Math 497C
Curves and Surfaces
Fall 2004, PSU

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Lecture Notes 2

1.5 Isometries of the Euclidean Space

Let M_1 and M_2 be a pair of metric space and d_1 and d_2 be their respective metrics. We say that a mapping $f\colon M_1 \to M_2$ is an *isometry* provided that

$$d_1(p,q) = d_2\big(f(p), f(q)\big),$$

for all pairs of points in p, $q \in M_1$. An *orthogonal transformation* $A\colon \mathbf{R}^n \to \mathbf{R}^n$ is a linear map which preserves the inner product, i.e.,

$$\big\langle A(p), A(q)\big\rangle = \langle p, q\rangle$$

for all p, $q \in \mathbf{R}^n$. One may immediately check that an orthogonal transformation is an isometry. Conversely, we have:

Theorem 1. *If $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is an isometry, then*

$$f(p) = f(o) + A(p),$$

where o is the origin of $\mathbf{R}^n$ and A is an orthogonal transformation.

Proof. Let

$$\overline{f}(p) := f(p) - f(o).$$

We need to show that $\overline{f}$ is a linear and $\langle \overline{f}(p), \overline{f}(q)\rangle = \langle p, q\rangle$. To see the latter note that

$$\langle x-y, x-y\rangle = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle.$$

Thus, using the definition of $\overline{f}$, and the assumption that f is an isometry, we obtain

$$\begin{aligned}
2\big\langle \overline{f}(p), \overline{f}(q)\big\rangle &= \|\overline{f}(p)\|^2 + \|\overline{f}(q)\|^2 - \|\overline{f}(p) - \overline{f}(q)\|^2 \\
&= \|f(p) - f(o)\|^2 + \|f(q) - f(o)\|^2 - \|f(p) - f(q)\|^2 \\
&= \|p\|^2 + \|q\|^2 - \|p - q\|^2 \\
&= 2\langle p, q\rangle.
\end{aligned}$$

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Next note that, since $\overline{f}$ preserves the inner product, if e_i, $i = 1 \dots n$, is an orthonormal basis for $\mathbf{R}^n$, then so is $\overline{f}(e_i)$. Further,

$$\begin{aligned}
\left\langle \overline{f}(p+q), \overline{f}(e_i) \right\rangle &= \langle p+q, e_i \rangle = \langle p, e_i \rangle + \langle q, e_i \rangle \\
&= \left\langle \overline{f}(p), \overline{f}(e_i) \right\rangle + \left\langle \overline{f}(q), \overline{f}(e_i) \right\rangle \\
&= \left\langle \overline{f}(p) + \overline{f}(q), \overline{f}(e_i) \right\rangle.
\end{aligned}$$

Thus if follows that

$$\overline{f}(p+q) = \overline{f}(p) + \overline{f}(q).$$

Similarly, for any constant c,

$$\left\langle \overline{f}(cp), \overline{f}(e_i) \right\rangle = \langle cp, e_i \rangle = \left\langle c\overline{f}(p), \overline{f}(e_i) \right\rangle,$$

which in turn yields that $\overline{f}(cp) = \overline{f}(p)$, and completes the proof $\overline{f}$ is linear. $\square$

If $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is an isometry with $f(o) = o$ we say that it is a *rotation*, and if $A = f - f(o)$ is identity we say that f is a *translation*. Thus another way to state the above theorem is that an isometry of the Euclidean space is the composition of a rotation and a translation.

Any mapping $f\colon \mathbf{R}^n \to \mathbf{R}^m$ given by $f(p) = q + A(p)$, where $q \in \mathbf{R}^n$, and A is any linear transformation, is called an *affine map* with translation part q and linear part A. Thus yet another way to state the above theorem is that any isometry $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is an affine map whose linear part is orthogonal

An isometry of Euclidean space is also referred to as a *rigid motion*. Recall that if A^T denotes the transpose of matrix A, then

$$\left\langle A^T(p), q \right\rangle = \left\langle p, A(q) \right\rangle.$$

This yields that if A is an orthogonal transformation, then $A^T A$ is the identity matrix. In particular

$$1 = \det(A^T A) = \det(A^T)\det(A) = \det(A)^2.$$

So $\det(A) = \pm 1$. If $\det(A) = 1$, then we say that A is a *special* orthogonal transformation, $A\colon \mathbf{R}^n \to \mathbf{R}^n$ is a *proper* rotation, and any isometry $f\colon \mathbf{R}^n \to \mathbf{R}^n$ given by $f(p) = q + A(p)$ is a *proper* rigid motion.

Exercise 2 (Isometries of $\mathbf{R}^2$). Show that if $A\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a proper rotation, then it may be represented by a matrix of the form

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Further, any improper rotation is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Conclude then that any isometry of $\mathbf{R}^2$ is a composition of a translation, a proper rotation, and possibly a reflection with respect to the y-axis.

In the following exercise you may use the following fact: any continuous mapping of $f\colon \mathbf{S}^2 \to \mathbf{S}^2$ of the sphere to itself has a fixed point or else sends some point to its antipodal reflection. Alternatively you may show that every 3×3 orthogonal matrix has a nonzero real eigenvalue.

Exercise 3 (Isometries of $\mathbf{R}^3$). (a) Show that any proper rotation $A\colon \mathbf{R}^3 \to \mathbf{R}^3$ fixes a line ℓ through the origin. Further if Π is a plane which is orthogonal to ℓ, then A maps Π to itself by rotating it around the point $\ell \cap \Pi$ by an angle which is the same for all such planes. (b) Show that any rotation of $\mathbf{R}^3$ is a composition of rotations about the x, and y-axis. (c) Find a pair of proper rotations A_1, A_2 which do not commute, i.e., $A_1 \circ A_2 \neq A_2 \circ A_1$. (d) Note that any improper rotation becomes proper after multiplication by an orthogonal matrix with negative determinant. Use this fact to show that any rotation of $\mathbf{R}^3$ is the composition of a proper rotation with reflection through the origin, or reflection through the xy-plane. (e) Conclude that any isometry of $\mathbf{R}^3$ is a composition of the following isometries: translations, rotations about the x, or y-axis, reflections through the origin, and reflections through the xy-plane.

Exercise 4. Show that if $\alpha\colon I \to \mathbf{R}^2$ is a C^1 curve, then for any $p \in I$ there exists and open neighborhood U of p in I and a rigid motion $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ such that α restricted to U has a reparametrization $\beta\colon J \to \mathbf{R}^2$, where $J \subset \mathbf{R}$ is a neighborhood of the origin, and $B(t) = (t, h(t))$ for some C^1 function $f\colon J \to \mathbf{R}$ with $h(0) = h'(0) = 0$.

1.6 Invariance of length under isometries

Recalling the definition of length as the limit of polygonal approximations, one immediately sees that

Exercise 5. Show that if $\alpha\colon [a,b] \to \mathbf{R}^n$ is a rectifiable curve, and $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is an isometry, then $\text{length}[\alpha] = \text{length}[f \circ \alpha]$.

Recall that earlier we had shown that the length of a curve was invariant under reparametrization. The above exercise further confirms that length is indeed a 'geometric quantity'. In the case where α is C^1, it is useful to give also an analytic proof for the above exercise, mainly as an excuse to recall and apply some basic concepts from multivariable calculus.

Let $U \subset \mathbf{R}^n$ be an open subset, and $f\colon U \to \mathbf{R}^m$ be a map. Note that f is a list of m functions of n variables:

$$f(p) = f(p^1, \dots, p^n) = \big(f^1(p^1, \dots, p^n), \dots, f^m(p^1, \dots, p^n)\big).$$

The first order partial derivatives of f are given by

$$D_j f^i(p) := \lim_{h \to 0} \frac{f^i(p^1, \dots, p^j + h, \dots, p^n) - f^i(p^1, \dots, p^j, \dots, p^n)}{h}.$$

If all the functions $D_j f^i\colon U \to \mathbf{R}$ exist and are continuous, then we say that f is C^1. The *Jacobian* of f at p is the $m \times n$ matrix defined by

$$J_p(f) := \begin{pmatrix} D_1 f^1(p) & \cdots & D_n f^1(p) \\ \vdots & & \vdots \\ D_1 f^m(p) & \cdots & D_n f^m(p) \end{pmatrix}.$$

The *derivative* of f at p is the linear transformation $Df(p)\colon \mathbf{R}^n \to \mathbf{R}^m$ given by the above matrix, i.e.,

$$\Big(Df(p)\Big)(x) := \Big(J_p(f)\Big)(x).$$

Exercise 6 (Derivative of linear maps). Show that if $A\colon \mathbf{R}^n \to \mathbf{R}^m$ is a linear map, then

$$DA(p) = A$$

for all $p \in \mathbf{R}^n$. In other words, for each $p \in \mathbf{R}^n$, $(DA(p))(x) = A(x)$, for all $x \in \mathbf{R}^n$.(*Hint:* Let a_{ij}, $i = 1 \dots n$, and $j = 1 \dots m$, be the coefficients of the matrix representation of A. Then $A^j(p) = \sum_{i=1}^n a_{ij} p_i$.)

Another basic fact is the chain rule which states that if $g\colon \mathbf{R}^m \to \mathbf{R}^\ell$ is a differentiable function, then

$$D(f \circ g)(p) = Df\big(g(p)\big) \circ Dg(p).$$

Now let $\alpha\colon I \to \mathbf{R}^n$ be a C^1 curve and $f\colon \mathbf{R}^n \to \mathbf{R}^n$, given by $f(p) = f(o) + A(p)$ be an isometry. Then

$$\begin{aligned}
\text{length}[f \circ \alpha] &= \int_I \|D(f \circ \alpha)(t)\|\, dt && (1)\\
&= \int_I \|Df\big(\alpha(t)\big) \circ D\alpha(t)\|\, dt && (2)\\
&= \int_I \|DA\big(\alpha(t)\big) \circ D\alpha(t)\|\, dt && (3)\\
&= \int_I \|A\big(D\alpha(t)\big)\|\, dt && (4)\\
&= \int_I \|D\alpha(t)\|\, dt && (5)\\
&= \text{length}[\alpha] && (6)
\end{aligned}$$

The six equalities above are due respectively to (1) definition of length, (2) the chain rule, (3) definition of isometry f, (4) Exercise 6, (5) definition of orthogonal transformation, and (6) finally definition of length applied again.

1.7 Curvature of $\mathbf{C^2}$ regular curves

The curvature of a curve is a measure of how fast it is turning. More precisely, it is the speed, with respect to the arclength parameter, of the unit tangent vector of the curve. The unit tangent vector, a.k.a. *tangential indicatrix*, or *tantrix* for short, of a regular curve $\alpha\colon I \to \mathbf{R}^n$ is defined as

$$T(t) := \frac{\alpha'(t)}{\|\alpha'(t)\|}.$$

Note that the tantrix is itself a curve with parameter ranging in I and image lying on the unit sphere $\mathbf{S}^{n-1} := \{x \in \mathbf{R}^n \mid \|x\| = 1\}$. If α is parametrized with respect to arclength, i.e., $\|\alpha'(t)\| = 1$, then the curvature is given by

$$\kappa(t) = \|T'(t)\| = \|\alpha''(t)\| \quad \text{(provided } \|\alpha'\| = 1).$$

Thus the curvature of a road is the amount of centripetal force which you would feel, if you traveled on it in a car which has unit speed; the tighter the turn, the higher the curvature, as is affirmed by the following exercise:

Exercise 7. Show that the curvature of a circle of radius r is $\frac{1}{r}$, and the curvature of the line is zero (First you need to find arclength parametrizations for these curves).

Recall that, as we showed earlier, there exists a unique way to reparametrize a curve $\alpha\colon [a,b] \to \mathbf{R}^n$ by arclength (given by $\alpha \circ s^{-1}(t)$). Thus the curvature does not depend on parametrizations. This together with the following exercise shows that κ is indeed a 'geometric quantity'.

Exercise 8. Show that κ is invariant under isometries of the Euclidean space (*Hint:* See the computation at the end of the last subsection).

As a practical matter, we need to have a definition for curvature which works for all curves (not just those already parametrized by arclength), because it is often very difficult, or even impossible, to find explicit formulas for unit speed curves.

To find a general formula for curvature of C^2 regular curve $\alpha\colon I \to \mathbf{R}^n$, let $T\colon I \to \mathbf{S}^{n-1}$ be its tantrix. Let $s\colon I \to [0,L]$ be the arclength function. Since, as we discussed earlier s is invertible, we may define

$$\overline{T} := T \circ s^{-1}$$

to be a reparametrization of T. Then curvature may be defined as

$$\kappa(t) := \|\overline{T}'(s(t))\|.$$

By the chain rule,

$$\overline{T}'(t) = T'\Big(s^{-1}(t)\Big) \cdot \Big(s^{-1}\Big)'(t).$$

Further recall that $(s^{-1})'(t) = 1/\|\alpha'(s^{-1}(t))\|$. Thus

$$\kappa(t) = \frac{\|T'(t)\|}{\|\alpha'(t)\|}.$$

Exercise 9. Use the above formula, together with definition of T, to show that

$$\kappa(t) = \frac{\sqrt{\|\alpha'(t)\|^2\|\alpha''(t)\|^2 - \langle \alpha'(t), \alpha''(t)\rangle^2}}{\|\alpha'(t)\|^3}.$$

In particular, in $\mathbf{R}^3$, we have

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}.$$

(*Hint*: The first identity follows from a straight forward computation. The second identity is an immediate result of the first via the identity $\|v \times w\|^2 = \|v\|^2\|w\|^2 - \langle v, w\rangle^2$.)

Exercise 10. Show that the curvature of a planar curve which satisfies the equation $y = f(x)$ is given by

$$\kappa(x) = \frac{|f''(x)|}{\left(\sqrt{1 + \left(f'(x)\right)^2}\right)^3}.$$

(*Hint:* Use the parametrization $\alpha(t) = (t, f(t), 0)$, and use the formula in previous exercise.) Compute the curvatures of $y = x$, x^2, x^3, and x^4.

Exercise 11. Let α, $\beta\colon (-1, 1) \to \mathbf{R}^2$ be a pair of C^2 curves with $\alpha(0) = \beta(0) = (0, 0)$. Further suppose that α and β both lie on or above the x-axis, and β lies higher than or at the same height as α. Show that the curvature of β at $t = 0$ is not smaller than that of α at $t = 0$(*Hint:* Use exercise 4, and a Taylor expansion).

Exercise 12. Show that if $\alpha\colon I \to \mathbf{R}^2$ is a C^2 closed curve which is contained in a circle of radius r, then the curvature of α has to be bigger than $1/r$ at some point. In particular, closed curves have a point of nonzero curvature. (*Hint*: Shrink the circle until it contacts the curve, and use Exercise 11).

Exercise 13. Let $\alpha\colon I \to \mathbf{R}^2$ be a closed planar curve, show that

$$\text{length}[\alpha] \geq \frac{2\pi}{\max \kappa}.$$

(*Hint:* Recall that the width w of α is smaller than or equal to its length divided by π to show that a piece of α should lie inside a circle of diameter at least w).