Math 497C
Curves and Surfaces
Fall 2004, PSU

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Lecture Notes 6

1.15 The four vertex theorem

A *vertex* of a planar curve $\alpha\colon I \to \mathbf{R}^2$ is a point where the curvature of α has a local max or min.

Exercise 1. Show that an ellipse has exactly 4 vertices, unless it is a circle.

The main aim of this section is to show that:

Theorem 2. *Any C^3 simple closed planar curve has (at least) four vertices.*

On the other hand if the curve is not simple, then the 4 vertex property may no longer be true:

Exercise 3. Sketch the limacon $\alpha\colon [0, 2\pi] \to \mathbf{R}^2$ given by

$$\alpha(t) := (2\cos t + 1)(\cos t, \sin t)$$

and show that it has only two vertices. (*Hint*: It looks like a loop with a smaller loop inside)

If the signed curvature of a closed curve changes sign, then it must have two points where κ vanishes. Since $\kappa \geq 0$, it follows then that κ has at least two local minimums. But there is a local maximum between any pairs of local minimums, so, we conclude that if the signed curvature changes sign then we have 4 vertices. It remains then to consider the case where the signed curvature does not change sign. By the result at the end of the previous section, if the signed curvature of a simple closed curve does not change sign, then the curve is convex. So we need only to prove the above theorem for convex curves.

We proceed by contradiction. Suppose that α has fewer than 4 vertices, then it must have exactly 2. Suppose that these two vertices occur at t_0

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and t_1. Then $\kappa'(t)$ will have one sign on (t_1, t_2) and the opposite sign on $I - [t_1, t_2]$. Let ℓ be the line passing through $\alpha(t_1)$ and $\alpha(t_2)$. Then, since α is convex, α restricted to (t_1, t_2) lies entirely in one of the closed half planes determined by ℓ and α restricted to $I - [t_1, t_2]$ lies in the other closed half plane.

Exercise 4. Verify the last sentence, i.e., show that if $\alpha\colon I \to \mathbf{R}^2$ is a simple closed convex planar curve, and ℓ is any line in the plane which intersects $\alpha(I)$, then ℓ intersects α in exactly two points, or $\alpha(I)$ lies on one side of ℓ.(*Hint*: Show that if α intersects ℓ at 3 points, then it lies on one side of ℓ.)

Let p be a point of ℓ and v be a vector orthogonal to ℓ, then $f\colon I \to \mathbf{R}$, given by $f(t) := \langle \alpha(t) - p, v\rangle$ has one sign on (t_1, t_2) and has the opposite sign on $I - [t_1, t_2]$. Consequently, $\kappa'(t) f(t)$ is always nonnegative. So

$$0 < \int_I \kappa'(t)\langle \alpha(t) - p, v\rangle dt.$$

On the other hand

$$\begin{aligned}
\int_I \kappa'(t)\langle \alpha(t) - p, v\rangle dt &= \kappa(t)\langle \alpha(t) - p, v\rangle\big|_a^b - \int_I \kappa(t)\langle T(t) - p, v\rangle dt \\
&= 0 - \int_I \langle -N'(t) - p, v\rangle dt \\
&= \langle -N(t) - p, v\rangle\big|_a^b \\
&= 0.
\end{aligned}$$

So we have a contradiction, as desired. It only remains to justify the implicit assumption above that κ is a C^1 function. In general this is not something that we can take for granted:

Exercise 5. Show that there exists a C^∞ regular planar curve whose curvature is not differentiable (*Hint:* Consider $\alpha\colon (-1, 1) \to \mathbf{R}^2$, $\alpha(t) := (t, t^3)$).

On the other hand the signed curvature is always well behaved:

Exercise 6. Show that the signed curvature of a C^3 regular curve in the plane is C^1.

So if the signed curvature does not change sign, then, either $\kappa = \overline{\kappa}$ or $\kappa = -\overline{\kappa}$, and hence, by the above exercise, κ is C^1.

The 4-vertex theorem we proved here may also be generalized to signed curvature, but the proof is more involved.

1.16 Area of planar regions and the Isoperimetric inequality

The area of a rectangle is defined as the product of lengths of two of its adjacent sides. Let $X \subset \mathbf{R}^2$ be any set, R be a collection of rectangles which cover X, and $Area(X, R)$ be the sum of the areas of all rectangles in R. Then area of X is defined as the infimum of $Area(X, R)$ where R ranges over all different ways to cover X by rectangles. In particular it follows that, if $X \subset Y$, then $Area(X) \leq Area(Y)$, and if $X = X_1 \cup X_2$, then $Area(X) = Area(X_1) + Area(X_2)$. These in turn quickly yield the areas of triangles and polygons.

Exercise 7 (Invariance under isometry and the Special linear group). Show that area is invariant under rigid motions of $\mathbf{R}^2$, and that dilation by a factor of r, i.e., multiplying each point $\mathbf{R}^2$ by r, changes the area by a factor of r^2. More generally show that any linear transformation $A\colon \mathbf{R}^2 \to \mathbf{R}^2$ changes the area by a factor of $\det(A)$.

Exercise 8 (Area of circle by plogonal approximation). Compute the area of a circle (*Hint:* For any n compute the area of regular n-gons which are inscribed in the circle, and take the limit. Each of these areas is the sum of n isoceles triangles with an angle $2\pi/n$, and adjacent sides of length equal to the radius of the circle. This gives a lower bound for the area. Un upper bound may also be obtained by taking the limit of regular polygons which circumscribe the circle.)

Recalling the defintion of Riemann sums, one quickly observes that

$$Area(X) = \int\int_X 1\, dxdy.$$

We say that a subset X of $\mathbf{R}^n$ is *connected* provided that the only subsets of X which are both open and closed in X are the X and the emptyset. Every subset of X which is connected and is different from X and the empty set is called a *component* of X.

Let $\alpha\colon I \to \mathbf{R}^2$ be a simple closed planar curve. By the Jordan curve theorem (which we will not prove here), $\mathbf{R}^2 - \alpha(I)$ consists of exactly two connected components, and the boundary of each component is $\alpha(I)$. Further, one of these components, which we call the *interior* of α, is contained in some large sphere, while the other is unbounded. By area of α we mean the area of its interior.

Theorem 9. *For any simple closed planar curve* $\alpha\colon I \to \mathbf{R}^2$*,*

$$Area[\alpha] \leq \frac{Length[\alpha]^2}{4\pi}.$$

Equality holds only when α *is a circle.*

Our proof of the above theorem hinges on the following subtle fact whose proof we leave out

Lemma 10. *Of all simple closed curves of fixed length* L*, there exists at least one with the biggest area. Further, every such curve is* C^1*.*

Exercise* 11. Show that the area maximizer (for a fixed length) must be convex. (*Hint*: It is enough to show that if the maximizer, say α, is not convex, then there exist a line ℓ with respect to which $\alpha(I)$ lies on one side, and intersects $\alpha(I)$ at two points p and q but not in the intervening open segment of ℓ determined by p and q. Then reflecting one of the segments of $\alpha(I)$, determined by p and q, through ℓ increases area while leaving the length unchanged.)

We say that α is symmetric with respect to a line ℓ provided that the image of α is invariant under reflection with respect to ℓ.

Exercise 12. Show that a C^1 convex planar curve $\alpha\colon I \to \mathbf{R}^2$ is a circle, if and only if for every unit vector $u \in \mathbf{S}^1$ there exists a line perpendicular to u with respect to which α is symmetric (*Hint* Suppose that α has a line of symmetry in every direction. First show that each line of symmetry is unique in the corresponding direction. After a translation we may assume that α is symmetric with respect to both the x-axis and the y-axis. Show that this yields that α is symmetric with respect to the origin, i.e. rotation by 180°. From this and the uniqueness of the lines of symmetry conclude that every line of symmetry passes through the origin. Finally show that each line of symmetry must meet the curve orthogonally at the intersection points. This shows that $\langle \alpha(t), \alpha'(t) \rangle = 0$, which in turn yields that $\|\alpha(t)\| = const.$)

Let $\alpha\colon I \to \mathbf{R}^2$ be an area maximizer. By Exercise 11 we may assume that α is convex. We claim that α must have a line of symmetry in every direction, which would show, by Exercise 12, that α is a circle, and hence would complete the proof.

Suppose, towards a contradiction, that there exists a direction $u \in \mathbf{S}^1$ such that α has no line of symmetry in that direction. Ater a rigid motion, we may assume that $u = (0, 1)$.

Let $[a, b]$ be the projection of $\alpha(I)$ to the x-axis. Then, since α is convex, every vertical line which passes through an interior point of (a, b) intersects $\alpha(I)$ at precisely two points. Let $f(x)$ be the y-coordinate of the higher point, and $g(x)$ be the y-coordinate of the other points. Then

$$Area[\alpha] = \int_a^b f(x) - g(x)\, dx.$$

Further note that if α is C^1 then f and g are C^1 as well, thus

$$Length[\alpha] = f(a) - g(a) + \int_a^b \sqrt{1 + f'(x)^2}\, dx + \int_a^b \sqrt{1 + g'(x)^2}\, dx + f(b) - g(b).$$

Now we are going to define a new curve $\overline{\alpha}$ which is bounded above by the graph of the function $\overline{f}\colon [a, b] \to \mathbf{R}$ given by

$$\overline{f}(x) := \frac{f(x) - g(x)}{2},$$

is bounded below by the graph of $-\overline{f}$, and is bouned on the left and right by vertical segments, which may consist only of a single point. One immdediately checks that

$$Area[\overline{\alpha}] = Area[\alpha].$$

Further, note that since by assumption α is not symmetric with respect to the x-axis, $\overline{f}$ is strictly positive on (a, b). This may be used to show that

$$Length[\overline{\alpha}] < Length[\alpha].$$

Exercise 13. Verify the last inequlaity above (*Hint*: It is enough to check that $\int_a^b \sqrt{1 + \overline{f}'(x)^2} dx$ is strictly smaller than either of the integrals in the above formula for the length of α).