Math 497C
Curves and Surfaces
Fall 2004, PSU

Dec 8, 2004[1]

Lecture Notes 16

3 The Gauss-Bonnet theorem

3.1 The Local Gauss-Bonnet theorem

Let $M \subset \mathbf{R}^3$ be a surface which is oriented by a choice of Gauss map $n\colon M \to \mathbf{S}^2$. Recall that n then induces a rotation on each tangent plane T_pM by $JV := n \times V$ for all $V \in T_pM$, which we call *counter clockwise rotation* (in analogy with the operation $JV := (0,0,1)\times V$ for vectors in $\mathbf{R}^2 = \mathbf{R}^2 \times \{0\} \subset \mathbf{R}^3$). A *regular region* $\Omega \subset M$ is a compact set whose boundary $\partial\Omega$ consists of a finite number of piecewise C^2 simple closed curves. We orient $\partial\Omega$ by choosing continuous unit tangent vectors T along each of its C^2 segments so that JT points into Ω. Then we say that Ω is *positively oriented*, and it is in this sense that the geodesic curvature of $\partial\Omega$ is to be measured. The points where the C^2 segments of $\partial\Omega$ meet are called the *corners* of Ω. The *turning angle* $\alpha \in [-\pi, \pi]$ of Ω at a corner is defined as $\pi - \alpha'$ where $\alpha' \in [0, 2\pi]$ is the interior angle of Ω at that corner. We say Ω is *simply connected* if it is homeomorphic to a disk, or equivalently, Ω can be covered by a coordinate chart $X\colon U \to M$ where U is a disk, and $\partial\Omega$ has only one component.

Theorem 1 (Local Gauss-Bonnet theorem)**.** *Let $\Omega \subset M$ be a positively oriented simply connected regular region with exterior angles α_i. Then*

$$\int_\Omega K + \int_{\partial\Omega} \kappa_g + \sum_{i=1}^{k} \alpha_i = 2\pi.$$

The following exercise shows that to prove the above theorem, we may assume that Ω is arbitrarily small

Exercise 2. Let Ω be a simply connected regular region in M and Ω_1, Ω_2 be a pair of subregions obtained by drawing a curve across Ω whose end points

[1]Last revised: December 11, 2021

lie on $\partial\Omega$. Show that if the local Gauss-Bonnet theorem holds on each of the regions Ω_1 and Ω_2, then it holds on Ω.

3.2 Proof of a special case of Gauss-Bonnet Theorem

Here we give a quick proof of the local Gauss-Bonnet Theorem in the special case where Ω has no corners, and $K > 0$ on Ω. Since $\det(dn) = K \neq 0$ on Ω, then, by the inverse function theorem, n is locally one-to-one on Ω. So we may subdivide Ω into a finite number of subregions so that n is one-to-one on each subregion. By the last exercise, it suffices to show that the Gauss-Bonnet theorem holds in each subregion. So we replace Ω with one of these subregions. Then, by the area formula,

$$\int_\Omega K = \int_\Omega \det(dn) = \mathrm{area}(n(\Omega)).$$

Let V be a vector field along $\partial\Omega$ obtained by taking a tangent vector of M at a point p of $\partial\Omega$ and parallel translating it all around $\partial\Omega$ until it comes back to p. Furthermore, let $\gamma\colon [a,b] \to M$ be a unit speed parametrization for $\partial\Omega$ which is consistent with the orientation of Ω, i.e., $J\gamma'$ points into Ω. There exists a continuous function $\theta\colon [a,b] \to \mathbf{R}$ such that $\theta(t)$ is an angle between $V(t)$ and $\gamma'(t)$, i.e.,

$$V(t) = \cos(\theta(t))\gamma'(t) + \sin(\theta(t))J\gamma'(t),$$

where $V(t) := V(\gamma(t))$. Then

$$\int_{\partial\Omega} \kappa_g = \theta(b) - \theta(a).$$

Next note that V may also be regarded as a tangent vector field along the curve $n(\partial\Omega)$ in $\mathbf{S}^2$, because $V(t) \perp n(\gamma(t))$, and thus $V(t) \in T_{n(\gamma(t))}\mathbf{S}^2$. Furthermore, since V is parallel along $\partial\Omega$, $V' || n$. So V is also a parallel vector field along $n(\partial\Omega)$ in $\mathbf{S}^2$. Conseqently,

$$\int_{n(\partial\Omega)} \kappa_g = \theta(b) - \theta(a) = \int_{\partial\Omega} \kappa_g.$$

So the Gauss-Bonnet theorem holds for Ω provided that

$$\mathrm{area}(n(\Omega)) + \int_{n(\partial\Omega)} \kappa_g = 2\pi.$$

It can be shown that, if we approximate $n(\partial\Omega)$ with geodesic polygonal curves, then $\int_{n(\partial\Omega)} \kappa_g$ is the limit of the sum of turning angles of the polygons. So it suffices to show that the area of a geodesic polygon in $\mathbf{S}^2$ is equal to 2π minus the sum of its exterior angles. Earlier we established this result for geodesic triangles in $\mathbf{S}^2$, which yields the general result for polygons via a subdivision. This completes the proof of the special case of the local Gauss-Bonnet theorem.

Exercise 3. Extend the above proof to the case where $K < 0$.

3.3 Proof of the Local Gauss-Bonnet Theorem

Let $\gamma\colon [a,b] \to \partial\Omega$ be a unit speed piecewise C^2 parametrization for $\partial\Omega$. We may extend γ to a periodic function on $\mathbf{R}$ by setting $\gamma(t+b-a) := \gamma(t)$. Then there are a finite number of points $t_1, \dots, t_k \in [a,b]$ such that $\gamma(t_i)$ are the corners of Ω and γ is C^2 on each of the intervals $[t_i, t_{i+1}]$, where we set $t_{k+1} := t_1 + b - a$. In particular γ' exists on the interior of each interval, and one sided derivatives $\gamma'_\pm$ are well-defined at each t_i. We assume that γ is consistent with the orientation of Ω, i.e., $J\gamma'$ points into Ω.

Let $X\colon U \to M$ be a local coordinate system with $\Omega \subset X(U)$. We set $\tilde\Omega := X^{-1}(\Omega)$ and $\tilde\gamma := X^{-1}(\gamma)$. Let $V\colon U \to \mathbf{R}^3$ be a smooth unit vector field Ω, and set $W := JV$. Then (V, W) is an orthonormal frame on Ω. It will be convenient to extend γ to a periodic function on $\mathbf{R}$ by setting $\gamma(t+b-a) := \gamma(t)$. Then for $i = 1, \dots, k$, there are continuous functions $\theta_i\colon [t_i, t_{i+1}] \to \mathbf{R}$ such that

$$\gamma'(t) := \cos(\theta_i(t))V(t) + \sin(\theta_i(t))W(t),$$

where $V(t) := V(\tilde\gamma(t))$, and $W(t) := W(\tilde\gamma(t))$. We are going to compute κ_g in terms of V, W as follows. First note that

$$J\gamma' = -\sin(\theta_i)V + \cos(\theta_i)W.$$

Then a straight forward computation shows that

$$\gamma'' = \theta_i'\, J\gamma' + \cos(\theta_i)V' + \sin(\theta_i)W'.$$

Another simple computation yields

$$\kappa_g = \langle \gamma'', J\gamma' \rangle = \theta_i' + \langle V', W \rangle.$$

Thus

$$\begin{aligned}
\int_{\partial\Omega} \kappa_g &= \sum_{i=1}^{k} \int_{t_i}^{t_{i+1}} \kappa_g \\
&= \sum_{i=1}^{k} \int_{t_i}^{t_{i+1}} \theta_i' + \sum_{i=1}^{k} \int_{t_i}^{t_{i+1}} \langle V', W \rangle \\
&= \sum_{i=1}^{k} \Delta\theta_i + \int_{\partial\Omega} \langle V', W \rangle,
\end{aligned}$$

where $\Delta\theta_i := \theta_i(t_{i+1}) - \theta_i(t_i)$. Now let us define the *total rotation* of $\partial\Omega$ with respect to V as

$$\mathrm{rot}_V(\partial\Omega) := \sum_{i=1}^{k} \Delta\theta_i + \sum_{i=1}^{k} \alpha_i.$$

Then we have

$$\int_{\partial\Omega} \kappa_g + \sum_{i=1}^{k} \alpha_i = \mathrm{rot}_V(\partial\Omega) + \int_{\partial\Omega} \langle V', W \rangle.$$

So to prove the local Gauss-Bonnet theorem, it suffices to show that

1. $\mathrm{rot}_V(\partial\Omega) = 2\pi$,
2. $\int_{\partial\Omega} \langle V', W \rangle = -\int_{\Omega} K$

3.3.1 First part of the proof

This part follows from Hopf's rotation index theorem for planar curves, through a series of reductions as we describe below.

Step 1. First we reduce the problem to the case where Ω has no corners. To see this let $\partial\Omega_i := \gamma([t_i, t_{i+1}])$ denote the *sides* of γ, and $\theta_i \colon [t_i, t_{i+1}] \to \mathbf{R}$ be the corresponding turning angle functions. We may deform each $\partial\Omega_i$ near its initial boundary point $\gamma(t_i)$, so that $\gamma_+'(t_i)$ turns around $\gamma(t_i)$ until it coincides with $\gamma_-'(t_i)$, and the exterior angle α_i vanishes. Specifically, if $\alpha_i > 0$, we perturb $\partial\Omega_i$ so that $\gamma_+'(t_i)$ turns in the counterclockwise direction, and if $\alpha_i < 0$, we perturb $\partial\Omega_i$ so that $\gamma_+'(t_i)$ turns in the clockwise direction. Then $\theta_i(t_i) \to \theta_i(t_i) - \alpha_i$. Consequently, $\Delta\theta_i \to \Delta\theta_i + \alpha_i$ and so the sum

$\Delta\theta_i + \alpha_i$ remains constant. Thus we may smoothen out all corners of Ω without effecting $\mathrm{rot}_V(\partial\Omega)$. So we may assume that $\partial\Omega$ is a C^1 closed curve, as claimed. Now, let $\gamma\colon [a,b] \to M$ be a C^1 closed curve which traces $\partial\Omega$ counterclockwise. Then there exists a continous function $\theta\colon [a,b] \to \mathbf{R}$ which yields an angle between $V(t)$ and $\gamma'(t)$. So $\mathrm{rot}_{\partial\Omega}(V) = \theta(b) - \theta(a)$. But since $V(a) = V(b)$, $\theta(b) = \theta(a) + 2k\pi$. So $\mathrm{rot}_V(\partial\Omega) = 2k\pi$. It remains then to show that $k = 1$.

Step 2. Now we show that Ω may be assumed to lie in $\mathbf{R}^2$. To see this note that $\mathrm{rot}_V(\partial\Omega)$ remains constant under any continuous deformation of $\partial\Omega$ and its tangent vectors through C^1 curves, or any continuous change of V, since $\mathrm{rot}_V(\partial\Omega)$ can assume only a discrete set of values, as discussed above. In particular, deforming the metric of M to a Euclidean metric is not going to effect $\mathrm{rot}_V(\partial\Omega)$. More explicitly, let $\tilde{\gamma} := X^{-1}(\gamma)$, $\tilde{V} := dX^{-1}(V)$, and $\tilde{\theta}(t)$ be the continuous angle function between $\tilde{\gamma}'(t)$ and $\tilde{V}(t)$ with respect to the metric g that X induces on U, i.e., $g_p(v,w) := \langle dX_p(v), dX_p(w)\rangle$, for all $p \in U$, and v, $w \in \mathbf{R}^2$. Then $\tilde{\theta}(t) = \theta(t)$. So

$$\mathrm{rot}_V(\partial\Omega) = \mathrm{rot}_{\tilde{V}}(\partial\tilde{\Omega}).$$

Now let g_E be the Euclidean metric on U, and consider the one parameter family of metrics

$$g^s := (1-s)g + s g_E.$$

Furthermore, let $\tilde{\theta}^s(t)$ be the angle function between $\tilde{\gamma}'(t)$ and $\tilde{V}(t)$ computed with respect to g^s. Since $\tilde{\theta}^s$ depends continuously on s, $\mathrm{rot}_{\tilde{V}}(\partial\tilde{\Omega})$ remains unchanged under the metric deformation. So we may assume that $\Omega \subset \mathbf{R}^2$.

Step 3. Finally we show that V may be assumed to be a constant vector field, which will complete the reduction to Hopf's theorem. To this end, we may assume that U is a disc centered at the origin o of $\mathbf{R}^n$. Let $\gamma_\lambda(t) := \lambda\gamma(t)$ be the rescaling of γ by a factor $\lambda < 1$. Then γ_λ remains in U and thus $V(\gamma_\lambda(t))$ will be well defined. Since γ_λ and its tangent vectors change continuously with λ, it follows that $\mathrm{rot}_V(\partial\Omega) = \mathrm{rot}_V(\partial\Omega_\lambda)$. Since V is continuous, we may choose λ so small that V on Ω_λ is arbitrarily close to the value of V at the origin, $V(o)$, as we may wish. In particular, we may assume that $V \neq -V(o)$ on Ω_λ. Then the one parameter family of vector fields

$$V^s(p) := \frac{(1-s)V(p) + sV(o)}{\|(1-s)V(p) + sV(o)\|}$$

will be well-defined on Ω_λ and will continuosly change V to the constant vector field $V(o)$ on Ω_λ. This completes the proof, since $\mathrm{rot}_V(\partial\Omega)$ remains constant under continous deformations of V.

Note 4. We may set $V := X_1/\|X_1\|$, which will eliminate the need for Step 3 of the proof above.

Note 5. If we assume that X is conformal, and set $V := X_1/\|X_1\|$, then the entire proof of the first part above follows immediately from Hopf's rotation theorem for piecewise C^1 curves which we had discussed earlier.

3.3.2 Second part of the proof

Suppose that $\tilde{\gamma}(t) = (x(t), y(t))$. Then $V'(t) = V_1\, x'(t) + V_2\, y'(t)$. So, using Green's theorem, we can compute that

$$\begin{aligned}
\int_a^b \langle V', W\rangle dt &= \int_a^b \big(\langle V_1, W\rangle x'(t) + \langle V_2, W\rangle y'(t)\big)\, dt \\
&= \int_{\partial\tilde{\Omega}} \langle V_1, W\rangle dx + \langle V_2, W\rangle dy \\
&= \int_{\tilde{\Omega}} \big(\langle V_2, W\rangle_1 - \langle V_1, W\rangle_2\big) dx\, dy \\
&= \int_{\tilde{\Omega}} \big(\langle V_2, W_1\rangle - \langle V_1, W_2\rangle\big) dx\, dy
\end{aligned}$$

So it remains to check that

$$\langle V_2, W_1\rangle - \langle V_1, W_2\rangle = -K\sqrt{g},$$

where recall that $g = \det(g_{ij})$. To establish the above formula we need to rewrite the left hand side in terms of X. To this end note that since X_1 and X_2 for a basis over Ω, we have

$$V = aX_1 + bX_2, \quad W = cX_1 + dX_2,$$

for some functions a, b, c, d on Ω. Before performing the above substitutions note that

$$\langle V_2, W_1\rangle = \langle\langle V_2, N\rangle N + \langle V_2, W\rangle W, W_1\rangle = \langle V_2, N\rangle\langle W_1, N\rangle.$$

Similarly,

$$\langle V_1, W_2\rangle = \langle\langle V_1, N\rangle N + \langle V_1, W\rangle W, W_2\rangle = \langle V_1, N\rangle\langle W_2, N\rangle.$$

So

$$\begin{aligned}
\langle V_2, W_1\rangle - \langle V_1, W_2\rangle &= \langle V_2, N\rangle\langle W_1, N\rangle - \langle V_1, N\rangle\langle W_2, N\rangle \\
&= \langle V, N_2\rangle\langle W, N_1\rangle - \langle V, N_1\rangle\langle W, N_2\rangle
\end{aligned}$$

Now the substitutions we had mentioned above yield that

$$\begin{aligned}
\langle V_2, W_1\rangle - \langle V_1, W_2\rangle &= (ad-bc)\big(\langle X_1, N_2\rangle\langle X_2, N_1\rangle - \langle X_1, N_1\rangle\langle X_2, N_2\rangle\big) \\
&= (ad-bc)\big(\langle X_{12}, N\rangle\langle X_{21}, N\rangle - \langle X_{11}, N\rangle\langle X_{22}, N\rangle\big) \\
&= -(ad-bc)\det(\ell_{ij}) \\
&= -(ad-bc)Kg,
\end{aligned}$$

Since $K = \det(\ell_{ij})/g$. Finally note that

$$1 = \|V \times W\| = (ad-bc)\|X_1 \times X_2\| = (ad-bc)\sqrt{g},$$

which completes the proof.

3.4 The general Gauss-Bonnet theorem

Here we generalize the local Gauss-Bonnet theorem to the case where the region Ω may not be simply connected. A regular region is *triagular* provided that it is simply connected and has exactly three corners, or *vertices*. Each segment of the boundary between a pair of vertices will be called an *edge* of the region. By a *triangulation* of a regular region Ω we mean a partition of Ω into a finite collection of triangular regions T_i, called *faces* of the triangulation, such that whenever $T_i \cap T_j \neq \emptyset$ then either $T_i \cap T_j$ is a common vertex or a common edge of T_i and T_j. The *Euler Characteritic* of Ω is defined as

$$\chi(\Omega) := V - E + F,$$

where V, E, and F are the number of (distinct) vertices, edges, and faces of a triangulation of Ω.

Theorem 6 (General Gauss-Bonnet theorem)**.** *Let $\Omega \subset M$ be a regular region with exterior angles α_i. Then*

$$\int_\Omega K + \int_{\partial\Omega} \kappa_g + \sum_{i=1}^{k} \alpha_i = 2\pi\chi(\Omega).$$

Note that the above theorem shows that the Euler characteristic is independent of the choice of triangulation. Also, the above theorem applies to the case where M is compact connected surface without boundary, and Ω is all of M, in which case we obtain:

$$\int_M K = 2\pi\chi(M).$$

The proof of the above theorem follows from applying the Gauss-Bonnet theorem to the faces of a triangulaion, and then summing over all the faces. Let T_i denote the faces of a triangulation of Ω. Then

$$\int_{T_i} K + \int_{\partial T_i} \kappa_g + \sum_{j=1}^{3} \alpha_{ij} = 2\pi.$$

Note that the geodesic curvatures are computed with respect to the counterclockwise orientation of the boundary of each face. Thus the geodesic curvature of each edge in the interior of Ω will have opposite signs when computed with respect to faces adjacent to that edge. Hence if we sum the above equality over all faces of the triangulation we obtain

$$2\pi F = \int_\Omega K + \int_{\partial\Omega} \kappa_g + \sum_{i=1}^{F}\sum_{j=1}^{3} \alpha_{ij}.$$

Hence it remains to show that

$$\sum_{i=1}^{F}\sum_{j=1}^{3} \alpha_{ij} = \sum_{i=1}^{V_{ext}} \beta_i + 2\pi(E - V),$$

where β_i are the turning angles of Ω at the corners of $\partial\Omega$, or the number of external vertices of the triangulation which we denote by V_{ext}. To see this note that

$$\begin{aligned}
\sum_{i=1}^{F}\sum_{j=1}^{3}\alpha_{ij} &= \sum_{i=1}^{F}\sum_{j=1}^{3}(\pi-\alpha'_{ij}) \\
&= 3\pi F - \sum_{i=1}^{F}\sum_{j=1}^{3}\alpha'_{ij} \\
&= 3\pi F - 2\pi V_{int} - \sum_{i=1}^{V_{ext}}(\pi-\beta_i) \\
&= (3F - 2V_{int} - V_{ext})\pi + \sum_{i=1}^{V_{ext}}\beta_i
\end{aligned}$$

where α'_{ij} are interior angles of the triangular regions, and V_{int} are the number of vertices in the interior of Ω. Next note that $3F = 2E_{int} + E_{ext}$ where E_{int} and E_{ext} denote respectively the number of interior and exterior edges of the triangulation. So

$$\begin{aligned}
3F - 2V_{int} - V_{ext} &= 2E_{int} + E_{ext} - 2V_{int} - V_{ext} \\
&= 2E_{int} + 2E_{ext} - 2V_{int} - 2V_{ext} \\
&= 2(E-V),
\end{aligned}$$

which completes the argument. Here we have used the fact that $E_{ext} = V_{ext}$. Thus $E_{ext} - V_{ext} = 0 = 2E_{ext} - 2V_{ext}$.

3.5 Applications of the Gauss-Bonnet theorem

All the following exercises follow fairly quickly from Gauss-Bonnet theorem.

Exercise 7. Show that the sum of the angles in a triangle is π.

Exercise 8. Show that the total geodesic curvature of a simple closed planar curve is 2π.

Exercise 9. Show that the Gaussian curvature of a surface which is homeomorphic to the torus must always be equal to zero at some point.

Exercise 10. Show that a simple closed curve with total geodesic curvature zero on a sphere bisects the area of the sphere.

Exercise 11. Show that there exists at most one closed geodesic on a cylinder with negative curvature.

Exercise 12. Show that the area of a geodesic polygon with k vertices on a sphere of radius 1 is equal to the sum of its angles minus $(k-2)\pi$.

Exercise 13. Let p be a point of a surface M, T be a geodesic triangle which contains p, and α, β, γ be the angles of T. Show that

$$K(p) = \lim_{T \to p} \frac{\alpha + \beta + \gamma - \pi}{Area(T)}.$$

In particular, note that the above proves Gauss's Theorema Egregium.

Exercise 14. Show that the sum of the angles of a geodesic triangle on a surface of positive curvature is more than π, and on a surface of negative curvature is less than π.

Exercise 15. Show that on a simply connected surface of negative curvature two geodesics emanating from the same point will never meet.

Exercise 16. Let M be a surface homeomorphic to a sphere in $\mathbf{R}^3$, and let $\Gamma \subset M$ be a closed geodesic. Show that each of the two regions bounded by Γ have equal areas under the Gauss map.

Exercise 17. Compute the area of the pseudo-sphere, i.e. the surface of revolution obtained by rotating a tractrix.