Math 497C
Curves and Surfaces
Fall 2004, PSU

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Lecture Notes 4

1.9 Curves of Constant Curvature

Here we show that the only curves in the plane with constant curvature are lines and circles. The case of lines occurs precisely when the curvature is zero:

Exercise 1. Show that the only curves with constant zero curvature in $\mathbf{R}^n$ are straight lines. (*Hint*: We may assume that our curve, $\alpha\colon I \to \mathbf{R}^n$ has unit speed. Then $\kappa = \|\alpha''\|$. So zero curvature implies that $\alpha'' = 0$. Integrating the last expression twice yields the desired result.)

So it remains to consider the case where we have a planar curve whose curvature is equal to some nonzero constant c. We claim that in this case the curve has to be a circle of radius $1/c$. To this end we introduce the following definition. If a curve $\alpha\colon I \to \mathbf{R}^n$ has nonzero curvature, the *principal normal* vector field of α is defined as

$$N(t) := \frac{T'(t)}{\|T'(t)\|},$$

where $T(t) := \alpha'(t)/\|\alpha'(t)\|$ is the tantrix of α as we had defined earlier. Thus the principal normal is the tantrix of the tantrix.

Exercise 2. Show that $T(t)$ and $N(t)$ are orthogonal. (*Hint*: Differentiate both sides of the expression $\langle T(t), T(t)\rangle = 1$).

So, if α is a planar curve, $\{T(t), N(t)\}$ form a *moving frame* for $\mathbf{R}^2$, i.e., any element of $\mathbf{R}^2$ may be written as a linear combination of $T(t)$ and $N(t)$ for any choice of t. In particular, we may express the derivatives of T and

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N in terms of this frame. The definition of N already yields that, when α is parametrized by arclength,

$$T'(t) = \kappa(t)N(t).$$

To get the corresponding formula for N', first observe that

$$N'(t) = aT(t) + bN(t).$$

for some a and b. To find a note that, since $\langle T, N\rangle = 0$, $\langle T', N\rangle = -\langle T, N'\rangle$. Thus

$$\alpha = \langle N'(t), T(t)\rangle = -\langle T'(t), N(t)\rangle = -\kappa(t).$$

Exercise 3. Show that $b = 0$. (*Hint:* Differentiate $\langle N(t), N(t)\rangle = 1$).

So we conclude that

$$N'(t) = -\kappa(t)T(t),$$

where we still assume that t is the arclength parameter. The formulas for the derivative may be expressed in the matrix notation as

$$\begin{bmatrix} T(t) \\ N(t) \end{bmatrix}' = \begin{bmatrix} 0 & \kappa(t) \\ -\kappa(t) & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \end{bmatrix}.$$

Now recall that our main aim here is to classify curves of constant curvature in the plane. To this end define the *center of the osculating circle* of α as

$$p(t) := \alpha(t) + \frac{1}{\kappa(t)}N(t).$$

The circle which is centered at $p(t)$ and has radius of $1/\kappa(t)$ is called the *osculating* circle of α at time t. This is the circle which best approximates α up to the second order:

Exercise 4. Check that the osculating circle of α is tangent to α at $\alpha(t)$ and has the same curvature as α at time t.

Now note that if α is a circle, then it coincides with its own osculating circle. In particular $p(t)$ is a fixed point (the center of the circle) and $\|\alpha(t) - p(t)\|$ is constant (the radius of the circle). Conversely:

Exercise 5. Show that if α has constant curvature c, then (i) $p(t)$ is a fixed point, and (ii) $\|\alpha(t) - p(t)\| = 1/c$ (*Hint*: For part (i) differentiate $p(t)$; part (ii) follows immediately from the definition of $p(t)$).

So we conclude that a curve of constant curvature $c \neq 0$ lies on a circle of radius $1/c$.

1.10 Signed Curvature and Turning Angle

As we mentioned earlier the curvature of a curve is a measure of how fast it is turning. When the curve lies in a plane, we may assign a sign of plus or minus one to this measure depending on whether the curve is rotating clockwise or counterclockwise. Thus we arrive at a more descriptive notion of curvature for planar curves which we call *signed curvature* and denote by $\overline{\kappa}$. Then we have

$$|\overline{\kappa}| = \kappa.$$

To obtain a formula for $\overline{\kappa}$, for any vector $V \in \mathbf{R}^2$, let JV be the counterclockwise rotation by 90 degrees. More formally,

$$JV := (0,0,1) \times V.$$

Then we set

$$\overline{\kappa}(t) := \frac{\big\langle T'(t), JT(t)\big\rangle}{\|\alpha'(t)\|}.$$

Exercise 6. Show that if α is a unit speed curve then

$$\overline{\kappa}(t) = \kappa(t)\big\langle N(t), JT(t)\big\rangle.$$

In particular, $|\overline{\kappa}| = \kappa$.

Exercise 7. Compute the signed curvatures of the clockwise circle $\alpha(t) = (\cos t, \sin t)$, and the counterclockwise circle $\alpha(t) = (\cos(-t), \sin(-t))$.

Exercise 8. Show that

$$\overline{\kappa}(t) := \frac{\big\langle \gamma'(t) \times \gamma''(t), (0,0,1)\big\rangle}{\|\gamma'(t)\|^3}.$$

Another simple and useful way to define the signed curvature (and the regular curvature) of a planar curve is in terms of the *turning angle* θ, which is defined as follows. We claim that for any planar curve $\alpha\colon I \to \mathbf{R}^2$ there exists a continuous function $\theta\colon I \to \mathbf{R}^2$ such that

$$T(t) = \big(\cos\theta(t), \sin\theta(t)\big).$$

Note that $\cos(\theta(t)) = \langle T(t), (1,0)\rangle$. Thus

$$\theta(t) = \angle\big(T(t), (1,0)\big) + 2k\pi,$$

where k is an integer depending on t. Assuming that t is the arclength parameter, or α has unit speed,

$$\overline{\kappa}(t) = \theta'(t).$$

Exercise 9. Check the above formula.

Now we check that θ indeed exists. To this end note that T may be thought of as a mapping from I to the unit circle $\mathbf{S}^1$. Thus it suffices to show that

Proposition 10. *For any continuous function $T\colon I \to \mathbf{S}^1$, where $I = [a, b]$ is a compact interval, there exists a continuous function $\theta\colon I \to \mathbf{R}$ such that $T(t) = (\cos(\theta(t)), \sin(\theta(t))$.*

Proof. Since T is continuous and I is compact, T is *uniformly* continuous, this means that for $\epsilon > 0$, we may find a $\delta > 0$ such that $\|T(t) - T(s)\| < \epsilon$, whenever $|t - s| < \delta$. In particular, we may set δ_0 to be equal to some constant less than one, and ϵ_0 to be the corresponding constant. Now choose a partition

$$a =: x_0 \leq x_1 \leq \cdots \leq x_n := b$$

of $[a, b]$ such that $|x_i - x_{i-1}| < \epsilon_0$, for $i = 1, \ldots, n$. Then $T([x_i, x_{i-1}])$ does not cover $\mathbf{S}^1$. So we may define $\theta_i\colon [x_{i-1}, x_i] \to \mathbf{R}$ by setting $\theta_i(x)$ to be the angle in $[0, 2\pi)$, measured counterclockwise, between $T(x_{i-1})$ and $T(x)$. Finally, θ may be defined as

$$\theta(x) := \theta_0 + \sum_{i=1}^{k-1} \theta_i(x_i) + \theta_k(x), \quad \text{if} \quad x \in [x_{k-1}, x_k].$$

□

1.11 Total Signed Curvature and Winding Number

The *total signed curvature* of a C^2 curve $\alpha\colon [a, b] \to \mathbf{R}^2$ is defined as

$$\operatorname{total} \overline{\kappa}[\alpha] := \int_a^b \overline{\kappa}(t) \|\alpha'(t)\| \, dt.$$

Recall that if α has unit speed, then $\overline{\kappa} = \theta'$, where θ is the turning angle of α provided by Proposition 10. So if we set $\Delta\theta := \theta(b) - \theta(a)$, then the fundamental theorem of calculus yields that

$$\mathrm{total}\,\overline{\kappa}[\alpha] = \int_a^b \overline{\kappa}(t)dt = \int_a^b \theta'(t)dt = \Delta\theta.$$

Note that $\Delta\theta$ is well-defined even when α is only C^1. Thus the total curvature of a C^1 curve may be defined as

$$\mathrm{total}\,\overline{\kappa}[\alpha] := \Delta\theta,$$

despite the fact that curvature of α may not be defined pointwise. We say that $\alpha\colon [a,b] \to \mathbf{R}^2$ is *closed* provided that $\alpha(a) = \alpha(b)$. In addition, if $\alpha'_-(a) = \alpha'_+(b)$, then α closes smoothly, and is called a C^1 *closed curve.* Then $\theta(b) = \theta(a) + 2k\pi$ for some integer k. So $\mathrm{total}\,\overline{\kappa}[\alpha] = 2k\pi$. The integer k is called the Hopf *rotation index* or *winding number* of α.

Exercise 11. (i) Compute the total curvature and rotation index of a circle which has been oriented clockwise, and a circle which is oriented counterclockwise. Sketch the *figure eight curve* $(\cos t, \sin 2t)$, $0 \leq t \leq 2\pi$, and compute its total signed curvature and rotation index.

We say that a *closed curve* $\alpha\colon [a,b] \to \mathbf{R}^2$ is *simple* if it is one-to-one on $[a,b)$. Furthermore, α is oriented *counter clockwise* provides that for every $t \in [a,b]$, the counterclockwise rotation $J\alpha'$ points into the compact region Ω bounded by $\alpha([a,b])$, i.e., there exists $\epsilon > 0$ such that $\alpha(t) + \epsilon J\alpha'(t) \in \Omega$. The following result proved by H. Hopf is one of the fundamental theorems in theory of planar curves.

Theorem 12 (Hopf's turning angle theorem)**.** *For any simple closed C^1 curve $\alpha\colon [a,b] \to \mathbf{R}^2$ which has counterclockwise orientation,* $\mathrm{total}\,\overline{\kappa}[\alpha] = 2\pi$.

Hopf proved the above result using analytic methods including Green's theorem. Here we outline a more elementary proof which will illustrate that the above theorem is simply a generalization of one of the most basic results in Euclidean geometry: the sum of the angles in a triangle is π, which is equivalent to the sum of the exterior angles being 2π.

First we will give another definition for $\mathrm{total}\,\overline{\kappa}$ which will establish the connection between the total signed curvature and the sum of the exterior angles in a *polygon*. By a polygon P we mean a closed curve formed by line segments joing an ordered set of points $\big(p_0, \dots, p_n\big)$ in $\mathbf{R}^2$, where $p_n = p_0$, but $p_i \neq p_{i-1}$, for $i = 1, \dots, n$. Furthermore we assume that the vectors p_ip_{i-i}, and $p_{i+1}p_i$ are not parallel. Each p_i is called a vertex of P. At each vertex p_i, $i = 1 \dots n$, we define the *turning angle* θ_i to be the angle in $(-\pi, \pi)$ determined by the vectors p_ip_{i-i}, and $p_{i+1}p_i$, and measured in the counterclockwise direction (we set $p_{n+1} := p_1$). More formaly,

$$\theta_i := \angle(p_ip_{i-i}, p_{i+1}p_i)\,\mathrm{sign}(p_{i-1}, p_i, p_{i+1}),$$

where we set $\mathrm{sign}(p_{i-1}, p_i, p_{i+1}) = 0$ if p_{i-1}, p_i p_{i+1} lie on a line; otherwise,

$$\mathrm{sign}(p_{i-1}, p_i, p_{i+1}) := \frac{\big\langle p_ip_{i-i} \times p_{i+1}p_i, (0, 0, 1)\big\rangle}{\|p_ip_{i-i} \times p_{i+1}p_i\|}.$$

So $\mathrm{sign}(p_{i-1}, p_i, p_{i+1}) = 1$ provided that p_{i+1} lies on the *left hand side* of the oriented line ℓ spanned by $p_{i-1}p_i$, i.e., the side where $J(p_{i-1}p_i)$ points, and $\mathrm{sign}(p_{i-1}, p_i, p_{i+1}) = -1$ if p_{i+1} lies on the *right hand side* of ℓ, i.e., the side where $-J(p_{i-1}p_i)$ points. Note that if P forms a simple closed curve which is oriented counterclockwise, then $\theta_i = \pi - \theta_i'$ where $\theta_i \in (0, \pi)$ are the interior angles of Ω. So θ_i are sometimes called *exterior angles*. The total curvature of P is defined as the sum of its turning angles:

$$\mathrm{total}\,\overline{\kappa}[P] := \sum_{i=1}^{n} \theta_i.$$

Now let $\alpha\colon [a, b] \to \mathbf{R}^2$ be a closed planar curve. For $i = 0, \dots, n$, set $t_i := a + i\frac{b-a}{n}$, and let $P_n[\alpha]$ be the closed polygon with vertices $\big(\alpha(t_0), \dots, \alpha(t_n)\big)$.

Lemma 13. *If n is sufficiently large, then* $\mathrm{total}\,\overline{\kappa}[\alpha] = \mathrm{total}\,\overline{\kappa}\big[P_n[\alpha]\big]$.

Proof. Let θ be the rotation angle of α, and θ_i be the turning angles of $P_n[\alpha]$. Since α is C^1, there exists, for $i = 0, \dots, n$, an element $\bar{t}_i \in [t_{i-1}, t_i]$ such that $T(t_i)$ is parallel to $\alpha(t_i) - \alpha(t_{i-1})$. Furthermore, choosing n large enough, we can make sure that $\theta(\bar{t}_i) - \theta(\bar{t}_{i-1}) < \pi$, since θ is continuous. Then it follows that $\theta_i = \theta(\bar{t}_i) - \theta(\bar{t}_{i-1})$. So

$$\sum \theta_i = \sum \big(\theta(\bar{t}_i) - \theta(\bar{t}_{i-1})\big) = \theta(b) - \theta(a) = \Delta\theta,$$

which completes the proof. ∎

Now to complete the proof of Theorem 12 we need only to verify:

Lemma 14. *For any simple closed polygonal curve P, oriented counter clockwise,* $\text{total}\,\overline{\kappa}[\alpha] = 2\pi$.

Proof. First note that the lemma holds for triangles. Then it holds for convex polygons as well, since they can be decomposed into triangles, by connecting a point in the interior of the region bounded by the polygon to all the vertices. The rest of the proof proceeds by induction. Suppose that the lemma holds for all polygons with n sides, and assume that P is a polygon with $n+1$ sides. If P is convex, then we are done. Otherwise, the boundary of the convex hull of P, which we call Q yields a convex simple closed polygonal curve whose vertices form a subset of the vertices of P. Any vertex of P which does not lie on Q is part of a polygonal path of P whose end points lie on a pair of adjacent vertices of Q. Joining the edge in between these vertices to the polygonal path, we obtain a simple closed polygonal curve. There are a finite number of such curves which we call $R_1, \ldots, R_m$. We claim that

$$\text{total}\,\overline{\kappa}[P] + \sum_{i=1}^{m} \text{total}\,\overline{\kappa}[R_i] = \text{total}\,\overline{\kappa}[Q] + 2m\pi,$$

which will complete the proof, since $\text{total}\,\overline{\kappa}[R_i] = 2\pi$ by the inductive hypothesis, and $\text{total}\,\overline{\kappa}[Q] = 2\pi$ as well since Q is convex. It is enough to establish the above equality for the case $m = 1$:

$$\text{total}\,\overline{\kappa}[P] + \text{total}\,\overline{\kappa}[R] = \text{total}\,\overline{\kappa}[Q] + 2\pi.$$

The general case for $m > 1$ then follows from repeated application of the last equality. To establish this equality, let v be a vertex of R in the interior of the region bounded by Q. Then v is also a vertex of P. Let α, β denote the turning angles of R and P at v. Then $\alpha + \beta = 2\pi - (\alpha' + \beta') = 0$, where α' and β' are the interior angles of R and P at v. So turning angles at vertices of P or R which are contained inside Q cancel each other. We just need to consider then the vertices v of P which lie on Q. In this case, if v is a vertex of P but not of R, then the turning angle of P at v is equal to that of Q. On the other hand if v also belongs to R, and α, β denote the turning angles of P and R at v, then $\alpha + \beta = 2\pi - (\alpha' + \beta') = 2\pi - \gamma' = \pi + \gamma$, where γ is the turning angle of Q at v, and $\gamma' = \pi - \gamma$ is the corresponding interior angle. Thus we pick up an extra π for each vertex of R on Q, which completes the proof. □

Our method of proof via polygonal approximations yields a genralization of Hopf's theorem to piecewise C^1 curves. We say α is piecewise C^1 provided that there are points $a =: t_0 < t_1 < \cdots < t_k := b$ such that α is C^1 on each subintervals $[t_{i-1}, t_i]$. Then the points $\alpha(t_i)$, $i = 1, \ldots, k$ will be called the *corners* of α or the corners of the region Ω bounded by α. Assuming that α is oriented counterclockwise, i.e., $J\alpha'(t)$ points into Ω at all differentiable points $t \in [a, b]$, then the turning angle at the corner θ_i is defined as $\pi - \theta_i'$ where $\theta_i' \in [0, 2\pi]$ is the interior angle of Ω at $\alpha(t_i)$. So $\theta_i \in [-\pi, \pi]$. Now we define

$$\operatorname{total}\overline{\kappa}[\alpha] := \sum_i \operatorname{total}\overline{\kappa}[\alpha|_{[t_{i-1}, t_i]}] + \sum_i \theta_i.$$

Theorem 15. *Let α be a simple closed piecewise C^1 planar curve oriented counter clockwise. Then*

$$\operatorname{total}\overline{\kappa}[\alpha] = 2\pi.$$

To prove the above theorem we just need to stipulate that the vertices of polygonal approximations $P_n[\alpha]$ we discussed earlier include all corners of α.

1.12 The fundamental theorem of planar curves

If $\alpha\colon [0, L] \to \mathbf{R}^2$ is a planar curve parametrized by arclength, then its signed curvature yields a function $\overline{\kappa}\colon [0, L] \to \mathbf{R}$. Now suppose that we are given a continuous function $\overline{\kappa}\colon [0, L] \to \mathbf{R}$. Is it always possible to find a unit speed curve $\alpha\colon [0, L] \to \mathbf{R}^2$ whose signed curvature is $\overline{\kappa}$? If so, to what extent is such a curve unique? In this section we show that the signed curvature does indeed determine a planar curve, and such a curve is unique up to proper rigid motions.

Recall that by a proper rigid motion we mean a composition of a translation with a proper rotation. A translation is a mapping $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$T(p) := p + v$$

where v is a fixed vector. And a proper rotation $\rho\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a linear mapping given by

$$\rho\Big(\begin{bmatrix} x \\ y \end{bmatrix}\Big) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Exercise 16. Show that the signed curvature of a planar curve is invariant under proper rigid motions.

Exercise 17 (Local Convexity)**.** Show that if the curvature of a planar curve $\alpha\colon I \to \mathbf{R}^2$ does not vanish at an interior point t_0 of I then there exists an open neighborhood U of t_0 in I such that $\alpha(U)$ lies on one side of the tangent line of α at t_0. (*Hint*: By the invariance of signed curvature under rigid motions, we may assume that $\alpha(t_0) = (0,0)$ and $\alpha'(0) = (1,0)$. Then we may reparametrize α as $(t, f(t))$ in a neighborhood of t_0. Recalling the formula for curvature for graphs, and applying the Taylor's theorem yields the desired result.)

Now suppose that we are given a function $\overline{\kappa}\colon [0,L] \to \mathbf{R}$. If there exist a curve $\alpha\colon [0,L] \to \mathbf{R}^2$ with signed curvature $\overline{\kappa}$, then

$$\theta' = \overline{\kappa}$$

where θ is the rotation angle of α. Integration yields

$$\theta(t) := \int_0^t \overline{\kappa}(s)ds + \theta(0).$$

By the definition of the turning angle

$$\alpha'(t) = \Big(\cos\theta(t), \sin\theta(t)\Big).$$

Consequently,

$$\alpha(t) = \Big(\int_0^t \cos\theta(s)ds, \int_0^t \sin\theta(s)ds\Big) + \alpha(0),$$

which gives an explicit formula for the desired curve.

Exercise 18 (Fundamental theorem of planar curves)**.** Let α, $\beta\colon [0,L] \to \mathbf{R}^2$ be unit speed planar curves with the same signed curvature function $\overline{\kappa}$. Show that there exists a proper rigid motion $m\colon \mathbf{R}^2 \to \mathbf{R}^2$ such that $\alpha(t) = m(\beta(t))$.

Exercise 19. Use the above formula to show that the only closed curves of constant curvature in the plane are circles.