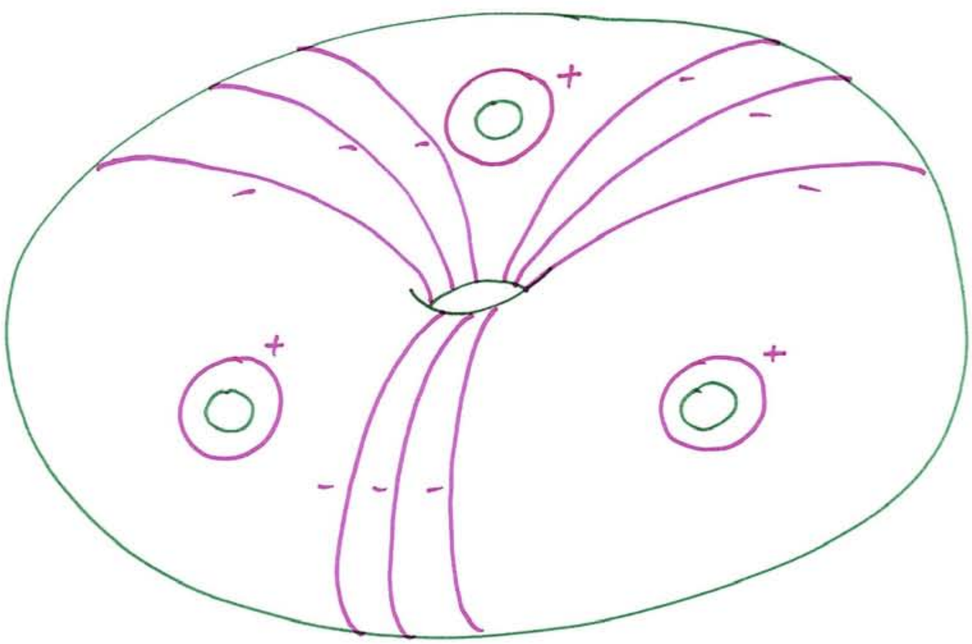
Open Book Decompositions
and the
Giroux Correspondence



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Open Book Decompositions

let V be a compact $(n-1)$-manifold w/ $\partial V \neq \emptyset$ and $\phi: V \to V$ a diffeomorphism of V such that

$$\phi|_N = \text{identity on } N$$

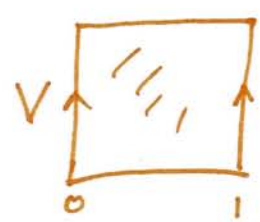
for some neighborhood N of ∂V.

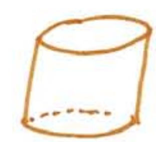
we denote the mapping torus of ϕ by

$$\boxed{V_\phi = V \times [0,1] \Big/ (p,1) \sim (\phi(p), 0)}$$

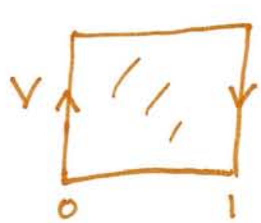
example: 1) $V = [-1,1]$

$\phi = id$



$V_\phi =$ 

$\phi = -id$

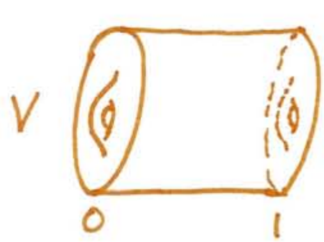


$V_\phi =$ 

2) V any mfd

$\phi = id$

$$V_\phi = V \times S^1$$



$=$ 

clearly

$$\partial(V_\phi) = (\partial V) \times [0,1] \Big/ _{\substack{(p,1) \sim (\phi(p),0) \\ = (p,0)}} = (\partial V) \times S^1$$

$$(p, \theta)$$

we now set

$$\boxed{M_{(V,\phi)} = V_\phi \cup (\partial V \times D^2)/\sim}$$

where

$$\big((p,\theta) \in \partial(V_\phi)\big) \sim \big((p,\theta) \in \underbrace{\partial(\partial V \times D^2)}_{= \partial V \times \partial D^2}\big)$$

we say (V,ϕ) is an open book decomposition of a closed n-manifold M if M is diffeomorphic to $M_{(V,\phi)}$.

Note: we could alternately define an obd of M to be a pair (B,π) where

1) B is an (oriented) codim 2 submfd of M and

2) $\pi : (M - B) \to S^1$ is a fibration such that $\partial\left(\overline{\pi^{-1}(\theta)}\right) = B$ for all $\theta \in S^1$.

(this is a stronger notion!)

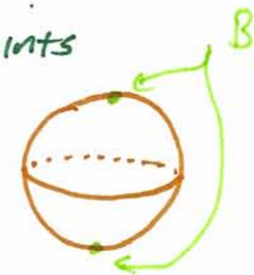
we call B the binding of the obd,

V (or equivalently $\overline{\pi^{-1}(\theta)}$) a page and

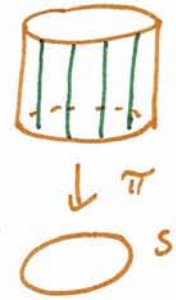
ϕ the monodromy.

examples: 1) $M = S^2$

B = 2 points



$M - B \approx$



2) $M = S^2 \times S^1$

$B = (2\text{ pts}) \times S^1$

so page $= I \times S^1$

monodromy = identity

3) $M = S^3$

B = unknot

$M - B = \mathbb{R}^2 \times S^1$

page $= D^2$

monodromy = identity

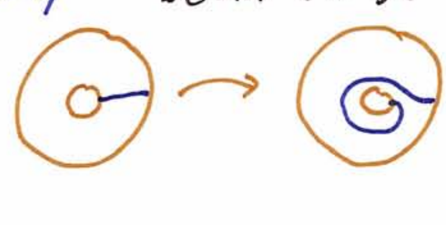
B = Hopf link

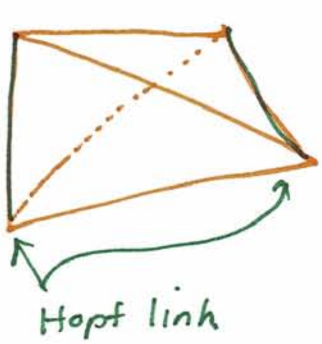
$M - B = T^2 \times (0,1)$

fibers many ways!

page $= I \times S^1$

monodromy = Dehn twist





4) let

$$f : \mathbb{C}^n \to \mathbb{C}$$

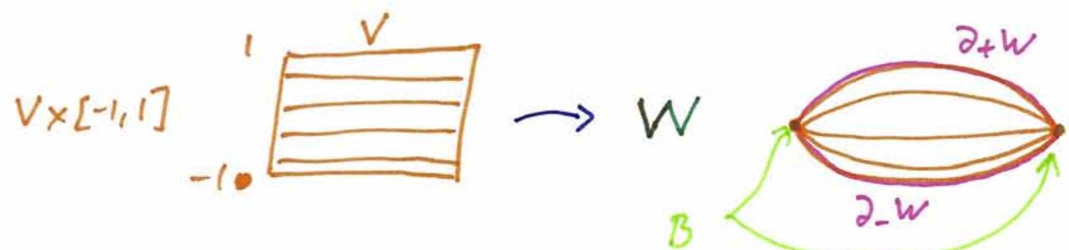
be a polynomial function with $f(0,0,..0) = 0$ and $(0,0,..0)$ an isolated critical point.

Th$^{\underline{m}}$ (Milnor '68):

> There is some $\varepsilon > 0$ such that if S^{2n-1} is the unit sphere of radius ε and $B = (f^{-1}(0) \cap S^{2n-1})$ then
>
> $$\pi : (S^{2n-1} - B) \to S^1$$
> $$(z_1 \dots z_n) \longmapsto \frac{f(z_1 .. z_n)}{|f(z_1 \dots z_n)|}$$
>
> is a fibration.

5) let V be any compact manifold w/ $\partial V \neq \emptyset$

set $W = V \times [-1,1] \Big/ \{(x,t) \mid t \in [-1,1]\}_{x \in \partial V}$



if $h : (\partial W, \partial_+ W, \partial_- W) \to (\partial W, \partial_+ W, \partial_- W)$ is a diffeomorphism then

$M = W \cup_h W$ has an o.b.d.

note: $\dim M = 3$ this gives a Heegaard splitting of M

Existence Of O.B.D.

Th$^{\underline{m}}$:

$\boxed{\text{Any closed oriented } (2n+1)\text{-manifold has an open book decomposition}}$

Remarks:

- the $n \geq 3$ case, with extra hypotheses, is due to Winkelnkemper '73 and Tamura '72.
- the general $n \geq 3$ case is due to Lawson '78.
- the $n = 2$ case for simply connected manifolds is due to A'Campo '72.
- the general $n \geq 2$ case is due to Quinn '78.
- the $n = 1$ case is due to Alexander '20.

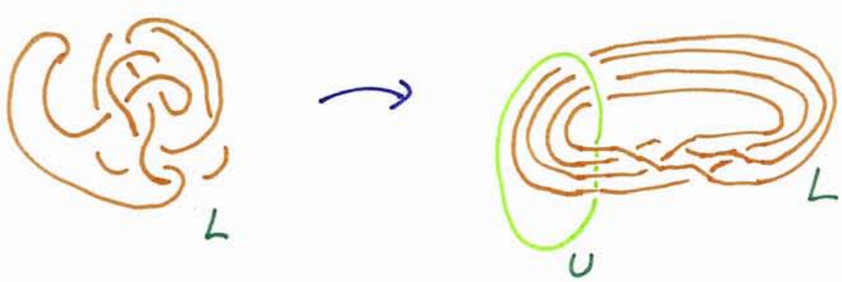
- quite a lot is known in the case of even dimensional manifolds. (need signature $=0, \ldots$)

Proof $n=1$ case:

Fact: Any closed oriented 3-manifold M is a branched cover of S^3 with branch locus a link L.

$$p : M \to S^3$$

Fact: Any link can be braided about the unknot U.

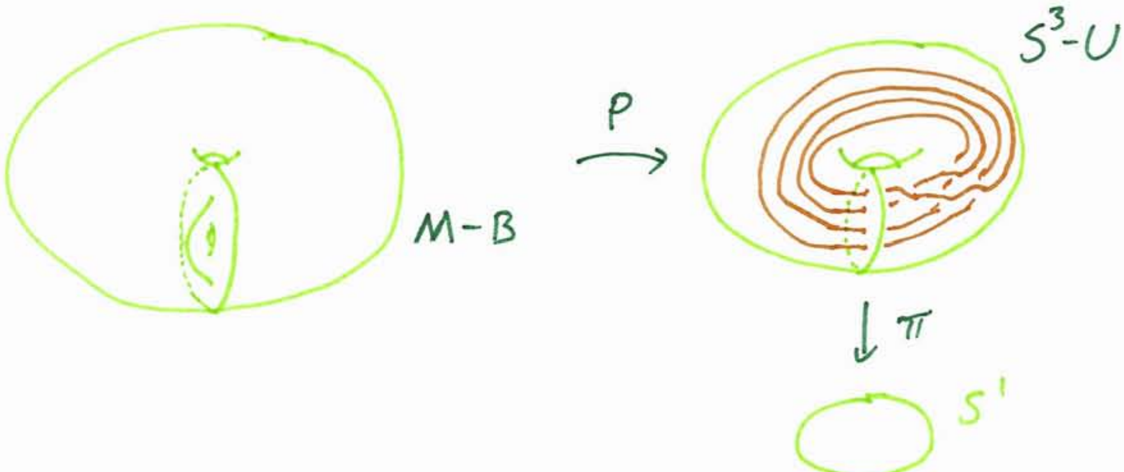


i.e. L can be isotoped to be transverse to the pages of the open book (U, π) of S^3.

Set $B = p^{-1}(U)$ then

$$\pi' = \pi \circ p : (M - B) \to S^1$$

is a fibration



O.B.D. and Contact Structures

Th$^{\underline{m}}$:

Suppose M^{2n+1} has an o.b.d. (V, ϕ) such that

1) V is a compact manifold admitting an exact symplectic form $\omega = d\beta$,

2) the vector field v defined by

$$\iota_v \omega = \beta$$

is transversely pointing out of V along ∂V, and

3) ϕ is a symplectomorphism of (V, ω).

Then M admits a unique contact structure $\xi_{(V,\phi)}$ for which there is a contact form α for $\xi_{(V,\phi)}$ satisfying

a) α induces a positive contact form on the binding and

b) $d\alpha$ induces a symplectic form on each page.

such a contact structure is said to be supported by or compatible with the O.B.D.

Remarks:

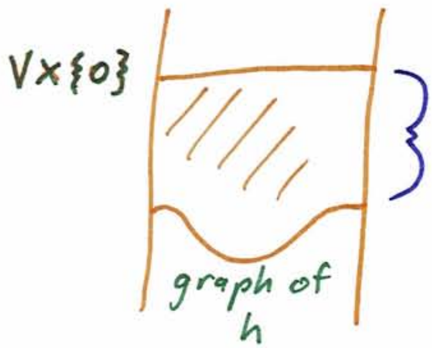
- when $n=1$ this is due to Thurston-Winkelnkemper '75
- the rest is due to Giroux ~ 2000

Sketch of Proof:

note: • $\tilde{\alpha} = dt + \beta$ is a contact form on $V \times \mathbb{R}_t$

• $d(\phi^*\beta - \beta) = \phi^*\omega - \omega = 0$

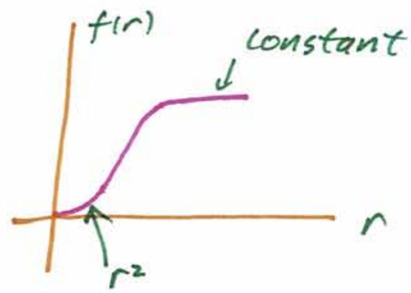
if $\phi^*\beta - \beta = -dh$ then



$(p,0) \sim (\phi(p), h(p))$ $\;\longrightarrow\;$ V_ϕ and $\tilde{\alpha}$ induces a contact form α on V_ϕ

on $\partial V \times D^2$ set $\alpha = \beta|_{\partial V} + f(r)\, d\theta$

where



for the correct choice of $f(r)$ this extends α over all of $M_{(V,\phi)}$.

(if $\phi^*\beta - \beta$ is not exact then let v be the vector field on V satisfying

$$\iota_v \omega = \beta - \phi^*\beta$$

if ψ_t is the flow of v then

$$\phi' = \phi \circ \psi_1$$

satisfies $(\phi')^*\beta - \beta = -dh$ for some h)

∎

Cor:

> All 3-manifolds admit a contact structure.

Proof:

given any surface V and diffeomorphism ϕ it is easy to find a 1-form β on V satisfying 1) – 3) of the theorem ∎

note: if ξ is a contact structure on M^{2n+1} then

$$T_x M = \xi_x \oplus \mathbb{R}$$

$d\alpha$ symplectic str on ξ_x
∴ can put a compatible almost complex structure

so TM has the structure of a $(U(n) \times 1)$ structure

such a reduction of the structure group of TM is called an almost contact structure on M.

Open Problem:

> Is there a contact structure in every homotopy class of almost contact structure?

Cor:

On a simply connected 5-manifold every homotopy class of almost contact structure admits a contact structure.

Remark: originally proven by Geiges '91.
reproved by van Koert '06 using open books.

Th$^{\underline{m}}$: (Martínez, Muñoz and Presas '02):

Any almost contact manifold M^{2n+1} admits an open book (V, ϕ) where V has the homotopy type of an $(n+1)$-dimensional CW-complex.

Question: Can this be improved so that we can use (V, ϕ) to construct a contact structure on M?

O.B.D. and Contact Structures II

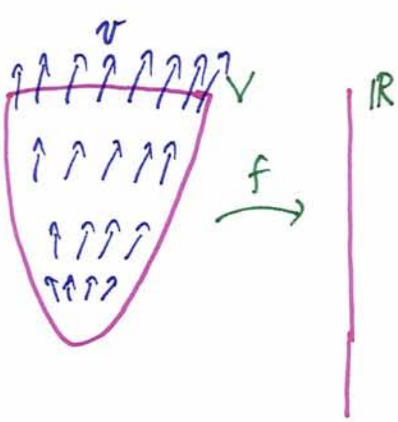
Thm:

Every oriented contact structure on a closed oriented manifold M^{2n+1} is compatible with some O.B.D. (V, ϕ).

Moreover, V may be assumed to be a Weinstein manifold and ϕ a symplectomorphism.

Remarks:

- when $n=1$ this is due to Giroux ~2000.
- when $n>1$ this is due to Giroux-Mohsen ~2002 (?).
- a symplectic manifold (V, ω) is a Weinstein manifold if there exists a function $f: V \to \mathbb{R}$ and vector field v such that
  1) f is positive, proper and Morse
  2) v is gradient like for f i.e. $df(v) \geq 0$ and $=0$ only at crit. pts.
  3) $\mathcal{L}_v \omega = \omega$
  4) ∂V is a regular level set of f.



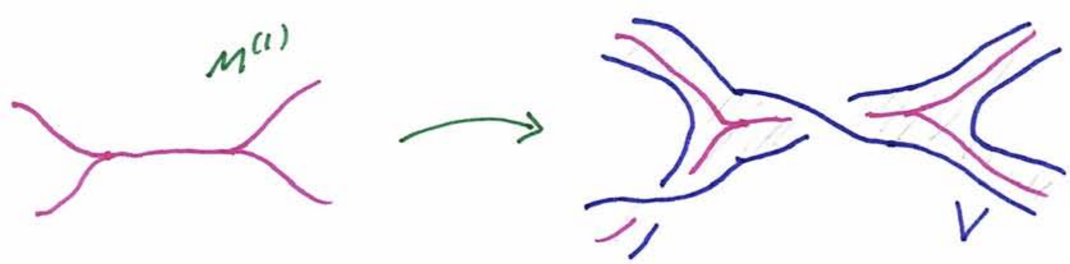
Proof n=1 case:

given a contact structure ξ on M^3
take a CW-decomposition of M so that

1) the 1-skeleton $M^{(1)}$ is Legendrian (i.e. tangent to ξ),
2) the "twisting" of ξ along the boundary of each 2-cell is -1, and
3) each 3-cell is contained in a Darboux chart.

(it is not hard to arrange such a CW-decomp.)

let V be the "ribbon" of $M^{(1)}$ (framed by ξ)



let $B = \partial V$

A theorem of Gabai (1983) says 2) & 3) $\Rightarrow (M - B)$ fibers over S^1

A theorem of Torisu (2000) says ξ is supported by this O.B.D. $\blacksquare$

Sketch of Proof $n>1$ case:

given a contact structure ξ on M^{2n+1}

let α be a contact 1-form for ξ and

J an almost complex structure on ξ compatible with $d\alpha$

this induces a metric g on M

(on ξ use $d\alpha(\cdot, J\cdot)$, on $(\ker d\alpha)$ use $\alpha\otimes\alpha$)

Th$\underline{m}$ (Ibort, Martínez-Torres and Presas, 00):

There are constants $C, \delta > 0$ and functions

$$s_k : M \to \mathbb{C} \qquad (k \geq 0)$$

such that

1) for each $p \in M$

$$|s_k(p)| \leq C$$

$$|ds_k(p) - ik\, s_k(p)\alpha(p)| \leq C k^{1/2}$$

$$|\bar\partial_\xi s_k(p)| \leq C$$

2) for each $p \in M$ with $|s_k(p)| \leq \delta$

$$|\partial_\xi s_k(p)| \geq \delta k^{1/2}$$

where
$$\bar\partial_\xi s_k = \frac{1}{2}\left(ds_k|_\xi + i\left(ds_k|_\xi \circ J\right)\right)$$
$$\partial_\xi s_k = \frac{1}{2}\left(ds_k|_\xi - i\left(ds_k|_\xi \circ J\right)\right)$$

Remark: can think of this as Donaldson's approximately holomorphic sections with controled transversality in this context.

if $w \in \mathbb{C}$ and $|w| \leq \delta$ then it is a regular value of s_k so

$$B_w = s_k^{-1}(w)$$

is a submanifold.

(take k so that $|\bar{\partial}_\xi s_k| < |\partial_\xi s_k|$ on B_w)

It is also not hard to check $d\alpha$ is non-degenerate on

$$\xi \cap TB_w = \ker(ds_k|_\xi)$$

thus B_w is a contact submanifold of M.

Moreover

$$\arg s_k : (M - B_0) \to S^1$$

is a fibration on which $d\alpha$ gives a symplectic form on the pages.

To see the pages are Weinstein takes a bit more work. $\blacksquare$

Applications

We have the following nice existence result

Th$\underline{\text{m}}$ (Bourgeois '02): If a close manifold M^{2n-1} admits a contact structure, then so does $M \times T^2$

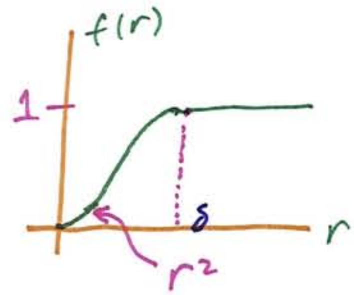
Proof: let (B, π) be an O.B.D. on M supporting the contact structure $\xi = \ker \alpha$.

choose a neighborhood $B \times D^2$ of B so that π extends to all of $B \times D^2$ as the angular coordinate on D^2 (let r denote the radial coordinate)

then

$$\tilde{\alpha} = f(r)(\cos \pi \, dx_1 + \sin \pi \, dx_2) + \alpha$$

is a contact form on $M \times T^2$ where x_1, x_2 are coordinates on T^2 and



δ sufficiently small

∎

Cor:

$\boxed{T^{2n+1} \text{ admits a contact structure.}}$

Remarks:

- T^3 is obvious

  $\xi = \ker(\cos z\, dx + \sin z\, dy)$

- T^5 case is due to Lutz '79

Cor:

If a closed manifold M admits a contact structure, then so does $M \times \Sigma_g$ for any surface Σ_g of genus $g \geq 1$.

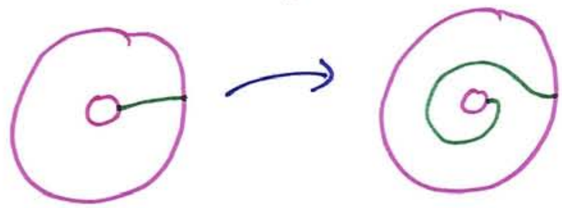
O.B.D. and Contact Structures III

Recall, if γ is an embedded curve on a surface Σ then it has a neighborhood $N = S^1 \times [0,1]$.

A **positive Dehn twist** along γ is the diffeomorphism

$$D_\gamma : \Sigma \to \Sigma$$

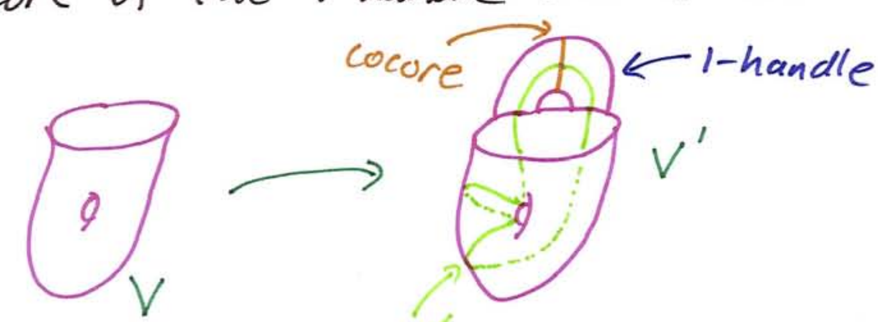
that is the identity outside N and on N is



$$(\theta, t) \longmapsto (\theta + 2\pi t, t)$$

a **negative** Dehn twist is D_γ^{-1}.

If (V, ϕ) is an open book decomposition with $\dim V = 2$ then set $V' = V \cup (1\text{-handle})$ and choose a curve $\gamma \subset V'$ that intersects the cocore of the 1-handle one time. Set $\phi' = \phi \circ D_\gamma$.



we call (V', ϕ') a **stabilization** of (V, ϕ)

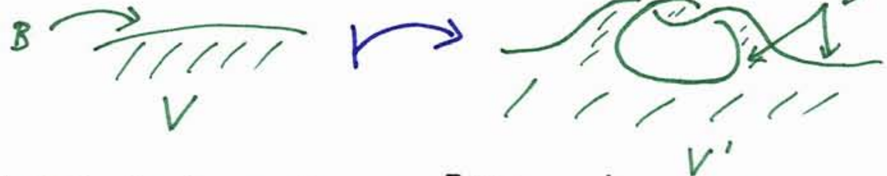
(if we take $\phi' = \phi \circ D_\gamma^{-1}$ then we get a **negative stabilization**).

Th$^{\underline{m}}$:

$\boxed{(M_{(V,\phi)}, \xi_{(V,\phi)}) \text{ is contactomorphic to } (M_{(V',\phi')}, \xi_{(V',\phi')}).}$

Remarks:

- $M_{(V,\phi)} \cong M_{(V',\phi')}$ is not hard to see and was known for some time (eg it is clear from Gabai '83)
- inside M the binding changes something like



  you can give an intrinsic definition of stabilization and then the theorem gives an isotopy.
- the full theorem is due to Giroux '00.
- the above theorems give a map

$$\left\{\begin{array}{l}\text{o.b.d. of } M \\ \text{upto (pos) stabilization} \\ \text{(and isotopy)}\end{array}\right\} \xrightarrow{\Phi} \left\{\begin{array}{l}\text{oriented contact} \\ \text{structures on } M \\ \text{upto isotopy}\end{array}\right\}$$

Th$^{\underline{m}}$ (Giroux '00):

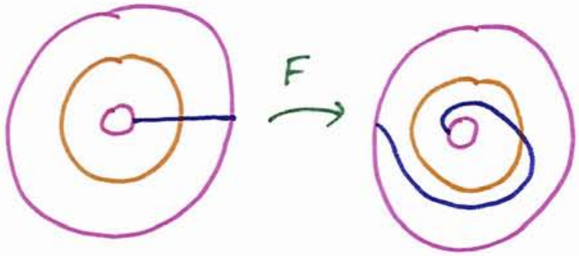
$\boxed{\Phi \text{ is a one-to-one correspondence}}$

Giroux Correspondence

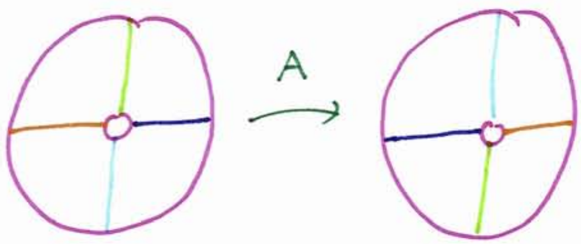
let (U, ω_0) be the unit cotangent bundle of S^n

let $F : U \to U$ be the time π map of the geodesic flow on S^n and

$A : U \to U$ the derivative of the antipodal map



Note: $A \circ F$ is the identity on ∂U



If (V, ω) is a symplectic manifold and L is a Lagrangian sphere in V then L has a neighborhood N symplectomorphic to $(U, \varepsilon \omega_0)$ for some ε.

A positive Dehn twist along L is the symplectomorphism

$$D_L : V \to V$$

that is the identity outside N and $A \circ F$ on N

a negative Dehn twist is D_L^{-1}.

Let (V, ϕ) be an open book with V a Weinstein manifold of dimension $2n$ and ϕ a symplecto.

If D^n is a Lagrangian disk properly embedded in V, then there is a canonical way to attach an n-handle to V along ∂D^n so that the Weinstein structure extends. Call the new manifold V'.

Moreover D^n extends to a Lagrangian sphere S in V'.

Set $\phi' = \phi \circ D_L$

We call (V', ϕ') a <u>stabilization</u> of (V, ϕ)

(if we take $\phi' = \phi \circ D_L^{-1}$ then we get a <u>negative stabilization</u>)

<u>Th$\underline{\text{m}}$ (Giroux-Mohsen ~2002):</u>

$(M_{(V,\phi)}, \xi_{(V,\phi)})$ is contactomorphic to $(M_{(V',\phi')}, \xi_{(V',\phi')})$.

<u>Th$\underline{\text{m}}$ (Giroux-Mohsen ~2002(?)):</u>

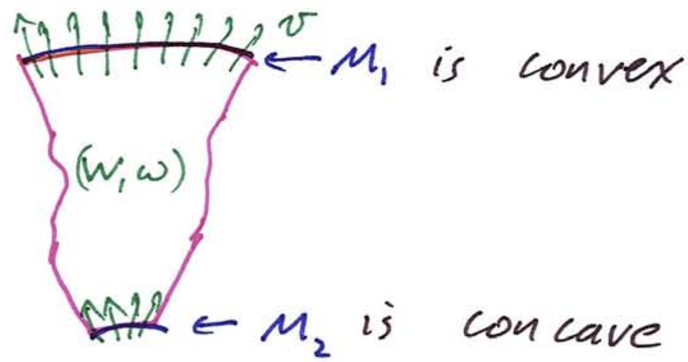
there is a one-to-one correspondence

$$\left\{\begin{array}{l}\text{O.b.d. of } M \text{ that} \\ \text{come from the} \\ \text{Ibort-Martinez-Presas} \\ \text{theorem, upto stabilization}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{oriented contact} \\ \text{structures on } M \\ \text{upto isotopy}\end{array}\right\}$$

Applications

let (W, ω) be a symplectic manifold

if M is a component of ∂W we say it is convex, resp. concave, if there exists a vector field (defined near M) v such that $\mathcal{L}_v \omega = \omega$ and v points out of, resp. into, W along M

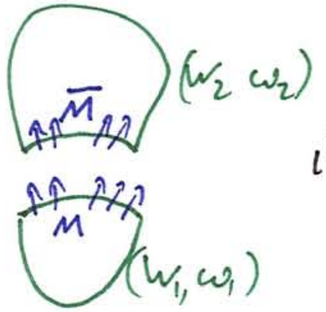


note that if M is convex/concave then

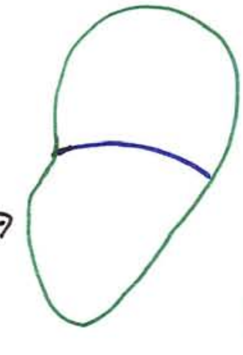
$$\alpha = (\iota_v \omega)_M$$

is a contact form on M.

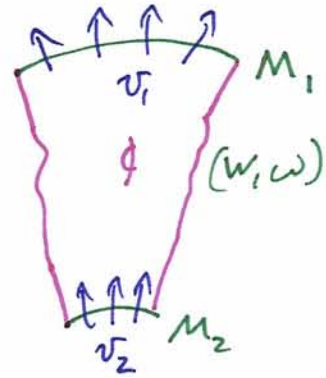
lemma:



if contact strs on M same then $\exists$ symp form ω on



If (W,ω) has boundary $\partial W = M_1 \cup \overline{M}_2$ with M_1 convex and $\overline{M}_2$ concave then we say W is a <u>symplectic cobordism</u> from $(M_2, \xi_2 = \ker(\iota_{v_2}\omega))$ to $(M_1, \xi_1 = \ker(\iota_{v_1}\omega))$.



If there is a Morse function $f: W \to \mathbb{R}$ with M_i, $i=1,2$, regular level sets and a gradient like vector field v for f extending v_1 and v_2 so that $\mathcal{L}_v \omega = \omega$ then we say (W,ω) is a <u>Weinstein cobordism</u> from (M_2, ξ_2) to (M_1, ξ_1).

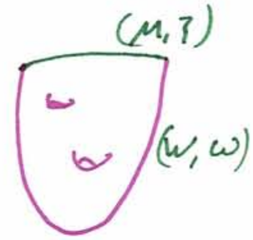
<u>Th$^{\underline{m}}$ (Eliashberg '90):</u>

Every Weinstein cobordism (W,ω) can be turned into a <u>Stein cobordism</u>.
(That is, there is a complex structure on W, denote its action on the tangent space by J, and a function $F: W \to \mathbb{R}$ such that

$$\omega_F(v, Jv) > 0 \quad \text{for all } v \neq 0$$

where $\omega_F = -d(dF \circ J)$. Such an F is called <u>strictly plurisubharmonic</u>.)

A symplectic cobordism (W,ω) from $\emptyset$ to (M,ξ) is called a <u>symplectic filling</u> of (M,ξ). Similarly for <u>Stein fillings</u>.



We say (W,ω) is a <u>weak symplectic filling</u> of (M,ξ) if $\partial W = M$ and $\omega|_\xi$ is non-degenerate.

(clearly Stein filling $\Rightarrow$ symp. filling $\Rightarrow$ weak symp filling)

in 3 dim's $\quad \not\Leftarrow$ (Ghiggini '05) $\quad \not\Leftarrow$ (Eliashberg '96)

<u>Theorem (Giroux '02):</u>

$\boxed{\text{A contact manifold } (M,\xi) \text{ is Stein fillable if and only if it is supported by an open book whose monodromy is a product of Dehn twists (positive)}}$

<u>Remark:</u> A weaker version of this theorem first appeared in Loi-Piergallini '01 (see also Akbulut-Ozbagci '01)

We sketch a proof of this below but we will restrict attention to dimension 3. We begin with some preliminary ideas.

let L be a Legendrian knot in (M^3, ξ)

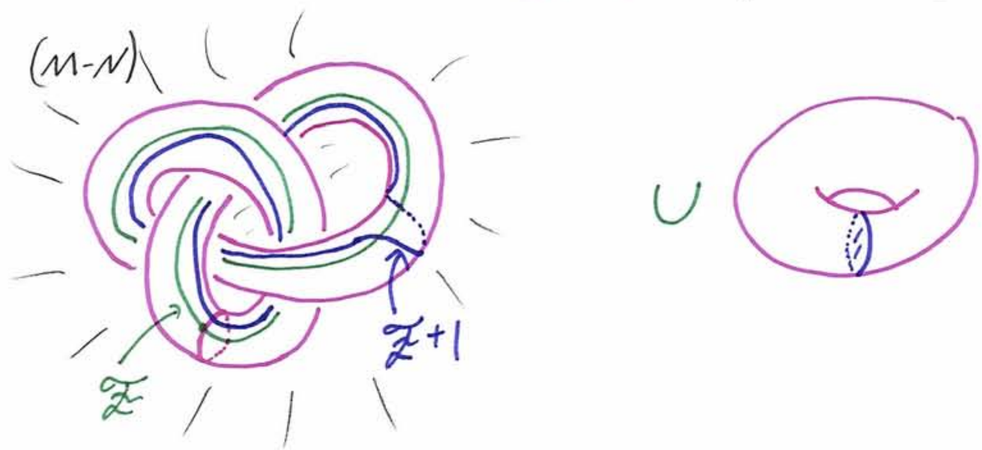
(i.e. $T_x L \subset \xi_x$ for all $x \in L$)

the contact planes give L a framing $\mathcal{F}$

L has a canonical neighborhood N

Set $M_\pm(L) = (\overline{M - N}) \cup (S^1 \times D^2)$

where $\{pt\} \times \partial D^2$ is glued to a curve in $\partial \overline{M - N}$ representing $\mathcal{F} \pm 1$



there is a canonical way to extend $\xi|_{\overline{M-N}}$ to $M_\pm(L)$

the resulting contact manifold is said to be obtained from (M, ξ) by <u>± 1-contact surgery</u> on L.

let (W, ω) be a Stein manifold.

- can always extend Stein structure over any 1-handle attached to W. Call result W'.
  the contact manifold $\partial W'$ is obtained from ∂W by connect summing with standard (tight) structure on $S^1 \times S^2$.
- can always extend Stein structure over any 2-handle attached to W along a Legendrian L with framing $\mathcal{F}-1$. Call result W'.
  the contact manifold $\partial W'$ is obtained from ∂W by -1-contact surgery on L.

Th$^{\underline{m}}$ (Eliashberg '90):

> Any Stein 4-manifold is obtained from B^4 by attaching 1- and 2-handles as above.

Proof of Giroux ($\Rightarrow$): let (W, ω) be a Stein filling of (M, ξ).

let W' be the union of B^4 and all 1-handles

$$\partial W' = \# S^1 \times S^2$$

let $L_1 \dots L_k$ be knots we need to attach 2-handles along by making $L_1 \ldots L_k$ part of a 1-skeleton of $\partial W'$ we can assume they sit on pages of an o.b.d. for $\partial W'$

(there is an obd for $\partial W'$ with trivial monodromy so we can, maybe after stabilizing, assume above obd. has monodromy = composition of $\underbrace{\ }_{\text{positive}}$ Dehn twists)

the result now follows from

lemma:

> let (V, ϕ) be an obd for (M, ξ)
> L a Legendrian knot on a page
> then $(M_{(V, \phi \circ D_L)}, \xi_{(V, \phi \circ D_L)})$ is obtained from (M, ξ) by $\underbrace{\text{Legendrian surgery}}_{-1\text{-contact surgery}}$ on L

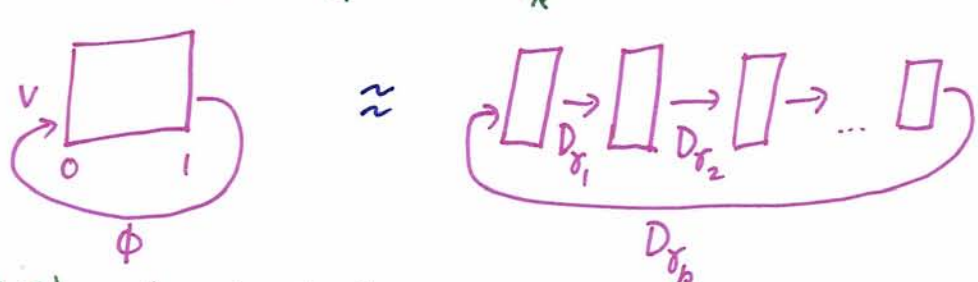
∎

Proof of Giroux($\Leftarrow$):

given (V, ϕ) supporting (M, ξ) with

$$\phi = D_{\gamma_1} \circ \ldots \circ D_{\gamma_k}$$

note

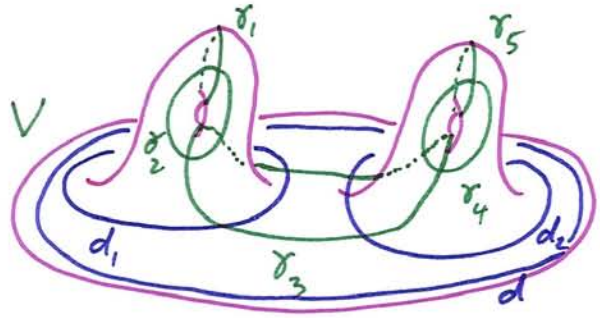


so (M, ξ) obtained from $(M_{(V, id_V)}, \xi_{(V, id_V)})$ by Leg. surgery. $\therefore$ Stein fillable ∎

$Th^{\underline{m}}$ (Baker - E - Van Horn-Morris; Wand '09):

> there are obd. for Stein fillable contact structures whose monodromy cannot be written as a composition of $\underbrace{\ }_{\text{positive}}$ Dehn twists.

Proof:



set $D_i = D_{\gamma_i}$

$\partial_i = D_{d_i}$

$$\phi = \underbrace{D_5\circ D_4\circ D_3\circ D_2\circ D_1\circ D_5\circ D_4\circ D_3\circ D_2\circ D_5\, D_4\circ D_3\circ D_5\circ D_4\circ D_5}_{r_2}\circ\partial_1^{-1}\circ\partial_2^{-1}\circ D_1\circ D_2$$

Fact:
- $r_2\circ r_2 = D_d$
- $(M_{(V,\phi)}, \xi_{(V,\phi)}) \cong (S^3, \xi_{std})$ ← Stein fillable
- any Stein filling of (S^3, ξ_{std}) is diffeomorphic to B^4

$\Rightarrow$ if ϕ is a comp of pos. Dehn twists then $\exists$ 4-curves $\delta_1 \dots \delta_4$ s.t. $\phi = \underbrace{D_{\delta_1}\circ\dots\circ D_{\delta_4}}_{F}$

so $D_d = F\circ\partial_1\circ\partial_2\circ D_1^{-1}\circ D_2^{-1}\circ F\circ\partial_1\circ\partial_2\circ D_1^{-1}\circ D_2^{-1}$

$= F\circ\partial_2\circ(D_1\circ D_2)^5\circ F\circ\partial_2\circ(D_1\circ D_2)^5 = (F\circ(D_4\circ D_5)^6\circ(D_1\circ D_2)^5)^2$

(length of D_d) = 52

recall $H_1(\mathrm{Map}^+(\hat{V})) = \mathbb{Z}/10\mathbb{Z}$

mapping class group of $\hat{V} = V$ capped off

so any representation of the identity map by Dehn twists along ~~separati~~ non-separating curves must have length divisible by 10 $\otimes$

$\blacksquare$

There are many topological applications of the following result.

Th$^{\underline{m}}$ (Eliashberg '04, E '04):

$$\boxed{\text{A weak symplectic filling of a 3-dimensional contact manifold can be symplectically embedded in a closed symplectic manifold.}}$$

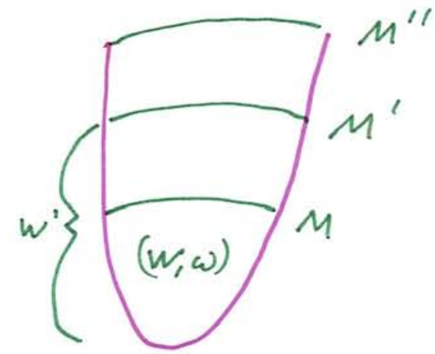
Remarks:

- both proofs use obd.
- for Stein fillings this was proven by Lisca-Matić '97 (also see Akbulut-Ozbagci '02)
- for (strong) symplectic fillings this was proven by Gay '02 and E-Honda '02.

Proof: let (V, ϕ) be an o.b.d. supporting (M^3, ξ) and (W, ω) a weak filling.

- by stabilizing we can assume ∂V connected.
- by attaching Stein 2-handles to ∂W along Legendrian knots on pages of (V, ϕ) we can embed (W, ω) in (W', ω') so that $\partial W'$ is a homology sphere.

  Fact: on a homology sphere a weak filling can be made strong.
- by attaching more 2-handles to $\partial W'$ we can embed (W', ω') in (W'', ω'') s.t. $\partial(W'', \omega'') = (M'', \xi'')$ is supported by (V, D_δ^k) where δ is a curve parallel to ∂V.

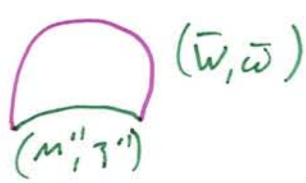


  in fact (after more stabilizations) can get monodromy D_δ.

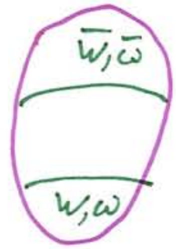
uses relations in mapping class group

- M'' is an S^1-bundle over $\hat{V}$ with Euler number -1

easy to find



so



is desired closed manifold.

∎

The above theorem is a key part of the following:

Th$^{\underline{m}}$ (Ozsváth-Szabó '06):

> The unknot, trefoil and figure eight knots are determined by surgery.
> That is if $S^3_K(r)$ is diffeomorphic (or$^{\underline{n}}$ pres) to
> r-surgery on one of the above then
> K must be the corresponding knot.

Remarks:
- the unknot case was conjectured by Gordon in the late 70's.
- the unknot case was originally proven by Kronheimer-Mrowka-Ozsváth-Szabó '04

Th$^{\underline{m}}$ (Kronheimer-Mrowka '04):

> All non-trivial knots satisfy Property P.
> (that is non-trivial surgery on them yields a manifold with non-trivial fundamental group)

Remarks:
- this was a conjecture of Bing-Martin and González-Acuña from the early 70's.
- the conjecture says there is no counter example to the Poincaré conjecture coming from surgery on a knot.

More 3D Contact Geometry

A contact 3-manifold (M, ξ) is called overtwisted if there is an embedded disk D in M such that ∂D is Legendrian and the twisting of ξ along ∂D is 0.

Otherwise (M, ξ) is called tight.

Thm (Eliashberg '89):

On a closed 3-manifold there is a one-to-one correspondence between homotopy classes of plane fields and isotopy classes of overtwisted contact structures.

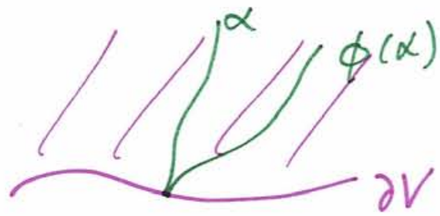
Tight structures seem to have a closer tie to the smooth topology of a 3-manifold (i.e. more geometric significance).

Thm (Gromov-Eliashberg late '80's):

if (M, ξ) is (weakly) fillable then (M, ξ) is tight.

Remarks:
- there are 3-manifolds that do not admit tight contact structures (E-Honda '01).
- there are tight but not fillable contact structures (E-Honda '02)

We say an open book (V, ϕ) is right-veering if for all properly embedded arcs α either $\phi(\alpha)$ and α are isotopic or (once $\phi(\alpha)$ isotoped to minimize intersections with α) we have



Th$^{\underline{m}}$ (Honda-Kazez-Matic '07):

$$\boxed{\text{A contact structure } \xi \text{ on } M \text{ is tight if and only if all of its o.b.d. } (V, \phi) \text{ have right-veering monodromy.}}$$

Remark: overtwisted contact structures can be supported by right-veering o.b.d.s.

Notice we have two interesting monoids in $Map^+(V, \partial V)$

$$Dehn^+(\Sigma) = \{ \phi \in Map^+(\Sigma, \partial\Sigma) : \phi \text{ comp of pos Dehn twists} \}$$

$$Veer(\Sigma) = \{ \phi \in Map^+(\Sigma, \partial\Sigma) : \phi \text{ is right-veering} \}$$

Are there others?

Th$^{\underline{m}}$ (Baker-E-Van Horn-Morris; Baldwin '09):

> If $\mathcal{P}$ is a property of contact structures that is preserved under connect sums and Legendrian surgery, then $Map^{\mathcal{P}}(\Sigma) = \{\phi \in Map^+(\Sigma, \partial\Sigma) : \xi_{(\Sigma,\phi)} \text{ has } \mathcal{P}\}$ is a monoid.

So

$$Stein(\Sigma) = \{\phi \in Map^+(\Sigma, \partial\Sigma) : \xi_{(\Sigma,\phi)} \text{ Stein fillable}\}$$
$$Strong(\Sigma) = \{\phi \in Map^+(\Sigma, \partial\Sigma) : \xi_{(\Sigma,\phi)} \text{ strongly fillable}\}$$
$$Weak(\Sigma) = \{\phi \in Map^+(\Sigma, \partial\Sigma) : \xi_{(\Sigma,\phi)} \text{ weakly fillable}\}$$
$$HFH(\Sigma) = \{\phi \in Map^+(\Sigma, \partial\Sigma) : \xi_{(\Sigma,\phi)} \text{ has non-zero Heegaard-Floer invariant}\}$$

are monoids.

$$\begin{tikzcd}
 & & & Weak(\Sigma) \arrow[dr, phantom, "\subsetneq"] & \\
Dehn^+(\Sigma) \arrow[r, phantom, "\subsetneq"] & Stein(\Sigma) \arrow[r, phantom, "\subsetneq"] & Strong(\Sigma) \arrow[ur, phantom, "\subsetneq"] \arrow[dr, phantom, "\subsetneq"] & & Veer(\Sigma) \\
 & & & HFH(\Sigma) \arrow[ur, phantom, "\subsetneq"] &
\end{tikzcd}$$

Eliashberg '96

Ghiggini '05

Baker-E-Van Horn-Morris
Wand '09

Ozsváth-Szabó '04

Honda-Kazez-Matić '07

Cor:

> $Tight(\Sigma) = \{\phi \in Map^+(\Sigma, \partial\Sigma) : \xi_{(\Sigma,\phi)} \text{ tight}\}$ is a monoid if and only if Legendrian surgery preserves tightness.

Invariants

given a contact manifold (M,ξ) we define the

support genus

$$sg(\xi) = \min\{\text{genus}(\text{page}(L,\pi)) : (L,\pi) \text{ supports } \xi\}$$

binding number

$$bn(\xi) = \min\{|\partial\Sigma| : \Sigma \text{ a page of an open book supporting } \xi \text{ and } g(\Sigma) = sg(\xi)\}$$

Th$^{\underline{m}}$ (E '04):

If (M,ξ) is overtwisted then $sg(\xi) = 0$

Th$^{\underline{m}}$ (E '04):

If (W,ω) is a symplectic filling of (M,ξ) and (M,ξ) has $sg = 0$ then

$$b_2^+(W) = b_2^0(W) = 0$$

and ∂W connected.
Moreover the intersection form of W embeds in a diagonalizable form.

example:

L

Legendrian surgery on L yields a contact manifold with $sg > 0$.

Remark: Onaran '09 has extended much of this to invariants of Legendrians.

Question: Does $bn(\xi)$ tell us anything? (Giroux torsion?)

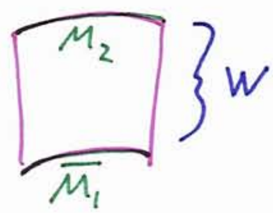
Heegaard - Floer Theory

given a 3-manifold M Ozsváth-Szabó define an invariant

$$\widehat{HF}(M) = \bigoplus_{t \in Spin^c(M)} \widehat{HF}(M,t)$$

$\mathbb{Z}_2$-vector space.

and given a bordism



a map $\qquad F_W : \widehat{HF}(M_1) \to \widehat{HF}(M_2)$

if $s \in Spin^c(W)$ then get

$$F_{W,s} : \widehat{HF}(M_1, s|_{M_1}) \to \widehat{HF}(M_2, s|_{M_2})$$

Remarks:

- $t \in Spin^c(M)$ like a plane field on M upto some equivalence.

  note: $t \in Spin^c(M) \Rightarrow c_1(t) \in H^2(M)$

- $s \in Spin^c(W)$ like an almost complex structure on $W - \{\text{some pts}\}$

  note:
  - $\exists$ map $Spin^c(W) \to Spin^c(\partial W)$ (take $\mathbb{C}$-tangencies)
  - $s \in Spin^c(W) \Rightarrow c_1(s) \in H^2(W)$

- theory can be defined over $\mathbb{Z}$
- there are other theories: $HF^{\pm}$, HF^{∞}

Th$^{\underline{m}}$ (Ozsváth-Szabó):

if $\begin{array}{c} V \to M \\ \downarrow p \\ S^1 \end{array}$ is a closed 3-manifold and $t_{can} \in Spin^c(M)$ is structure associated to tangents to fibers of p

then $\widehat{HF}(M, t_{can}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and there is a distinguished generator g

$\widehat{HF}(M, t) = 0$ for all $t \in Spin^c(M)$ such that $c_1(t)([V]) = c_1(t_{can})([V])$.

Now given a contact structure ξ on M, let (V, Φ) be a supporting o.b.d. with connected ~~boundary~~ binding

let $W = M \times [0,1] \cup (2\text{-handle})$ attached to binding with framing 0.

easy to see M' fibers over S^1 with fiber $\hat{V}$

note $F_{-W} : \widehat{HF}(-M', t_{can}) \to \widehat{HF}(-M, t_{can})$

$\widehat{HF}(-M', t_{can}) \cong \widehat{HF}(M', t_{can}) \ni g$

the Heegaard-Floer contact invariant of ξ is

$$c(\xi) \in \widehat{HF}(-M, t_{can})$$

$$c(\xi) = F_{-W}(g)$$

Th$^{\underline{m}}$ (Ozsváth-Szabó '05):

$$\boxed{c(\xi) \text{ is an invariant of } \xi.}$$

Remark: need to develop Heegaard-Floer knot invariants for the proof.

Properties (O-S):

- ξ the standard contact structure on S^3

  $c(\xi) \neq 0 \qquad (\widehat{HF}(S^3) = \mathbb{Z}_2)$

- 

  obtained by $+1$-contact surgery on $L \subset M_1$ $\Rightarrow F_{-W}(c(\xi_1)) = c(\xi_2)$.

- if (M, ξ) is Stein fillable then $c(\xi) \neq 0$.
- if (M, ξ) is overtwisted then $c(\xi) = 0$.

Th$^{\underline{m}}$ (Ozsváth-Stipsicz-Szabó '05):

> can use $c(\xi)$ (actually need $c_+(\xi) \in HF^+(-M)$)
> to get an obstruction to $sg(\xi) = 0$.

Much of the progress in understanding contact structures, especially tight not fillable ones, is due (in part) to this contact invariant.

Some sample results:

Th$^{\underline{m}}$ (Lisca-Stipsicz '04):

For each $n \in \mathbb{N}$ there is a Seifert fibered rational homology sphere M_n that has at least n distinct tight contact structures none of which are fillable.

Th$^{\underline{m}}$ (L-S '04):

Let K be a positive (p,q)-torus knot. For any $r \in \mathbb{Q}$, except $r = pq-p-q$, $S^3_K(r)$ has a tight contact structure.

There are also many classification results on Seifert fibered spaces (and others). Many of these are due to some combination of Lisca, Ghiggini and Stipsicz. (building on work of Wu and E-Honda.)

Sins of Omission

- **Reeb Dynamics**

  Abbas-Cieliebak-Hofer '05: sg(3) ⇒ Weinstein conjecture true.

  Colin-Honda '08: if monodromy pseudo-anosov (+ technical), then Weinstein conjecture true.

- **Legendrian Invariants via Heegaard-Floer**

  Lisca-Ozsváth-Stipsicz-Szabó '08

- **Deformations of Foliations**

  Mori '02, E '07: every contact structure on a 3-manifold is the deformation of a foliation.

- **Partial Open Books**

  Honda-Kazez-Matić '08

- **Boundary Conditions of a Lefschetz fibration**

- **Harer's Conjecture**

  Giroux-Goodman '06: any fibered link in S^3 can be obtained from the unknot by a sequence of plumbings and deplumbings of Hopf bands.