Math 497C Dec 8, 2004[1]
Curves and Surfaces
Fall 2004, PSU

Lecture Notes 16

2.14 Applications of the Gauss-Bonnet theorem

We talked about the Gauss-Bonnet theorem in class, and you may find the statement and prove of it in Gray or do Carmo as well. The following are all simple consequences of the Gauss-Bonnet theorem:

Exercise 1. Show that the sum of the angles in a triangle is π.

Exercise 2. Show that the total geodesic curvature of a simple closed planar curve is 2π.

Exercise 3. Show that the Gaussian curvature of a surface which is homeomorphic to the torus must alwasy be equal to zero at some point.

Exercise 4. Show that a simple closed curve with total geodesic curvature zero on a sphere bisects the area of the sphere.

Exercise 5. Show that there exists at most one closed geodesic on a cylinder with negative curvature.

Exercise 6. Show that the area of a geodesic polygon with k vertices on a sphere of radius 1 is equal to the sum of its angles minus $(k-2)\pi$.

Exercise 7. Let p be a point of a surface M, T be a geodesic triangle which contains p, and α, β, γ be the angles of T. Show that

$$K(p) = \lim_{T \to p} \frac{\alpha + \beta + \gamma - \pi}{Area(T)}.$$

In particular, note that the above proves Gauss's Theorema Egregium.

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Exercise 8. Show that the sum of the angles of a geodesic triangle on a surface of positive curvature is more than π, and on a surface of negative curvature is less than π.

Exercise 9. Show that on a simply connected surface of negative curvature two geodesics emanating from the same point will never meet.

Exercise 10. Let M be a surface homeomorphic to a sphere in $\mathbf{R}^3$, and let $\Gamma \subset M$ be a closed geodesic. Show that each of the two regions bounded by Γ have equal areas under the Gauss map.

Exercise 11. Compute the area of the pseudo-sphere, i.e. the surface of revolution obtained by rotating a tractrix.