# RIGIDITY OF NONPOSITIVELY CURVED MANIFOLDS WITH CONVEX BOUNDARY

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ABSTRACT. We show that a compact Riemannian 3-manifold $M$ with strictly convex simply connected boundary and sectional curvature $K \leq a \leq 0$ is isometric to a convex domain in a complete simply connected space of constant curvature $a$, provided that $K \equiv a$ on planes tangent to the boundary of $M$. This yields a characterization of strictly convex surfaces with minimal total curvature in Cartan-Hadamard 3-manifolds, and extends some rigidity results of Greene-Wu, Gromov, and Schroeder-Strake. Our proof is based on a recent comparison formula for total curvature of Riemannian hypersurfaces, which also yields some dual results for $K \geq a \geq 0$.

## 1. INTRODUCTION

A *Cartan-Hadamard* manifold $\mathcal{H}$ is a complete simply connected Riemannian $n$-space with sectional curvature $K \leq 0$. Greene and Wu [9, 10] and Gromov [3, Sec. 5] showed that, when $n \geq 3$, these spaces exhibit remarkable rigidity properties, analogous to those observed earlier by Mok, Siu, and Yau [13, 18] in Kähler geometry. In particular, a fundamental result is that if $K$ vanishes outside a compact set $C \subset \mathcal{H}$, then $\mathcal{H}$ is isometric to Euclidean space $\mathbf{R}^n$. More generally, if $K \leq a \leq 0$ on $\mathcal{H}$, and $K \equiv a$ on $\mathcal{H} \setminus C$, then $K \equiv a$ on $\mathcal{H}$ [9, p. 734] [17]. We extend this result when $n = 3$:

**Theorem 1.1.** *Let $M$ be a compact Riemannian 3-manifold with nonempty $\mathcal{C}^2$ boundary $\partial M$ and sectional curvature $K \leq a \leq 0$. Suppose that $\partial M$ is strictly convex, each component of $\partial M$ is simply connected, and $K \equiv a$ on planes tangent to $\partial M$. Then $M$ is isometric to a convex domain in a Cartan-Hadamard manifold of constant curvature $a$. In particular, $M$ is diffeomorphic to a ball.*

*Strictly convex* here means that the second fundamental form of $\partial M$ is positive definite with respect to the outward normal. For $n = 3$, this theorem immediately implies the rigidity results mentioned above, by letting $M$ be a geodesic ball in $\mathcal{H}$ containing $C$.

*Date*: March 3, 2023 (Last Typeset).
2010 *Mathematics Subject Classification.* Primary: 53C20, 58J05; Secondary: 53C42, 52A15.
*Key words and phrases.* Cartan-Hadamard manifold, Hyperbolic space, Gap theorem, Minimal total curvature, Asymptotic rigidity, Gauss-Kronecker curvature, Bounded sectional curvature.
The research of M.G. was supported by NSF grant DMS-2202337 and a Simons Fellowship. The research of J.S. was supported by a Simons Collaboration Grant.

Schroeder and Strake [15] had established this result for $a = 0$ (and only for $n = 3$) refining earlier work of Schroeder and Ziller [14]. The simply connected assumption on components of $\partial M$ is necessary, as can be seen by considering a tubular neighborhood of a closed geodesic in a hyperbolic manifold.

As an application of Theorem 1.1 we obtain the following characterization for strictly convex surfaces with minimal total curvature. We say that an oriented closed (compact, connected, without boundary) hypersurface $\Gamma \subset \mathcal{H}$ is *strictly convex* if its second fundamental form $\mathrm{I\!I}$ is positive definite. Then $\Gamma$ is embedded, bounds a convex domain, and is simply connected [1]. The *total Gauss-Kronecker curvature* of $\Gamma$ is given by $\mathcal{G}(\Gamma) := \int_\Gamma \det(\mathrm{I\!I})$, and $|\Gamma|$ denotes the area of $\Gamma$.

**Corollary 1.2.** *Let $\mathcal{H}$ be a 3-dimensional Cartan-Hadamard manifold with curvature $K \leq a \leq 0$, and $\Gamma \subset \mathcal{H}$ be a $\mathcal{C}^2$ closed strictly convex surface. Then*

$$\mathcal{G}(\Gamma) \geq 4\pi - a|\Gamma|, \tag{1}$$

*with equality only if $K \equiv a$ on the convex domain bounded by $\Gamma$.*

*Proof.* By Gauss' equation $\det(\mathrm{I\!I}_p) = K_\Gamma(p) - K(T_p\Gamma)$ for all $p \in \Gamma$, where $K_\Gamma$ is the intrinsic curvature of $\Gamma$, and $T_p\Gamma$ is the tangent plane of $\Gamma$ at $p$. Since $\Gamma$ is simply connected, $\int_\Gamma K_\Gamma = 4\pi$ by Gauss-Bonnet theorem. Thus

$$\mathcal{G}(\Gamma) = 4\pi - \int_{p\in\Gamma} K(T_p\Gamma) \geq 4\pi - a|\Gamma|. \tag{2}$$

If equality holds in (1), then it also holds in (2), which forces $K \equiv a$ on tangent planes of $\Gamma$. Theorem 1.1, applied to the convex domain bounded by $\Gamma$, completes the proof. $\square$

For $a = 0$, the last result is stated in [3, p. 66] and follows from [15, Thm. 2]. Gromov's approach to the rigidity theorems mentioned above [3, Sec. 5], which are further developed in [14, 15], was based on extension of isometric embeddings in locally symmetric spaces. In most of these results the rank of the space is required to be bigger than 1, which precludes negative upper bounds for curvature. The arguments of Greene and Wu [9] on the other hand involve volume comparison theory, which applies readily to various curvature bounds [9, p. 734]; see Seshadri [17]. Here we develop a different approach via recent work on total curvature of Riemannian hypersurfaces [7, 8], which also yields some results for the dual case $K \geq a \geq 0$; see Note 3.1. Generalizing (1) to dimensions $n > 3$ is an important open problem with applications to the isoperimetric inequality; see [7] for more references and background in this area.

## 2. Proof of Theorem 1.1

The proof consists of three parts. First we use the comparison formula developed in [7, 8] to show that $K \equiv a$ on a neighborhood of $\partial M$ (which we do not a priori assume to be connected). Then it follows that $M$ is isometric to a convex domain in a Cartan-Hadamard manifold $\overline{M}$, which has constant curvature $a$ outside $M$ (in particular $\partial M$ is connected). Finally enclosing $M$ in a geodesic ball $B \subset \overline{M}$ and shrinking the radius of $B$ completes the proof via the first part of the argument. The first part also involves the Gauss-Bonnet theorem, which is why we need to assume that $n = 3$. Other aspects of the proof work in all dimensions $n \geq 3$.

**(Part I).** Let $\Gamma$ be a component of $\partial M$, $d_\Gamma \colon M \to \mathbf{R}$ be the distance function of $\Gamma$, and $\Gamma_t := d_\Gamma^{-1}(t)$ be the parallel surface of $\Gamma$ at distance $t$. Since $\Gamma$ is $\mathcal{C}^2$, there exists $\varepsilon > 0$ such that $\Gamma_t$ is $\mathcal{C}^2$ for $t \in [0, \varepsilon]$, see [5]. In particular, for $t \in [0, \varepsilon]$, the principal curvatures $\kappa_i^t$ of $\Gamma_t$ with respect to the outward normal $\nu_t$ are well-defined. By assumption, $\kappa_i^0 > 0$, which yields that $\kappa_i^t > 0$, assuming $\varepsilon$ is sufficiently small. Let $e_i^t$ be a choice of orthogonal principal directions for $\kappa_i^t$, and $K_i^t$ be the sectional curvatures of $M$ for planes spanned by $\nu_t$ and $e_i^t$. Let $\Omega_\varepsilon \subset M$ be the domain bounded in between $\Gamma$ and $\Gamma_\varepsilon$, and recall that $\mathcal{G}(\Gamma_t)$ denote the total Gauss-Kronecker curvature of $\Gamma_t$, i.e., the integral of $\kappa_1^t \kappa_2^t$ over $\Gamma_t$. Since $|\nabla d_\Gamma|$ is constant on $\Gamma_t$, the comparison formula in [8, Thm. 3.1] reduces to

$$\mathcal{G}(\Gamma) - \mathcal{G}(\Gamma_\varepsilon) = -\int_{\Omega_\varepsilon} \big(\kappa_1^t K_2^t + \kappa_2^t K_1^t\big).$$

Let $H^t := \kappa_1^t + \kappa_2^t$ denote the mean curvature of $\Gamma_t$. Since $\kappa_i^t \geq 0$,

$$\mathcal{G}(\Gamma) - \mathcal{G}(\Gamma_\varepsilon) \geq -a \int_{\Omega_\varepsilon} H^t = -a\big(|\Gamma| - |\Gamma_\varepsilon|\big). \tag{3}$$

The last equality is due to Stokes theorem, since $H^t = \operatorname{div}(\nabla d_\Gamma)$ and $|\nabla d_\Gamma| = 1$ (more formally, the above inequality holds on $\Omega_\varepsilon \setminus \Omega_s$, for $0 < s < \varepsilon$, and we may take the limit as $s \to 0^+$). On the other hand, by Gauss' equation and Gauss-Bonnet theorem,

$$\begin{aligned}\mathcal{G}(\Gamma) &= 4\pi - \int_{p\in\Gamma} K(T_p\Gamma) = 4\pi - a|\Gamma|,\\ \mathcal{G}(\Gamma_\varepsilon) &= 4\pi - \int_{p\in\Gamma_\varepsilon} K(T_p\Gamma_\varepsilon) \geq 4\pi - a|\Gamma_\varepsilon|.\end{aligned} \tag{4}$$

Hence

$$\mathcal{G}(\Gamma) - \mathcal{G}(\Gamma_\varepsilon) \leq -a\big(|\Gamma| - |\Gamma_\varepsilon|\big).$$

So equality holds in (3) which forces $K_i^t \equiv a$ on $\Omega_\varepsilon$ for $i = 1, 2$. Furthermore, equality in (3) implies equality in (4), which yields that $K \equiv a$ on tangent planes of $\Gamma_\varepsilon$. So $K \equiv a$ on a triplet of mutually orthogonal planes at each point of $\Gamma_\varepsilon$. It follows that $K \equiv a$

with respect to all planes with footprint on $\Gamma_\varepsilon$, since $K \leq a$. As this argument holds for all $\varepsilon' \leq \varepsilon$, we conclude that $K \equiv a$ on $\Omega_\varepsilon$.

**(Part II).** Let $\mathcal{H}$ be the 3-dimensional Cartan-Hadamard manifold of constant curvature $a$. Then $\Omega_\varepsilon$ is locally isometric to $\mathcal{H}$. Furthermore, since $\Gamma$ is simply connected, so is $\Omega_\varepsilon$. Thus there exists an isometric immersion $f\colon \Omega_\varepsilon \to \mathcal{H}$, by a standard monodromy argument. In particular, $f(\Gamma)$ forms a closed immersed surface in $\mathcal{H}$ with positive principal curvatures. Consequently, by a result of Alexander [1, Thm. 1], see also [15, Lem. 1], $f$ embeds $\Gamma$ into the boundary of a convex domain $C \subset \mathcal{H}$. Let $C' \subset \mathcal{H}$ be the closure of $\mathcal{H} \setminus C$. Using the diffeomorphism $f$ between $f(\Gamma) = \partial C'$ and $\Gamma$, we may glue $C'$ to $M$ along $\Gamma$ to obtain a smooth manifold with one fewer boundary component. Repeating this procedure for each component $\Gamma$ of $\partial M$ yields an extension of $M$ to a complete manifold $\overline{M}$ of nonpositive curvature. Now pick a point $p \in \overline{M}$. By Cartan-Hadamard theorem, the exponential map $\exp_p\colon T_p\overline{M} \to \overline{M}$ is a covering. Let $X$ be a component of $\overline{M} \setminus M$. Note that $X$ is simply connected since, by Schoenflies theorem, it is homeomorphic to the complement of a ball in $\mathbf{R}^3$. Let $X'$ be a component of $\exp_p^{-1}(X)$. Since $X$ is simply connected, $X'$ is homeomorphic to $X$. In particular $\partial X'$ is an embedded topological sphere. Thus $X'$ is the complement of a bounded set in $T_p\overline{M}$. Since any two such sets must intersect, it follows that $\overline{M} \setminus M$ is connected, and $X' = \exp_p^{-1}(X)$. In particular $\overline{M} \setminus M = X$, which is simply connected. Consequently $\exp_p\colon X' \to X$ is one-to-one, which yields that it is one-to-one everywhere, since $\exp_p$ is a covering map. Hence $\overline{M}$ is simply connected, and therefore is a Cartan-Hadamard manifold. Finally, since $\overline{M} \setminus M$ is connected, $\partial M$ is connected. So $M$ forms a convex domain in $\overline{M}$ by Alexander's result [1, Thm. 1].

**(Part III).** It remains to show that $K \equiv a$ on $\overline{M}$. By construction $K \equiv a$ on $\overline{M} \setminus M$. So we just need to check that $K \equiv a$ on $M$. Fix a point $o$ in $\overline{M}$ and let $B_r \subset \overline{M}$ be the geodesic ball of radius $r$ centered at $o$. If $r$ is large enough, so that $M \subset B_r$, then $K \equiv a$ outside $B_r$. Let $r_0$ be the infimum of $r > 0$ such that $K \equiv a$ on $\overline{M} \setminus B_r$. If $r_0 = 0$ we are done. Otherwise, since $a \leq 0$, $\partial B_{r_0}$ forms a strictly convex surface by Hessian comparison (the principal curvatures of $\partial B_{r_0}$ are bigger than those of a sphere of the same radius in $\mathbf{R}^3$ [12, 1.7.3]). Thus we may apply the result of Part I to $B_{r_0}$ to obtain that $K \equiv a$ on a neighborhood of $\partial B_{r_0}$ in $B_{r_0}$, which contradicts the definition of $r_0$. So we conclude that $K \equiv a$ everywhere, which completes the proof.

## 3. Notes

**Note 3.1.** Part I of the proof of Theorem 1.1 works just as well for nonnegative curvature, i.e., suppose that $K \geq a \geq 0$ on $M$ and $K \equiv a$ on tangent planes of $\partial M$, then virtually the same argument shows that $K \equiv a$ on an open neighborhood of $\partial M$. Thus if $\partial M$ contracts to a point through strictly convex surfaces, then $K \equiv a$ on $M$ as we showed in Part III. This may be considered a dual version of Theorem 1.1. For instance if $M$ is a geodesic ball of radius $r$ in a space with $K \leq b$, then it satisfies the contraction property provided that $r \leq \pi/(2\sqrt{b})$, by Hessian comparison [12, 1.7.3]. More generally, the required contraction may be achieved via a curvature flow when maximum value of $K$ is not too large compared to principal curvatures of $\partial M$ [2]. Furthermore note that when $a = 0$, and $\partial M$ is simply connected, $M$ may be extended to a nonnegatively curved manifold $\overline{M}$ which is flat outside $M$, as discussed in Part II of the above proof. Then $\overline{M}$ is isometric to $\mathbf{R}^3$ by [9, Thm. 1]. So $M$ will be flat, as had been observed earlier in [15, p. 486]. See [14, 16] for more rigidity results for nonnegative curvature

**Note 3.2.** Once Part I of the proof of Theorem 1.1 has been established, and it is known a priori that $M$ is simply connected with connected boundary $\Gamma$, one may complete the argument more directly by covering $M$ with a continuous family of strictly convex surfaces $\Gamma_t$ with $\Gamma_0 = \Gamma$. For instance, we may let $\Gamma_t$ be level sets of a strictly convex function on $M$, see [4, Lem. 1]. Alternatively, one may use harmonic mean curvature flow, i.e., set $\Gamma_t := X(\Gamma, t)$ for $X\colon \Gamma \times [0, T) \to M$ given by

$$\frac{\partial}{\partial t} X(p,t) = \frac{-1}{1/\kappa_1^t(p) + 1/\kappa_2^t(p)} \nu_t(p), \qquad X(p,0) = p,$$

where $\nu_t$ is the outward normal of $\Gamma_t$ and $\kappa_i^t$ are its principal curvatures. Xu showed [19], see also Gulliver and Xu [11], that $\Gamma_t$ converges to a point $o$ as $t \to T$, and remains strictly convex throughout [19, Prop. 19]. So $\Gamma_t$ always moves inward, foliating the region $M \setminus \{o\}$. The stated regularity requirement in [19] is that $\Gamma$ be $\mathcal{C}^\infty$, which we may assume is the case after a perturbation of $\Gamma$ [6, Lem. 3.3], since by Part I we have $K \equiv a$ near $\Gamma$.

## Acknowledgments

We thank Igor Belegradek and Joe Hoisington for useful communications.

## References

[1] S. Alexander, *Locally convex hypersurfaces of negatively curved spaces*, Proc. Amer. Math. Soc. **64** (1977), no. 2, 321–325. MR448262 ↑2, 4

[2] B. Andrews, *Contraction of convex hypersurfaces in Riemannian spaces*, J. Differential Geom. **39** (1994), no. 2, 407–431. MR1267897 ↑5

[3] W. Ballmann, M. Gromov, and V. Schroeder, *Manifolds of nonpositive curvature*, Progress in Mathematics, vol. 61, Birkhäuser Boston, Inc., Boston, MA, 1985. MR823981 ↑1, 2

[4] A. Borbély, *On the total curvature of convex hypersurfaces in hyperbolic spaces*, Proc. Amer. Math. Soc. **130** (2002), no. 3, 849–854. MR1866041 ↑5

[5] R. L. Foote, *Regularity of the distance function*, Proc. Amer. Math. Soc. **92** (1984), no. 1, 153–155. MR749908 ↑3
[6] M. Ghomi and J. Spruck, *Minkowski inequality in cartan-hadamard manifolds*, arXiv:2206.06554 (2022). ↑5
[7] ———, *Total curvature and the isoperimetric inequality in Cartan-Hadamard manifolds*, J. Geom. Anal. **32** (2022), no. 2, Paper No. 50, 54pp. MR4358702 ↑2, 3
[8] ———, *Total mean curvatures of Riemannian hypersurfaces*, arXiv:2204.07624, to appear in Adv. Nonliner Stud. (2022). ↑2, 3
[9] R. E. Greene and H. Wu, *Gap theorems for noncompact Riemannian manifolds*, Duke Math. J. **49** (1982), no. 3, 731–756. MR672504 ↑1, 2, 5
[10] ———, *On a new gap phenomenon in Riemannian geometry*, Proc. Nat. Acad. Sci. U.S.A. **79** (1982), no. 2, 714–715. MR648065 ↑1
[11] R. Gulliver and G. Xu, *Examples of hypersurfaces flowing by curvature in a Riemannian manifold*, Comm. Anal. Geom. **17** (2009), no. 4, 701–719. MR2601350 ↑5
[12] H. Karcher, *Riemannian comparison constructions*, Global differential geometry, 1989, pp. 170–222. MR1013810 ↑4, 5
[13] N. Mok, Y. T. Siu, and S. T. Yau, *The Poincaré-Lelong equation on complete Kähler manifolds*, Compositio Math. **44** (1981), no. 1-3, 183–218. MR662462 ↑1
[14] V. Schroeder and W. Ziller, *Local rigidity of symmetric spaces*, Trans. Amer. Math. Soc. **320** (1990), no. 1, 145–160. MR958901 ↑2, 5
[15] V. Schroeder and M. Strake, *Local rigidity of symmetric spaces of nonpositive curvature*, Proc. Amer. Math. Soc. **106** (1989), no. 2, 481–487. MR929404 ↑2, 4, 5
[16] ———, *Rigidity of convex domains in manifolds with nonnegative Ricci and sectional curvature*, Comment. Math. Helv. **64** (1989), no. 2, 173–186. MR997359 ↑5
[17] H. Seshadri, *An elementary approach to gap theorems*, Proc. Indian Acad. Sci. Math. Sci. **119** (2009), no. 2, 197–201. MR2526423 ↑1, 2
[18] Y. T. Siu and S. T. Yau, *Complete Kähler manifolds with nonpositive curvature of faster than quadratic decay*, Ann. of Math. (2) **105** (1977), no. 2, 225–264. MR437797 ↑1
[19] G. Xu, *Harmonic mean curvature flow in Riemannian manifolds and Ricci flow on noncompact manifolds*, ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–University of Minnesota. MR2941371 ↑5

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