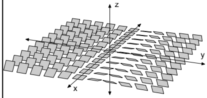
Concave symplectic embeddings and relations in mapping class groups of surfaces

Laura Starkston

University of Texas at Austin

December 6, 2014

Contact
3-manifolds
(M^3, ξ)



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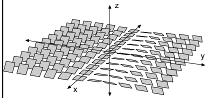
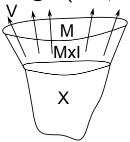


Symplectic convex fillings (X^4, ω)



$$\mathcal{L}_V \omega = \omega$$

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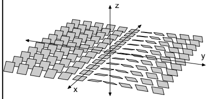
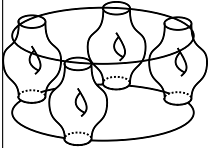


Open book decomposition $\pi : M \setminus L \to S^1$



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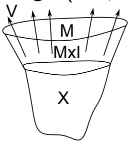

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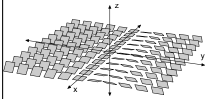

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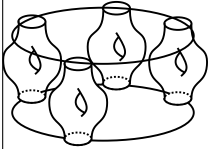

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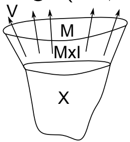

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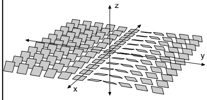

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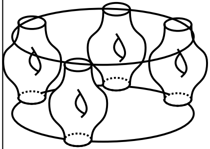

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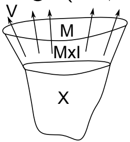

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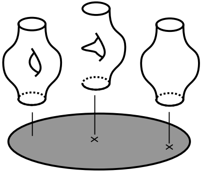
Lefschetz fibration



vanishing cycles $c_1, \cdots, c_n$

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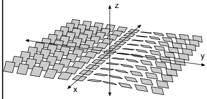

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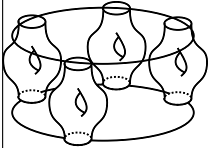

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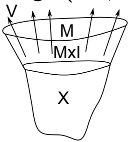

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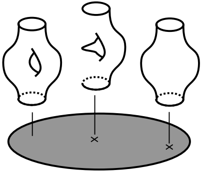

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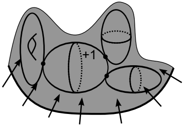
Right-handed Dehn twists

Concave boundary of $\nu(\Sigma_1 \cup \cdots \cup \Sigma_n)$



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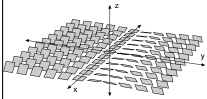

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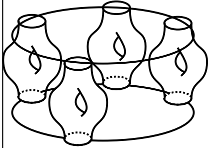

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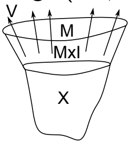

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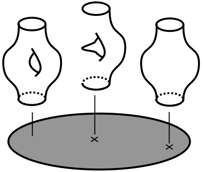

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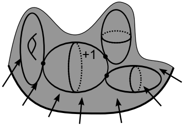

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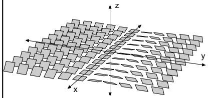

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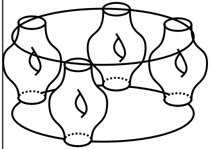

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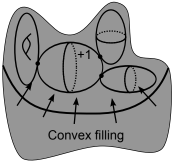
Complement of embedded surfaces in $\mathbb{CP}^2 \# N \overline{\mathbb{CP}^2}$



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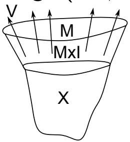

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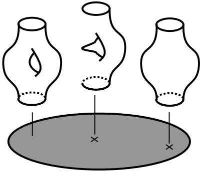

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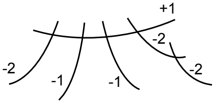
Concave Caps and Convex Fillings

$\Sigma_1, \cdots, \Sigma_n$
surfaces in a 4-manifold intersecting positively transversely



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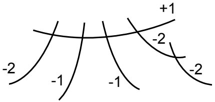

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There exists ω *on* $\nu(\Sigma_1 \cup \cdots \cup \Sigma_n)$ *with*

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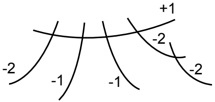

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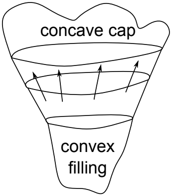

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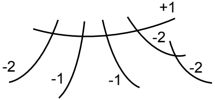

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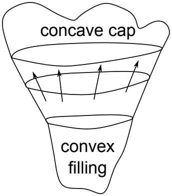
Idea: Use concave caps to find convex fillings.

Theorem [McDuff]
A closed symplectic manifold containing a symplectic positive S^2 is symplectomorphic to $\mathbb{C}P^2 \# N \overline{\mathbb{C}P^2}$.

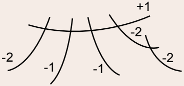


Concave embedding approach

Theorem (S.)

For a Seifert fibered space Y over S^2 with k singular fibers and $e_0 \leq -k-1$, with its canonical contact structure ξ_{can}:

1. *Every convex filling of (Y, ξ_{can}) is the complement of a symplectic embedding of a concave star-shaped plumbing of spheres into $\mathbb{C}P^2 \# N \overline{\mathbb{C}P^2}$.*



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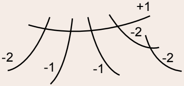

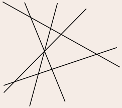

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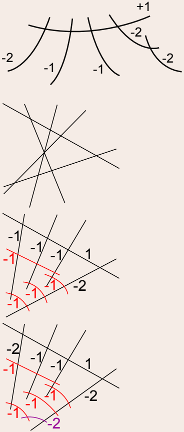

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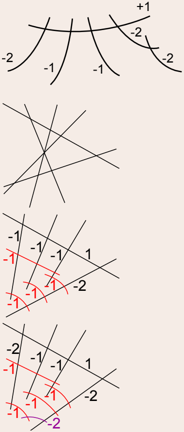
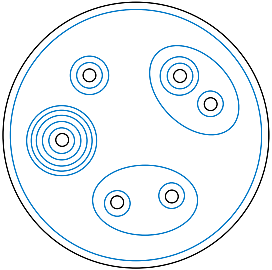
3. *For many such (Y, ξ_{can}) the isotopy class of the embedding is determined by combinatorial/homological data (sufficient conditions: $k \leq 5$ or $e_0 \leq -k-3$).*



Monodromy factorization approach

[Gay-Mark] For these Seifert fibered spaces, (Y, ξ_{can}), there are open book decompositions with

- Planar pages
- Monodromy $\phi = D_{c_1} D_{c_2} \cdots D_{c_n}$, $c_1, \cdots, c_n$ disjoint



Theorem (Wendl)

Because these contact structures are planar, each (minimal) convex filling of (Y, ξ_{can}) corresponds to a different positive factorization of ϕ.

Comparing and translating the two methods

Monodromy substitution approach

- Easy to write down
- Good for applications for cut and paste operations

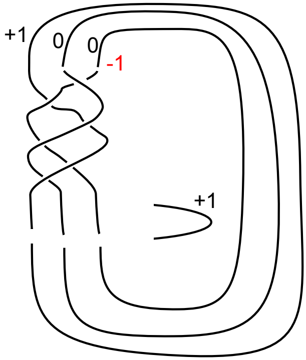
Concave embedding approach

- Easy to find interesting examples
- Easy (in many cases) to classify *all* fillings of certain contact manifolds (combinatorial reduction)

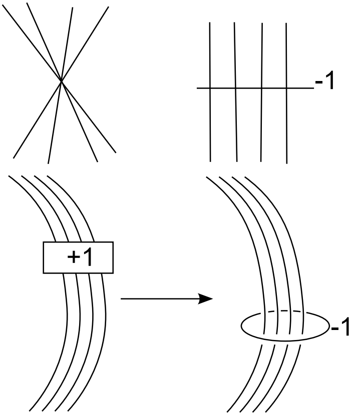
Translating between the two approaches: Handlebody decompositions for embedded surfaces and Lefschetz fibrations

How to draw an embedding of spheres in a 4-manifold

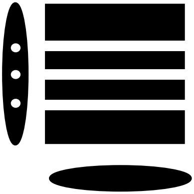
Unknotted circles in S^3	$\leftrightarrow$	equators of spheres
Framing	$\leftrightarrow$	self-intersection number
Linking	$\leftrightarrow$	Pairwise intersections



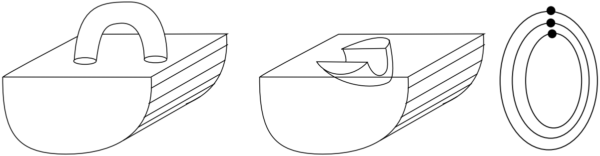
Blowing up and down

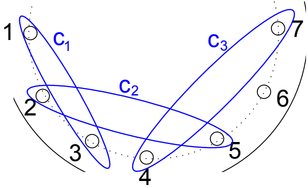


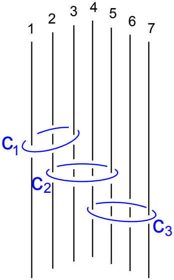
How to draw a planar Lefchetz fibration

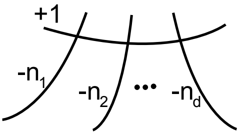


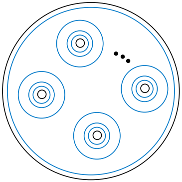
Dotted circle notation: Attaching a 1-handle is equivalent to carving out a 2-handle – a dotted circle denotes the boundary of this core disk.





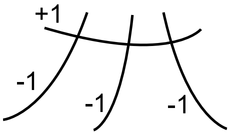


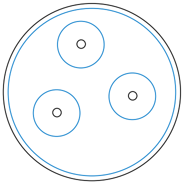




Concave embeddings: Start with $d+1$ pseudoholomorphic $\mathbb{C}P^1$'s in $\mathbb{C}P^2$.

$\mathbb{C}P^2$





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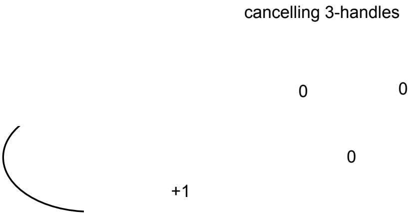

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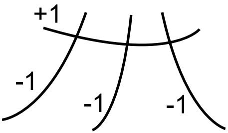

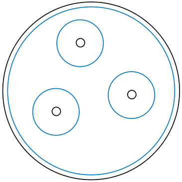

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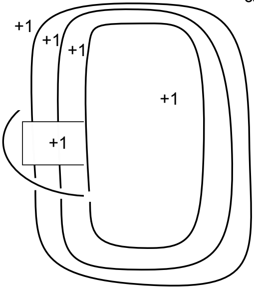
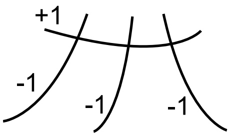
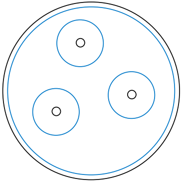
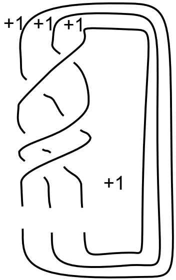


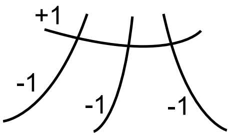
$\mathbb{C}P^2$

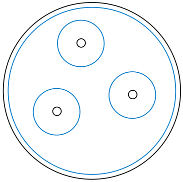




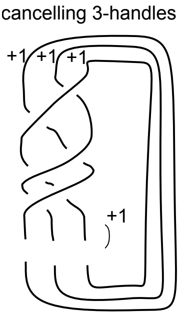
Concave embeddings: Start with $3+1$ pseudoholomorphic $\mathbb{C}P^1$'s in $\mathbb{C}P^2$.

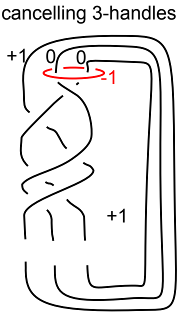




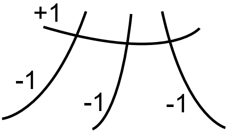


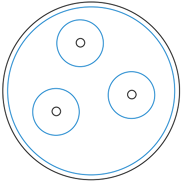
Concave embeddings: Start with $3+1$ pseudoholomorphic $\mathbb{C}P^1$'s in $\mathbb{C}P^2$.





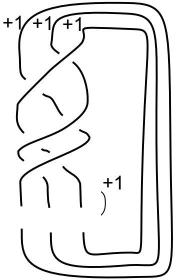
Blow up to remove intersections and change framings. $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$



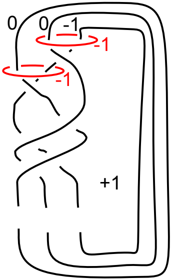


Concave embeddings: Start with $3+1$ pseudoholomorphic $\mathbb{C}P^1$'s in $\mathbb{C}P^2$.

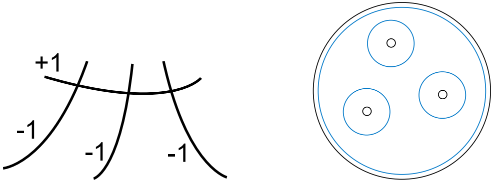
cancelling 3-handles



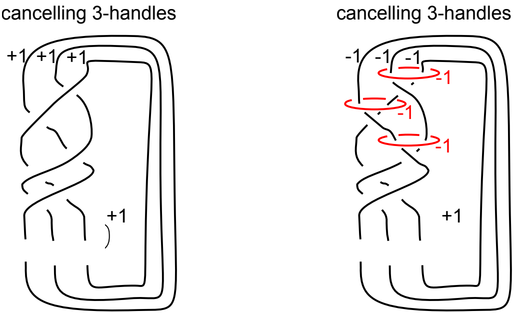
cancelling 3-handles



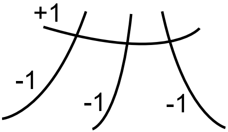
Blow up to remove intersections and change framings. $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$

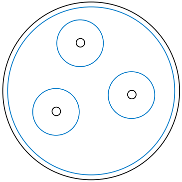


Concave embeddings: Start with $3 + 1$ pseudoholomorphic $\mathbb{C}P^1$'s in $\mathbb{C}P^2$.



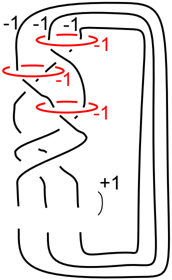
Blow up to remove intersections and change framings. $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$



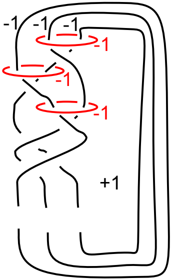


Concave embeddings: Start with $3+1$ pseudoholomorphic $\mathbb{C}P^1$'s in $\mathbb{C}P^2$.

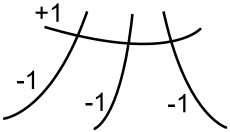
cancelling 3-handles

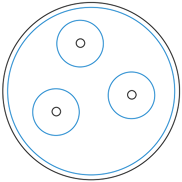


cancelling 3-handles

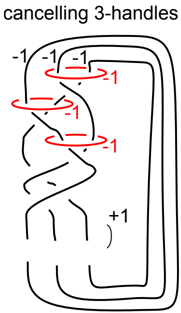


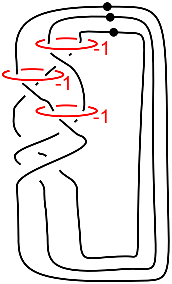
Notice the embedding of the dual graph into $\mathbb{C}P^2 \# 3\,\overline{\mathbb{C}P^2}$



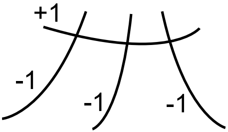


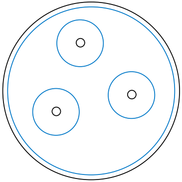
Concave embeddings: Start with $3+1$ pseudoholomorphic $\mathbb{C}P^1$'s in $\mathbb{C}P^2$.



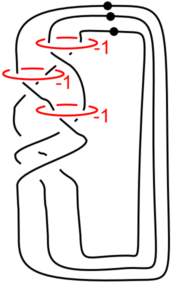


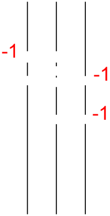
Cut out concave cap from $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$. Turn upsidedown. Simplify



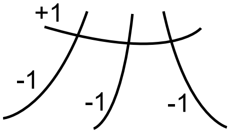


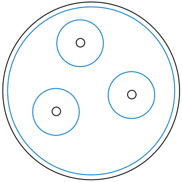
Concave embeddings: Start with $3+1$ pseudoholomorphic $\mathbb{C}P^1$'s in $\mathbb{C}P^2$.



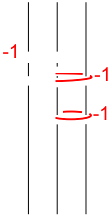


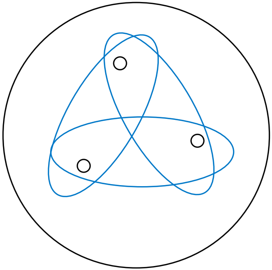
Lefschetz Fibration!





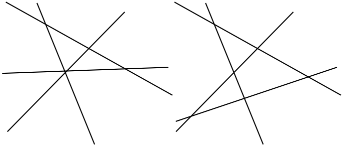
Concave embeddings: Start with $3 + 1$ pseudoholomorphic $\mathbb{C}P^1$'s in $\mathbb{C}P^2$.



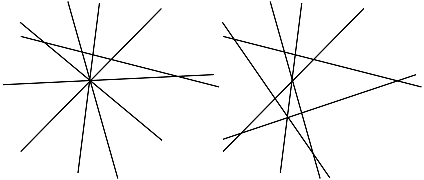


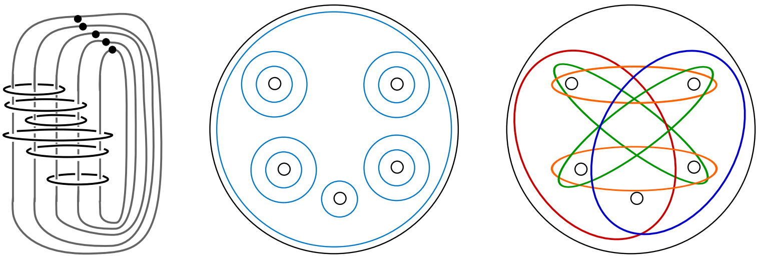
Lantern Relation! $D_1 D_2 D_3 D_{1,2,3} = D_{1,2} D_{1,3} D_{2,3}$

The lantern relation concave embedding came from blowing up:

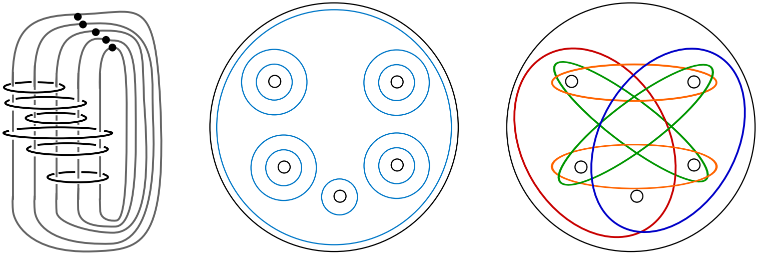


By blowing up more interesting configurations of lines, we get new relations:





$$D_1^2 D_2^2 D_3 D_4^2 D_5^2 D_{1,2,3,4,5} = D_{1,2,3} D_{1,4} D_{1,5} D_{2,4} D_{2,5} D_{3,4,5}$$



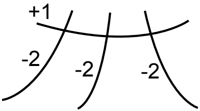
$$D_1^2 D_2^2 D_3 D_4^2 D_5^2 D_{1,2,3,4,5} = D_{1,2,3} D_{1,4} D_{1,5} D_{2,4} D_{2,5} D_{3,4,5}$$

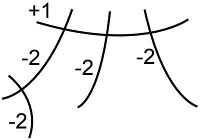
Concave embedding strategy shows: no other fillings $\Rightarrow$ no other $+$ factorizations.

Such *indecomposable* relations are essential relators for elements in Dehn$^+$.

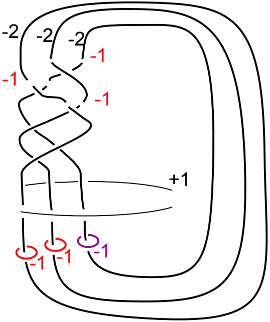
There is an infinite family of indecomposable relations generalizing this example.

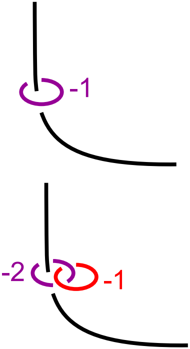
Longer Arms



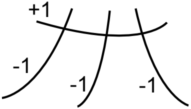


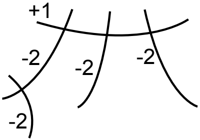
cancelling 3-handles



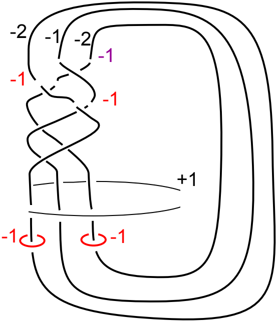


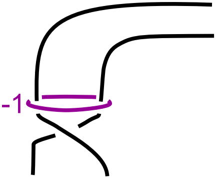
Longer Arms

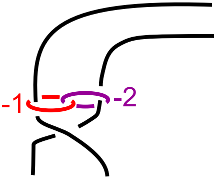


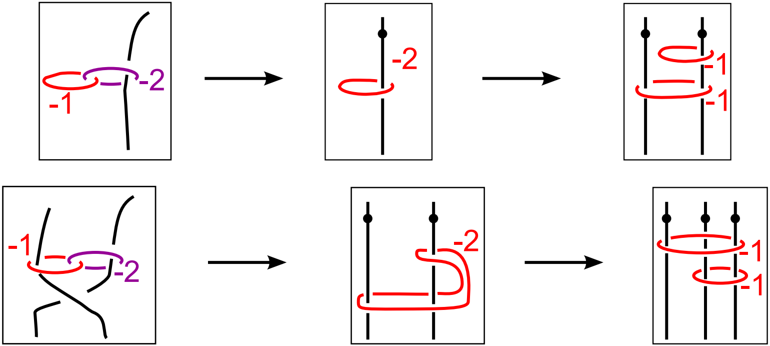


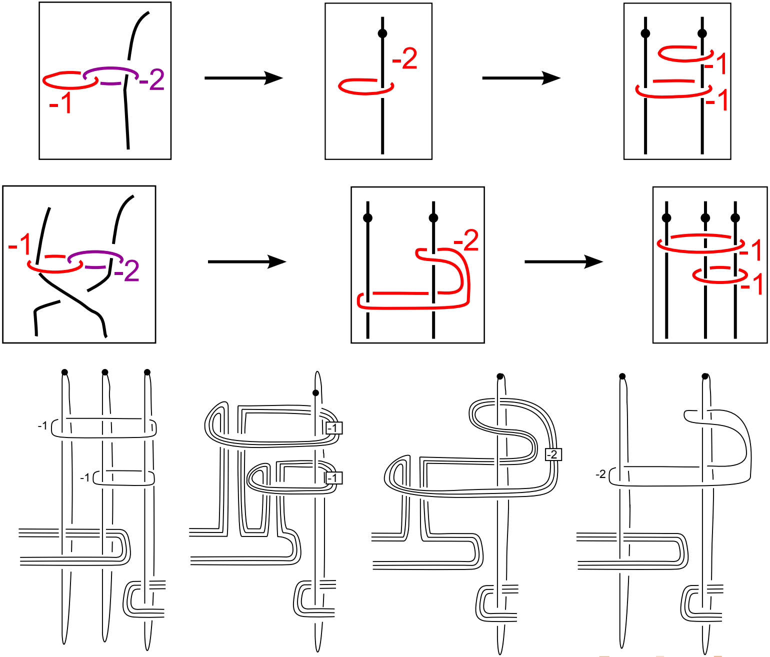
cancelling 3-handles








-1
-2
-2
-1
-1
-1
-2
-2
-1
-1


-1
-2
-2
-1
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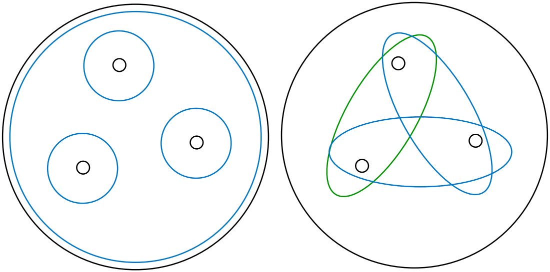
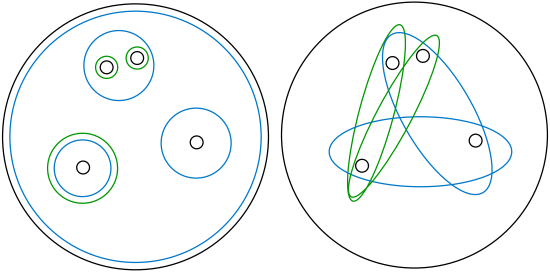
$$D_1 D_2 D_3 D_{1,2,3} = D_{1,2} D_{1,3} D_{2,3}$$



$$D_{1_a} D_{1_b} D_{1a1b} D_2^2 D_3 D_{1_a,1_b,2,3} = D_{1_a,2} D_{1_b,2} D_{1_a,1_b,3} D_{2,3}$$



Moving towards a complete dictionary: other moves

- What is a complete list of embedding moves, and how do they each translate to moves on mapping class group relations?
- How does a sequence of embedding moves translate to a sequence of mapping class group relations?