Math 497C
Curves and Surfaces
Fall 2004, PSU

Sep 17, 2004[1]

Lecture Notes 3

1.8 The general definition of curvature; Fox-Milnor's Theorem

Let $\alpha\colon [a,b] \to \mathbf{R}^n$ be a curve and $P = \{t_0, \dots, t_n\}$ be a partition of $[a,b]$, then (the approximation of) the total curvature of α with respect to P is defined as

$$\mathit{total}\,\kappa[\alpha, P] := \sum_{i=1}^{n-1} \text{angle}\Big(\alpha(t_i) - \alpha(t_{i-1})\ ,\ \alpha(t_{i+1}) - \alpha(t_i)\Big),$$

and the *total curvature* of α is given by

$$\mathit{total}\,\kappa[\alpha] := \sup\{\,\kappa[\alpha, P] \mid P \in \mathit{Partition}[a,b]\,\}.$$

Our main aim here is to prove the following observation due to Ralph Fox and John Milnor:

Theorem 1 (Fox-Minor)**.** *If $\alpha\colon [a,b] \to \mathbf{R}^n$ is a C^2 unit speed curve, then*

$$\mathit{total}\,\kappa[\alpha] = \int_a^b \|\alpha''(t)\|\, dt.$$

This theorem implies, by the mean value theorem for integrals, that for any $t \in (a,b)$,

$$\kappa(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \mathit{total}\,\kappa\Big[\alpha\big|_{t-\epsilon}^{t+\epsilon}\Big].$$

The above formula may be taken as the definition of curvature for general (not necessarily C^2) curves. To prove the above theorem first we need to develop some basic spherical geometry. Let

$$\mathbf{S}^n := \{p \in \mathbf{R}^{n+1} \mid \|p\| = 1\}.$$

[1]Last revised: September 30, 2021

denote the n-dimensional unit sphere in $\mathbf{R}^{n+1}$. Define a mapping from $\mathbf{S}^n \times \mathbf{S}^n$ to $\mathbf{R}$ by

$$\mathrm{dist}_{\mathbf{S}^n}(p, q) := \mathrm{angle}(p, q).$$

Exercise 2. Show that $(\mathbf{S}^n, \mathrm{dist}_{\mathbf{S}^n})$ is a metric space.

The above metric has a simple geometric interpretation described as follows. By *a great circle* $C \subset \mathbf{S}^n$ we mean the intersection of $\mathbf{S}^n$ with a two dimensional plane which passes through the origin o of $\mathbf{R}^{n+1}$. For any pair of points p, $q \in \mathbf{S}^2$, there exists a plane passing through them and the origin. When $p \neq \pm q$ this plane is given by the linear combinations of p and q and thus is unique; otherwise, p, q and o lie on a line and there exists infinitely many two dimensional planes passing through them. Thus through every pairs of points of $\mathbf{S}^n$ there passes a great circle, which is unique whenever $p \neq \pm q$.

Exercise 3. For any pairs of points p, $q \in \mathbf{S}^n$, let C be a great circle passing through them. If $p \neq q$, let ℓ_1 and ℓ_2 denote the length of the two segments in C determined by p and q, then $\mathrm{dist}_{\mathbf{S}^n}(p, q) = \min\{\ell_1, \ell_2\}$. (*Hint:* Let $p^\perp \in C$ be a vector orthogonal to p, then C may be parametrized as the set of points traced by the curve $p\cos(t) + p^\perp \sin(t)$.)

Let $\alpha \colon [a, b] \to \mathbf{S}^n$ be a spherical curve, i.e., a Euclidean curve $\alpha \colon [a, b] \to \mathbf{R}^{n+1}$ with $\|\alpha\| = 1$. For any partition $P = \{t_0, \ldots, t_n\}$ of $[a, b]$, the spherical length of α with respect the partition P is defined as

$$\mathrm{length}_{\mathbf{S}^n}[\alpha, P] = \sum_{i=1}^{n} \mathrm{dist}_{\mathbf{S}^n}\big(\alpha(t_i), \alpha(t_{i-1})\big).$$

The norm of any partition P of $[a, b]$ is defined as

$$|P| := \max\{t_i - t_{i-1} \mid 1 \leq i \leq n\}.$$

If P^1 and P^2 are partitions of $[a, b]$, we say that P^2 is a *refinement* of P^1 provided that $P^1 \subset P^2$.

Exercise 4. Show that if P^2 is a refinement of P^1, then

$$\mathrm{length}_{\mathbf{S}^n}[\alpha, P^2] \geq \mathrm{length}_{\mathbf{S}^n}[\alpha, P^1].$$

(*Hint:*Use the fact that $\mathrm{dist}_{\mathbf{S}^n}$ satisfies the triangle inequality, see Exc. 2).

The spherical length of α is defined by

$$\text{length}_{\mathbf{S}^n}[\alpha] = \sup\{\,\text{length}_{\mathbf{S}^n}[\alpha, P] \mid P \in \mathit{Partition}[a,b]\,\}\,.$$

Lemma 5. *If $\alpha\colon [a,b] \to \mathbf{S}^n$ is a unit speed spherical curve, then*

$$\text{length}_{\mathbf{S}^n}[\alpha] = \text{length}[\alpha].$$

Proof. Let $P^k := \{t_0^k, \ldots, t_n^k\}$ be a sequence of partitions of $[a,b]$ with

$$\lim_{k\to\infty} |P^k| = 0,$$

and

$$\theta_i^k := \text{dist}_{\mathbf{S}^n}\big(\alpha^k(t_i), \alpha^k(t_{i-1})\big) = \text{angle}\big(\alpha^k(t_i), \alpha^k(t_{i-1})\big)$$

be the corresponding spherical distances. Then, since α has unit speed,

$$2\sin\left(\frac{\theta_i^k}{2}\right) = \|\alpha(t_i^k) - \alpha(t_{i-1}^k)\| \le t_i^k - t_{i-1}^k \le |P^k|.$$

In particular,

$$\lim_{k\to\infty} 2\sin\left(\frac{\theta_i^k}{2}\right) = 0.$$

Now, since $\lim_{x\to 0}\sin(x)/x = 1$, it follows that, for any $\epsilon > 0$, there exists $N > 0$, depending only on $|P^k|$, such that if $k > N$, then

$$(1-\epsilon)\theta_i^k \le 2\sin\left(\frac{\theta_i^k}{2}\right) \le (1+\epsilon)\theta_i^k,$$

which yields that

$$(1-\epsilon)\,\text{length}_{\mathbf{S}^n}[\alpha, P^k] \le \text{length}[\alpha, P^k] \le (1+\epsilon)\,\text{length}_{\mathbf{S}^n}[\alpha, P^k].$$

The above inequalities are satisfied by any $\epsilon > 0$ provided that k is large enough. Thus

$$\lim_{k\to\infty} \text{length}_{\mathbf{S}^n}[\alpha, P^k] = \text{length}[\alpha].$$

Further, note that if P is any partitions of $[a,b]$ we may construct a sequence of partitions by successive refinements of P so that $\lim_{k\to\infty}|P^k| = 0$. By Exercise 4, $\text{length}_{\mathbf{S}^n}[\alpha, P^k] \le \text{length}_{\mathbf{S}^n}[\alpha, P^{k+1}]$. Thus the above expression shows that, for any partition P of $[a,b]$,

$$\text{length}_{\mathbf{S}^n}[\alpha, P] \le \text{length}[\alpha].$$

The last two expressions now yield that

$$\sup\{\,\mathrm{length}_{\mathbf{S}^n}[\alpha, P] \mid P \in Partition[a,b]\,\} = \mathrm{length}[\alpha],$$

which completes the proof. $\square$

Exercise 6. Show that if P^2 is a refinement of P^1, then

$$total\kappa[\alpha, P^2] \geq total\kappa[\alpha, P^1].$$

Now we are ready to prove the theorem of Fox-Milnor:

Proof of Theorem 1. As in the proof of the previous lemma, let $P^k = \{t_0^k, \dots, t_n^k\}$ be a sequence of partitions of $[a,b]$ with $\lim_{k\to\infty}|P^k| = 0$. Set

$$\theta_i^k := \mathrm{angle}\Big(\alpha(t_i^k) - \alpha(t_{i-1}^k)\ ,\ \alpha(t_{i+1}^k) - \alpha(t_i^k)\Big),$$

where $i = 1, \dots, n-1$. Further, set

$$\bar{t}_i^k := \frac{t_i^k + t_{i-1}^k}{2}$$

and

$$\phi_i^k := \mathrm{angle}\Big(\alpha'(\bar{t}_i^k), \alpha'(\bar{t}_{i+1}^k)\Big).$$

Recall that, by the previous lemma,

$$\lim_{k\to\infty}\sum_i \phi_i^k = \mathrm{length}_{\mathbf{S}^{n-1}}[\alpha'] = \mathrm{length}[\alpha'] = \int_a^b \|\alpha''(t)\|\,dt.$$

Thus to complete the proof it suffices to show that, for every $\epsilon > 0$, there exists N such that for all $k \geq N$,

$$|\theta_i^k - \phi_i^k| \leq \epsilon(t_{i+1}^k - t_{i-1}^k); \tag{1}$$

for then it would follow that

$$2\epsilon[a,b] \leq \sum_i \theta_i^k - \sum_i \phi_i^k \leq 2\epsilon[a,b],$$

which would in turn yield

$$\lim_{k\to\infty} total\,\kappa[\alpha, P^k] = \lim_{k\to\infty}\sum_i \theta_i^k = \lim_{k\to\infty}\sum_i \phi_i^k = \int_a^b \|\alpha''(t)\| dt.$$

Now, similar to the proof of Lemma 5, note that given any partition P of $[a, b]$, we may construct by subsequent refinements a sequence of partitions P^k, with $P^0 = P$, such that $\lim_{k\to\infty} |P^k| = 0$. Thus the last expression, together with Exercise 6, yields that

$$total\kappa[\alpha, P] \leq \int_a^b \|\alpha''(t)\|\, dt.$$

The last two expressions complete the proof; so it remains to establish (1). To this end let

$$\beta_i^k := \text{angle}\Big(\alpha'(\bar{t}_i^k),\, \alpha(t_i^k) - \alpha(t_{i-1}^k)\Big).$$

By the triangle inequality for angles (Exercise 2).

$$\phi_i^k \leq \beta_i^k + \theta_i^k + \beta_{i+1}^k\ , \quad \text{and} \quad \theta_i^k \leq \beta_i^k + \phi_i^k + \beta_{i+1}^k,$$

which yields

$$|\phi_i^k - \theta_i^k| \leq \beta_i^k + \beta_{i+1}^k.$$

So to prove (1) it is enough to show that for every $\epsilon > 0$

$$\beta_i^k \leq \frac{\epsilon}{2}(t_i - t_{i-1})$$

provided that k is large enough. See Exercise 7. □

Exercise* 7. Let $\alpha\colon [a, b] \to \mathbf{R}^n$ be a C^2 curve. For every t, $s \in [a, b]$, $t \neq s$, define

$$f(t, s) := \text{angle}\left(\alpha'\left(\frac{t+s}{2}\right),\, \alpha(t) - \alpha(s)\right).$$

Show that

$$\lim_{t\to s} \frac{f(t, s)}{t - s} = 0.$$

In particular, if we set $f(t, t) = 0$, then the resulting function $f\colon [a, b] \times [a, b] \to \mathbf{R}$ is continuous. So, since $[a, b]$ is compact, f is uniformly continuous, i.e., for every $\epsilon > 0$, there is a δ such that $\|f(t) - f(s)\| \leq \epsilon$, whenever $|t - s| \leq \delta$. Does this result hold for C^1 curves as well?