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Lecture Notes 15

2.13 The Geodesic Curvature

Let $\alpha\colon I \to M$ be a unit speed curve lying on a surface $M \subset \mathbf{R}^3$. Then the *absolute geodesic curvature* of α is defined as

$$|\kappa_g| := \big\|(\alpha'')^\top\big\| = \big\|\alpha'' - \langle\alpha'', n(\alpha)\rangle n(\alpha)\big\|,$$

where n is a local Gauss map of M in a neighborhood of $\alpha(t)$. In particular note that if $M = \mathbf{R}^2$, then $|\kappa_g| = \kappa$, i.e., absolute geodesic curvature of a curve on a surface is a gneralization of the curvature of curves in the plane.

Exercise 1. Show that the absolute geodesic curvature of great circles in a sphere and helices on a cylinder are everywhere zero.

Similarly, the *(signed) geodesic curvature* generalizes the notion of the signed curvature of planar curves and may be defined as follows.

We say that a surface $M \subset \mathbf{R}^3$ is *orientable* provided that there exists a (global) Gauss map $n\colon M \to \mathbf{S}^2$, i.e., a *continuous* mapping which satisfies $n(p) \in T_pM$, for all $p \in M$. Note that if n is a global Gauss map, then so is $-n$. In particular, any orientable surface admits precisely two choices for its global Gauss map. Once we choose a Gauss map n for an orientable surface, then M is said to be *oriented*.

If M is an oriented surface (with global Gauss map n), then, for every $p \in M$, we define a mapping $J\colon T_pM \to T_pM$ by

$$JV := n \times V.$$

Exercise 2. Show that if $M = \mathbf{R}^2$, and $n = (0, 0, 1)$, then J is counter clockwise rotation about the origin by $\pi/2$.

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Then the *geodesic curvature* of a unit speed curve $\alpha\colon I \to M$ is given by

$$\kappa_g := \big\langle \alpha'', J\alpha' \big\rangle.$$

Note that, since $J\alpha'$ is tangent to M,

$$\big\langle \alpha'', J\alpha' \big\rangle = \big\langle (\alpha'')^\top, J\alpha' \big\rangle.$$

Further, since $\|\alpha'\| = 1$, α'' is orthogonal to α', which in turn yields that the projection of α'' into the tangent plane is either parallel or antiparallel to $J\alpha'$. Thus $\kappa_g > 0$ when the projection of α'' is parallel to $J\alpha'$ and is negative otherwise.

Note that if the curvature of α does not vanish (so that the principal normal N is well defined), then

$$\kappa_g = \kappa \big\langle N, JT \big\rangle.$$

In particular geodesic curvature is invariant under reparametrizations of α.

Exercise 3. Let $\mathbf{S}^2$ be oriented by its outward unit normal, i.e., $n(p) = p$, and compute the geodesic curvature of the circles in $\mathbf{S}^2$ which lie in planes $z = h$, $-1 < h < 1$. Assume that all these circles are oriented consistently with respect to the rotation about the z-axis.

Next we derive an expression for κ_g which does not require that α have unit speed. To this end, let $s\colon I \to [0, L]$ be the arclength function of α, and recall that $\overline{\alpha} := \alpha \circ s^{-1}\colon [0, L] \to M$ has unit speed. Thus

$$\kappa_g = \overline{\kappa}_g(s) = \big\langle \overline{\alpha}''(s), J\overline{\alpha}'(s) \big\rangle.$$

Now recall that $(s^{-1})' = 1/\|\alpha'\|$. Thus by chain rule.

$$\overline{\alpha}'(t) = \alpha'\big(s^{-1}(t)\big) \cdot \frac{1}{\|\alpha'(s^{-1}(t))\|}.$$

Further, differentiating both sides of the above equation yields

$$\overline{\alpha}'' = \alpha''\big(s^{-1}\big) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2} + \alpha'\big(s^{-1}\big) \cdot \frac{-\Big\langle \alpha''\big(s^{-1}\big), \alpha'\big(s^{-1}\big) \Big\rangle}{\|\alpha'(s^{-1})\|^3}.$$

Substituting these values into the last expression for $\overline{\kappa}_g$ above yiels

$$\kappa_g = \frac{\big\langle \alpha'', J\alpha' \big\rangle}{\|\alpha'\|^3}.$$

Exercise 4. Verify the last two equations.

Next we show that the geodesic curvature is intrinsic, i.e., it is invariant under isometries of the surface. To this end define $\tilde{\alpha}' \colon \alpha(I) \to \mathbf{R}^3$ be the vectorfiled along $\alpha(I)$ given by

$$\tilde{\alpha}'\big(\alpha(t)\big) = \alpha'(t).$$

Then one may immediatley check that

$$\alpha''(t) = \overline{\nabla}_{\alpha'(t)}\tilde{\alpha}'.$$

Thus

$$\big\langle \alpha'', J\alpha' \big\rangle = \big\langle (\alpha'')^{\top}, J\alpha' \big\rangle = \big\langle \nabla_{\alpha'}\tilde{\alpha}', J\alpha' \big\rangle.$$

and it follows that

$$\kappa_g = \frac{\big\langle \nabla_{\alpha'}\tilde{\alpha}', J\alpha' \big\rangle}{\|\alpha'\|^3}.$$

We say that a curve $\alpha \colon I \to M$ is a geodesic provided that its geodesic curvature $\kappa_g \equiv 0$.

Exercise 5. Show that if α is a geodesic, then it must have constant speed.

Exercise 6. Show that if α is parametrized by arclength, then

$$|\kappa_g| = \|\nabla_{\alpha'}\tilde{\alpha}'\|.$$

Exercise 7. Show that α is a geodesic if and only if $\nabla_{\alpha'}\tilde{\alpha}' \equiv 0$.

Now recall that ∇ is intrinic, which immediately implies that $|\kappa_g|$ is intrinsic by the last exercise. Thus to complete the proof that κ_g is intrinsic it remains to show that J is intrinsic. To see this let $X \colon U \to M$ be a local patch, then

$$JX_i = \sum_{j=1}^{2} b_{ij} X_j.$$

In particular,

$$JX_1 = b_{11}X_1 + b_{12}X_2.$$

Now note that

$$0 = \langle JX_1, X_1 \rangle = b_{11}g_{11} + b_{12}g_{21}.$$

Further,

$$g_{11} = \langle X_1, X_1 \rangle = \langle JX_1, JX_1 \rangle = b_{11}^2 g_{11} + 2b_{11}b_{12}g_{12} + b_{12}^2 g_{22}.$$

Solving for b_{21} in the next to last equation, and substituting in the last equation yields

$$g_{11} = b_{11}^2 g_{11} - 2b_{11}^2 g_{11} + b_{11}^2 \frac{g_{11}^2}{g_{21}^2} g_{22} = b_{11}^2(-g_{11} + \frac{g_{11}^2}{g_{21}^2} g_{22}).$$

Thus b_{11} may be computed in terms of g_{ij} which in turn yields that b_{12} may be computed in terms of g_{ij} as well. So JX_1 nay be expressed intrinsically. Similarly, JX_2 may be expressed intrinsically as well. So we conclude that J is intrinsic.

2.14 Geodesics in Local Coordinates

Here we will derive a system of ordinary differential equations, in terms of any local coordinates, whose solutions yield geodesics.

To this end let $X : U \to M$ be a patch, and $\alpha : I \to X(U)$ be a unit speed one-to-one curve. Then we may write

$$X(u(t)) = \alpha(t),$$

by letting $u(t) := X^{-1}(\alpha(t))$. Next note that, if u_i denotes the coordinates of u, i.e., $u(t) = (u_1(t), u_2(t))$, then by the chain rule,

$$\alpha' = \sum_{i=1}^{2} X_i u_i',$$

which in turn yields

$$\alpha'' = \sum_{i,j=1}^{2} X_{ij} u_i' u_j' + X_i u_i'' = \sum_{i,j,k=1}^{2} (\Gamma_{ij}^k X_k + \ell_{ij} N) u_i' u_j' + X_i u_i'',$$

by Gauss's formula. Consequently,

$$(\alpha'')^\top = \sum_{i,j,k=1}^{2} (\Gamma_{ij}^k X_k) u_i' u_j' + X_i u_i'' = \sum_{i,j,k=1}^{2} (\Gamma_{ij}^k u_i' u_j' + u_k'') X_k.$$

So, since $|\kappa_g| = \|(\alpha'')^\top\|$, we conclude that α is a geodesic if and only if

$$\sum_{i,j=1}^{2} (\Gamma_{ij}^k u_i' u_j' + u_k'') = 0$$

for $k = 1, 2$. In other words, for α to be a geodesic the following two equations must be satisfied:

$$u_1'' + \Gamma_{11}^1 (u_1')^2 + 2\Gamma_{12}^1 u_1' u_2' + \Gamma_{22}^1 (u_2')^2 = 0$$
$$u_2'' + \Gamma_{11}^2 (u_1')^2 + 2\Gamma_{12}^2 u_1' u_2' + \Gamma_{22}^2 (u_2')^2 = 0$$

Exercise 8. Write down the equations of the geodesic in a surface of revolution. In particular, verify that the great circles in a sphere are geodesics.

2.15 Parallel Translation

Here we will give another interpretation for the concept of geodesic curvature. Let $\alpha\colon I \to M$ be a simple curve and V be a vector field on M. We say that V is *parallel* along α provided that

$$\nabla_{\alpha'(t)} V = \big((V \circ \alpha)'\big)^\top = 0.$$

for all $t \in I$. Recall that α is a geodesic if and only if its velocity is parallel (i.e., $\nabla_{\alpha'(t)} \tilde{\alpha}' \equiv 0$)

Exercise 9. Show that if V is parallel along α, then its length is constant.

Exercise 10. Show that if V and W are a pair of parallel vector fields along α, then the angle between them is constant.

Proposition 11. *If α is a unit speed curve on a surface, and V is a parallel vector field along α, which makes an angle ϕ with the tangent vector of α, then $\kappa_g = \phi'$.*

Proof. We may assume that V has unit length. Then we may write:

$$V = \cos(\phi)T + \sin(\phi)JT.$$

Computing $\|\nabla_T\|$, and setting it equal to zero yields the desired result. □

Corollary 12. *The total geodesic curvature of a curve on a surface is equal to the total rotation of a parallel vector field along the curve, i.e.*

$$\int_a^b \kappa_g = \phi(b) - \phi(a),$$

where ϕ is an angle function between T and V.