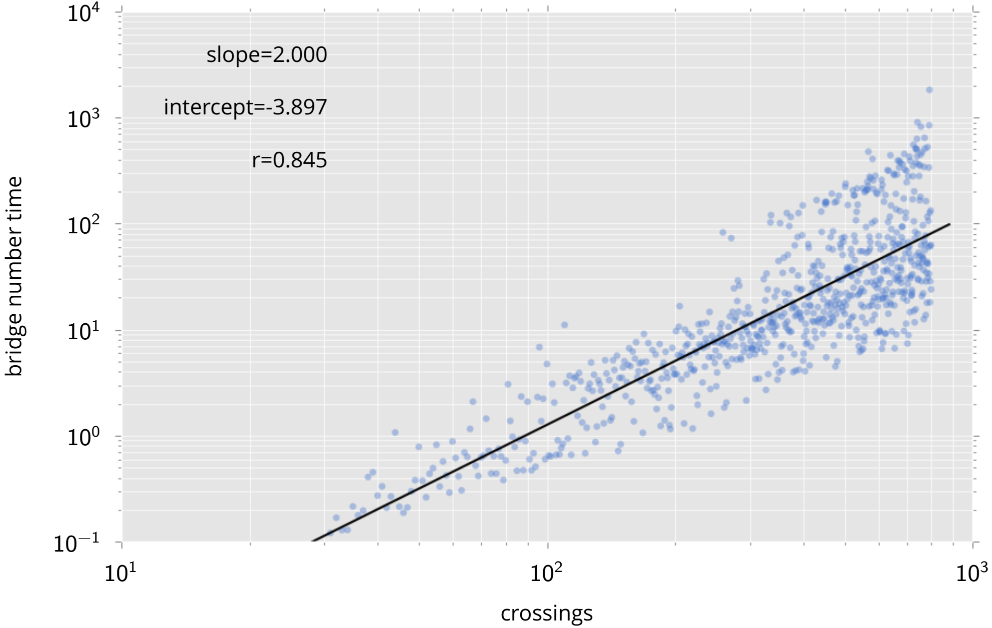
Conj [Agol-H-T] If $H_1(M) = 0$, then KnotGenus is in coNP.

Conj: For $M = S^3$, then KnotGenus is in P.

[AHT] KnotArea is NP complete.

[D-Hirani] If $H_1(M) = 0$, then KnotArea is in P.

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intercept=-3.897
r=0.845
bridge number time
crossings
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}