

Lecture Notes 5

1.13 Osculating Circle and Radius of Curvature

Recall that in a previous section we defined the osculating circle of a planar curve $\alpha: I \rightarrow \mathbf{R}^2$ at a point $\alpha(t)$ of nonvanishing curvature $t \in I$ as the circle with radius $r(t)$ and center at

$$\alpha(t) + r(t)N(t)$$

where

$$r(t) := \frac{1}{\kappa(t)}$$

is called the *radius of curvature* of α . If we had a way to define the osculating circle independently of curvature, then we could define curvature simply as the reciprocal of the radius of the osculating circle, and thus obtain a more geometric definition for curvature.

Proposition 1. *Let $r(s, t)$ be the radius of the circle which is tangent to α at $\alpha(t)$ and is also passing through $\alpha(s)$. Show that*

$$\kappa(t) = \lim_{s \rightarrow t} r(s, t).$$

Proof. Since curvature is invariant under rigid motions, we may assume that $\alpha(t) = (0, 0)$ and $\alpha'(t)$ is parallel to the x -axis. Then, we may assume that $\alpha(t) = (t, f(t))$, for some function $f: \mathbf{R} \rightarrow \mathbf{R}$ with $f(0) = 0$ and $f'(0) = 0$. Further, recall that

$$\kappa(t) = \frac{|f''(t)|}{(\sqrt{1 + f'(t)^2})^3}.$$

Thus

$$\kappa(0) = |f''(0)|.$$

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Next note that the center of the circle which is tangent to α at $(0, 0)$ must lie on the y -axis at some point $(0, r)$, and for this circle to also pass through the point $(s, f(s))$ we must have:

$$r^2 = s^2 + (r - f(s))^2.$$

Solving the above equation for r and taking the limit as $s \rightarrow 0$, via the L'Hopital's rule, we have

$$\lim_{s \rightarrow 0} \frac{2|f(s)|}{f^2(s) + s^2} = |f''(0)| = \kappa(0),$$

which is the desired result. □

Note 2. The above limit can be used to define a notion of curvature for curves that are not twice differentiable. In this case, we may define the *upper curvature* and *lower curvature* respectively as the upper and lower limit of

$$\frac{2|f(s)|}{f^2(s) + s^2}.$$

as $s \rightarrow 0$. We may even distinguish between right handed and left handed upper or lower curvature, by taking the right handed or left handed limits respectively.

Exercise* 3. Let $\alpha: I \rightarrow \mathbf{R}^2$ be a planar curve and $t_0, t_1, t_2 \in I$ with $t_1 \leq t_0 \leq t_2$. Show that $\kappa(t_0)$ is the reciprocal of the limit of the radius of the circles which pass through $\alpha(t_0)$, $\alpha(t_1)$ and $\alpha(t_2)$ as $t_1, t_2 \rightarrow t_0$.

1.14 Total Curvature and Convexity

We say that a simple closed curve $\alpha: I \rightarrow \mathbf{R}^2$ is *convex* provided that for every $t \in I$ there exists a line $\ell \subset \mathbf{R}^2$ passing through $\alpha(t)$ such that $\alpha(I)$ lies on one side of ℓ .

The *boundary* of a subset $X \subset \mathbf{R}^n$, which we denote by $\text{bd } X$, is defined as the intersection of the closure of X with the closure of its complement. A subset of \mathbf{R}^n is *convex* if it contains the line segment joining each pairs of its points. Clearly the intersection of convex sets is convex.

Exercise 4. Show that if a closed planar curve $\alpha: I \rightarrow \mathbf{R}^2$ is convex then it lies on the boundary of a convex set. (*Hint:* Let $\Gamma := \alpha(I)$. By definition,

through each point p of Γ there passes a line ℓ_p with respect to which Γ lies on one side. Thus each ℓ_p defines a closed half plane H_p which contains Γ . Show that Γ lies on the boundary of the intersection of all these half planes).

Exercise 5 (Total curvature of convex curves). Show that the total curvature of any simple closed convex planar curve is 2π . (*Hint:* It is enough to check that the signed exterior angles of polygonal approximations of a convex curve do not change sign; because, as we showed in a previous section, the sum of these angles is the total signed curvature, and the sum of their absolute values is the total curvature by Fox-Milnor's theorem. So it would follow that the total signed curvature of α is equal to its total curvature up to a sign. Since by definition the curve is simple, however, the total signed curvature is $\pm 2\pi$ by Hopf's rotation index theorem.)

Theorem 6. For any closed planar curve $\alpha: I \rightarrow \mathbf{R}^2$,

$$\int_I \kappa(t) dt \geq 2\pi,$$

with equality if and only if α is convex.

First we show that the total curvature of any curve is at least 2π . To this end recall that when t is the arclength parameter $\kappa(t) = \|T'(t)\|$. Thus the total curvature is simply the total length of the tantrix curve $T: I \rightarrow \mathbf{S}^1$. Since T is a closed curve, to show that its total length is bigger than 2π , it suffices to check that the image of T does not lie in any semicircle.

Exercise 7. Verify the last sentence.

To see that the image of T does not lie in any semicircle, let $u \in \mathbf{S}^1$ be a unit vector and note that

$$\int_a^b \langle T(t), u \rangle dt = \int_a^b \langle \alpha'(t), u \rangle dt = \langle \alpha(b) - \alpha(a), u \rangle = 0.$$

Since $T(t)$ is not constant (why?), it follows that the function $t \mapsto \langle T(t), u \rangle$ must change sign. So the image of T must lie on both sides of the line through the origin and orthogonal to u . Since u was chosen arbitrarily, it follows that the image of T does not lie in any semicircle, as desired.

Next we show that the total curvature is 2π if and only if α is convex. The “if” part has been established already in exercise 5. To prove the “only

if' part, suppose that α is not convex, then there exists a tangent line ℓ_0 of α , say at $\alpha(t_0)$, with respect to which the image of α lies on both sides. Then α must have two more tangent lines parallel to ℓ_0 .

Exercise 8. Verify the last sentence (*Hint:* Let u be a unit vector orthogonal to ℓ and note that the function $t \mapsto \langle \alpha(t) - \alpha(t_0), u \rangle$ must have a minimum and a maximum different from 0. Thus the derivative at these two points vanishes.)

Now that we have established that α has three distinct parallel lines, it follows that it must have at least two parallel tangents. This observation is worth recording:

Lemma 9. *If $\alpha: I \rightarrow \mathbf{R}^2$ is a closed curve which is not convex, then it has a pair of parallel tangent vectors which generate distinct parallel lines.*

Next note that

Exercise 10. If $\alpha: I \rightarrow \mathbf{R}^2$ is closed curve whose tantrix $T: I \rightarrow \mathbf{S}^1$ is not onto, then the total curvature is bigger than 2π . (*Hint:* This is an immediate consequence of the fact that T is a closed curve and it does not lie in any semicircle.)

So if T is not onto then we are done (recall that we are trying to show that if α is not convex, then its total curvature is bigger than 2π). We may assume, therefore, that T is onto. This together with the above lemma yields that the total curvature is bigger than 2π . To see this note that let $t_1, t_2 \in I$ be the two points such that $T(t_1)$ and $T(t_2)$ are parallel and the corresponding tangent lines are distinct. Then T restricted to $[t_1, t_2]$ is a closed nonconstant. So either $T([t_1, t_2])$ (i) covers some open segment of the circle twice or (ii) covers the entire circle. Since we have established that T is onto, the first possibility implies that the length of T is bigger than 2π . Further, since, T restricted to $I - (t_1, t_2)$ is not constant, the second possibility (ii) would imply the again the first case (i). Hence we conclude that if α is not convex, then its total curvature is bigger than 2π , which completes the proof of Theorem 6.

Corollary 11. *Any simple closed C^2 regular curve $\alpha: I \rightarrow \mathbf{R}^2$ is convex if and only if its signed curvature $\bar{\kappa}$ does not change sign. In particular, if κ never vanishes then α is convex.*

Proof. Since α is simple, its total signed curvature is $\pm 2\pi$ by Hopf's theorem. After switching the orientation of α , if necessary, we may assume that the total signed curvature is 2π . Suppose, towards a contradiction, that the signed curvature does change sign. The integral of the signed curvature over the regions where it is positive must be bigger than 2π , which in turn implies that the total curvature is bigger than 2π , which contradicts the previous theorem. So if α is convex, then $\bar{\kappa}$ does not change sign.

Next suppose that $\bar{\kappa}$ does not change sign. Then the total signed curvature is equal to the total curvature (up to a sign), which, since the curve is simple, implies, via the Hopf's theorem, that the total curvature is 2π . So by the previous theorem the curve is convex. \square