Math 497C
Curves and Surfaces
Spring 2004, Penn State

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Lecture Notes 0

Basics of Euclidean Geometry

By $\mathbf{R}$ we shall always mean the set of real numbers. The set of all n-tuples of real numbers $\mathbf{R}^n := \{(p^1, \dots, p^n) \mid p^i \in \mathbf{R}\}$ is called the *Euclidean n-space*. So we have

$$p \in \mathbf{R}^n \iff p = (p^1, \dots, p^n), \quad p^i \in \mathbf{R}.$$

Let p and q be a pair of points (or vectors) in $\mathbf{R}^n$. We define $p + q := (p^1 + q^1, \dots, p^n + q^n)$. Further, for any scalar $r \in \mathbf{R}$, we define $rp := (rp^1, \dots, rp^n)$. It is easy to show that the operations of addition and scalar multiplication that we have defined turn $\mathbf{R}^n$ into a vector space over the field of real numbers. Next we define the standard *inner product* on $\mathbf{R}^n$ by

$$\langle p, q \rangle = p^1 q^1 + \ldots + p^n q^n.$$

Note that the mapping $\langle \cdot, \cdot \rangle \colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is linear in each variable and is symmetric. The standard inner product induces a norm on $\mathbf{R}^n$ defined by

$$\|p\| := \langle p, p \rangle^{\frac{1}{2}}.$$

If $p \in \mathbf{R}$, we usually write $|p|$ instead of $\|p\|$.

The first nontrivial fact in Euclidean geometry, and an exercise which every geometer should do, is

Exercise 1. (The Cauchy-Schwartz inequality) Prove that

$$|\langle p, q \rangle| \leqslant \|p\| \, \|q\|,$$

for all p and q in $\mathbf{R}^n$ (*Hints:* If p and q are linearly dependent the solution is clear. Otherwise, let $f(\lambda) := \langle p - \lambda q, p - \lambda q \rangle$. Then $f(\lambda) > 0$. Further, note that $f(\lambda)$ may be written as a quadratic equation in λ. Hence its discriminant must be negative).

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The standard Euclidean distance in $\mathbf{R}^n$ is given by

$$\mathrm{dist}(p,q) := \|p-q\|.$$

Exercise 2. (The triangle inequality) Show that

$$\mathrm{dist}(p,q) + \mathrm{dist}(q,r) \geqslant \mathrm{dist}(p,r)$$

for all p, q in $\mathbf{R}^n$. (*Hint:* use the Cauchy-Schwartz inequality).

By a *metric* on a set X we mean a mapping $d\colon X \times X \to \mathbf{R}$ such that

1. $d(p,q) \geqslant 0$, with equality if and only if $p = q$.
2. $d(p,q) = d(q,p)$.
3. $d(p,q) + d(q,r) \geqslant d(p,r)$.

These properties are called, respectively, positive-definiteness, symmetry, and the triangle inequality. The pair (X,d) is called a *metric space*. Using the above exercise, one immediately checks that $(\mathbf{R}^n, \mathrm{dist})$ is a metric space. *Geometry*, in its broadest definition, is the study of metric spaces, and *Euclidean Geometry*, in the modern sense, is the study of the metric space $(\mathbf{R}^n, \mathrm{dist})$.

Finally, we define the *angle* between a pair of nonzero vectors in $\mathbf{R}^n$ by

$$\mathrm{angle}(p,q) := \cos^{-1}\frac{\langle p,q\rangle}{\|p\|\,\|q\|}.$$

Note that the above is well defined by the Cauchy-Schwartz inequality. Now we have all the necessary tools to prove the most famous result in all of mathematics:

Exercise 3. (The Pythagorean theorem) Show that in a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the length of the sides (*Hint:* First prove that whenever $\langle p,q\rangle = 0$, $\|p\|^2 + \|q\|^2 = \|p-q\|^2$. Then show that this proves the theorem.).

The next exercise is concerned with another corner stone of Euclidean Geometry; however, the proof requires the use of some trignometric identities and is computationally intensive.

Exercise* 4. (Sum of the angles in a triangle) Show that the sum of the angles in a triangle is π.