MINKOWSKI INEQUALITY IN CARTAN-HADAMARD MANIFOLDS

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Abstract. Using harmonic mean curvature flow, we establish a sharp Minkowski type lower bound for total mean curvature of convex surfaces with a given area in Cartan-Hadamard 3-manifolds. This inequality also improves the known estimates for total mean curvature in hyperbolic 3-space. As an application, we obtain a Bonnesen-style isoperimetric inequality for surfaces with convex distance function in nonpositively curved 3-spaces, via monotonicity results for total mean curvature. This connection between the Minkowski and isoperimetric inequalities is extended to Cartan-Hadamard manifolds of any dimension.

1. Introduction

Complete simply connected Riemannian spaces of nonpositive curvature, or *Cartan-Hadamard manifolds*, form a natural generalization of Euclidean and hyperbolic spaces. A *strictly convex* hypersurface Γ of a Cartan-Hadamard space M is a closed embedded submanifold of codimension one which, when properly oriented, has positive definite second fundamental form $\mathrm{I\!I}_\Gamma$. The *mean curvature* of Γ is then given by $H := \mathrm{trace}(\mathrm{I\!I}_\Gamma)$, and its *total mean curvature* is defined as $\mathcal{M}(\Gamma) := \int_\Gamma H d\mu$. A celebrated result of Minkowski [29] states that in Euclidean space $\mathbf{R}^3$

$$\mathcal{M}(\Gamma) \geq \sqrt{16\pi|\Gamma|}, \tag{1}$$

where $|\Gamma|$ denotes the area of Γ, and equality holds only when Γ is a sphere. Extension of this inequality to hyperbolic space $\mathbf{H}^3$ has been a long standing problem [35], which has been intensively studied [13, 31], specially with the aid of curvature flows [2, 14, 37, 44] in recent years; however, the sharp inequality remains unknown. Here we generalize Minkowski's inequality to Cartan-Hadamard manifolds via harmonic mean curvature flow. By *smooth* we mean $\mathcal{C}^\infty$, *curvature* means sectional curvature unless specified otherwise, and a *domain* is a connected open set with compact closure.

Theorem 1.1. *Let Γ be a smooth strictly convex surface in a Cartan-Hadamard 3-manifold M with curvature $K \leq a \leq 0$. Then*

$$\mathcal{M}(\Gamma) \geq \sqrt{16\pi|\Gamma| - 2a|\Gamma|^2}, \tag{2}$$

with equality only if the domain bounded by Γ is isometric to a ball in $\mathbf{R}^3$.

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Inequality (2) appears to be new even in hyperbolic space $\mathbf{H}^3(a)$ of constant curvature $a<0$ [31, p. 109], where the previous best estimate was $\mathcal{M}(\Gamma) \geq \sqrt{-a}\,|\Gamma|$ by Gallego-Solanes [4,13] (note that in [13], $H := \mathrm{trace}(\mathrm{I\!I}_\Gamma)/(n-1)$). Santalo asked [35], see [36, p. 78], whether the sharp inequality in $\mathbf{H}^3(a)$ could be

$$\mathcal{M}(\Gamma) \geq \sqrt{16\pi|\Gamma| - 4a|\Gamma|^2}, \tag{3}$$

as the lower bound would then correspond to the total mean curvature of a sphere with the same area as Γ; however, an example by Naveira-Solanes [36, p. 815], see [31, p. 109], shows that (3) cannot in general hold. In Note 1.3 we will analyze this example to show that (2) is not far from optimal. Under the additional hypothesis that Γ is *h-convex* (or *horo-convex*), i.e., supported at each point by a horosphere, (3) does hold in $\mathbf{H}^3(a)$ [14,44]. In Note 2.1 we will discuss a possible improvement of (2) in the h-convex case, and in Theorem 3.2 we extend (2) to nonsmooth surfaces.

Since total mean curvature is the first variation of area, Minkowski's inequality is closely related to isoperimetric problems in Euclidean space [39, Sec. 7.2] [32, p. 1191]. Here we apply the extension of (2) to nonsmooth surfaces, together with recent monotonicity results for mean curvature [16], to establish an isoperimetric inequality in Cartan-Hadamard manifolds in the style of Bonnesen [33]. This gives a refinement, for convex surfaces, of a theorem of Kleiner [27] who first generalized the isoperimetric inequality to 3-dimensional Cartan-Hadamard manifolds. The *inradius*, $\mathrm{inrad}(\Omega)$, of a domain $\Omega \subset M$ is the supremum of radii of spheres which are contained in Ω. A closed embedded hypersurface Γ, bounding a domain Ω in a Cartan-Hadamard manifold, is *d-convex* (or *distance-convex*) provided that its distance function is convex on Ω. This condition is weaker than h-convexity [15, Sec. 3]. We let $|\Omega|$ denote volume of Ω.

Theorem 1.2. *Let Γ be a smooth d-convex surface in a Cartan-Hadamard 3-manifold, and Ω be the domain bounded by Γ. Then*

$$|\Omega| \leq \frac{4\pi}{3}\left(\left(\frac{|\Gamma|}{4\pi}\right)^{\frac{3}{2}} - \left(\sqrt{\frac{|\Gamma|}{4\pi}} - \mathrm{inrad}(\Omega)\right)^3\right), \tag{4}$$

with equality only if Ω is isometric to a ball in $\mathbf{R}^3$.

The isoperimetric inequality has been established in Cartan-Hadamard manifolds only up to dimension 4 [8,27,46], and a Bonnesen-style inequality was also established recently in dimension 2 [24]. The *Cartan-Hadamard conjecture* states that the isoperimetric inequality should hold in all dimensions [15,28]. Kleiner's approach to this problem was based on estimating the total Gauss-Kronecker curvature, which was further studied in [15]; see also [7,40] and [34, Sec. 3.3.2] for other proofs or variations in dimension 3. Proof of Theorem 1.2 provides an alternative approach based on total mean curvature.

An advantage of this approach is that mean curvature satisfies a monotonicity property, see Lemma 3.4, whereas Gauss-Kronecker curvature does not [11]. In Section 5 below we will show that this method may be deployed in any dimension, as stated in Theorem 5.1.

Minkowski's inequality in Euclidean space is a special case of Alexandrov-Fenchel inequalities for generalized mean curvatures of convex bodies, which may be proved via Brunn-Minkowski theory of mixed volumes; see [39, Thm. 7.2.1 & Note 2, p. 387], and [41, 42] for more recent treatments. Differential geometric proofs using the isoperimetric inequality and Steiner formulas may be found in [31] and [30, p. 201]. Ge-Wang-Wu [14] and Wang-Xia [44] extended the inequality to h-convex surfaces in hyperbolic space via curvature flows; see also [2, 45]. In addition, there has been substantial work on weakening the convexity condition [6, 9, 22], and extensions to other spaces [4, 38] due to applications in general relativity [10, 17].

Note 1.3. Here we examine the Naveira-Solanes example [31] mentioned above to estimate the optimality of (2). This object is constructed by taking a disc D of radius r in a totally geodesic surface in hyperbolic space $\mathbf{H}^3(a)$, and letting $\Gamma = \Gamma(\varepsilon, r)$ be the outer parallel surface of D at a small distance ε. We seek the largest value of λ so that

$$\phi_\lambda(\Gamma) := \mathcal{M}(\Gamma)^2 - 16\pi|\Gamma| + \lambda a|\Gamma|^2$$

is nonnegative for all Γ. As ϕ_λ is invariant under rescaling of the metric, we may assume for convenience that $a = -1$. Then

$$\lim_{\varepsilon\to 0} |\Gamma| = 2|D| = 4\pi(\cosh(r) - 1).$$

Note that Γ consists of a pair of topological disks parallel to D plus a half tube T about ∂D. The mean curvature of the disks vanish as $\varepsilon \to 0$. On the other hand, $|T| \to |\partial D|\pi\sinh(\varepsilon)$ up to first order, since the full tube about ∂D is fibrated by (geodesic) circles of radius ε. So $\mathcal{M}(\Gamma) \to \partial|T|/\partial\varepsilon = |\partial D|\pi\cosh(\varepsilon)$. Thus

$$\lim_{\varepsilon\to 0} \mathcal{M}(\Gamma) = |\partial D|\pi = 2\pi^2\sinh(r),$$

and we conclude that

$$\lim_{\varepsilon\to 0} \phi_\lambda(\Gamma) = 4\pi^4\sinh^2(r) - 64\pi^2(\cosh(r) - 1) - 16\pi^2\lambda(\cosh(r) - 1)^2.$$

Setting this quantity ≥ 0 yields

$$\lambda \leq \frac{\pi^2\sinh^2(r) - 16(\cosh(r) - 1)}{4(\cosh(r) - 1)^2},$$

which tends to $\pi^2/4$, as $r \to \infty$. Together with Theorem 1.1, this shows that if the optimal Minkowski's inequality in a Cartan-Hadamard 3-space, with curvature $K \leq$

$a \leq 0$, is of the form

$$\mathcal{M}(\Gamma) \geq \sqrt{16\pi|\Gamma| - \lambda a|\Gamma|^2},$$

then $2 \leq \lambda \leq \pi^2/4 \simeq 2.47$. Hence the constant 2 in (2) is within 80% of the largest value which it might possibly have.

2. Smooth Strictly Convex Surfaces

Here we prove Theorem 1.1 via harmonic mean curvature flow. A *geometric flow* of a hypersurface Γ in a Riemannian n-manifold M [1, 18, 25] is a one parameter family of immersions $X\colon \Gamma \times [0,T) \to M$, $X_t(\cdot) := X(\cdot, t)$, given by

$$X_t'(p) = -F_t(p)\nu_t(p), \qquad X_0(p) = p, \tag{5}$$

where $(\cdot)' := \partial/\partial t(\cdot)$, ν_t is a normal vector field along $\Gamma_t := X_t(\Gamma)$, and the *speed function* F_t depends on *principal curvatures* or eigenvalues κ_i^t of the second fundamental form $\mathrm{I\!I}_t := \mathrm{I\!I}_{\Gamma_t}$. More precisely, $\nu_t(p)$ is the normal and $\kappa_i^t(p)$ are the principal curvatures of Γ_t at the point $X_t(p)$. When F_t is the harmonic mean of κ_i^t, i.e.,

$$F_t = \left(\sum \frac{1}{\kappa_i^t}\right)^{-1},$$

X is called the *harmonic mean curvature flow* of Γ. In particular when $n = 3$,

$$F_t = \frac{G_t}{H_t},$$

where $G_t := \det(\mathrm{I\!I}_t)$ and $H_t := \mathrm{trace}(\mathrm{I\!I}_t)$ are the *Gauss-Kronecker curvature* and *mean curvature* of Γ_t respectively. Xu showed that [23; 47, Thm. 1.2] when Γ is a smooth strictly convex hypersurface in a Cartan-Hadamard manifold M and F_t is the harmonic mean curvature, X exists for $t \in [0,T)$, is $\mathcal{C}^\infty$, and Γ_t are strictly convex hypersurfaces converging to a point as $t \to T$. This is the only geometric flow known to preserve the convexity of a hypersurface in M while contracting it to a point.

Proof of Theorem 1.1. Let Γ_t, $t \in [0,T)$, be the surfaces generated by the harmonic mean curvature flow of Γ, converging to a point o in M. Set $\mathcal{M}_t := \mathcal{M}(\Gamma_t)$, and

$$\phi(t) := \mathcal{M}_t^2 - 16\pi|\Gamma_t| + 2a|\Gamma_t|^2.$$

We need to show that $\phi(0) \geq 0$. To this end, we compute ϕ' as follows. It is well-known that [25, Thm. 3.2(v)] for any geometric flow

$$(H_t)' = \Delta_t F_t + \left(|\mathrm{I\!I}_t|^2 + \mathrm{Ric}(\nu_t)\right) F_t,$$

where $|\mathrm{I\!I}_t| := \sqrt{\sum(\kappa_i^t)^2}$, Δ_t is the Laplace-Beltrami operator induced on Γ by X_t, and $\mathrm{Ric}(\nu_t)$ is the Ricci curvature of M at the point $X_t(p)$ in the direction of $\nu_t(p)$, i.e., the sum of sectional curvatures of M with respect to a pair of orthogonal planes which contain $\nu_t(p)$. Let $d\mu_t$ be the area element induced on Γ by X_t. By [25, Lem. 7.4],

$$(d\mu_t)' = -F_t H_t d\mu_t = -G_t d\mu_t.$$

Using the above formulas, we compute that

$$\begin{aligned} \mathcal{M}_t' &= \int_\Gamma \big((H_t)' d\mu_t + H_t (d\mu_t)'\big) \\ &= \int_\Gamma \Big(\Delta_t F_t + \big(|\mathrm{I\!I}_t|^2 - (H_t)^2\big) F_t + \mathrm{Ric}(\nu_t) F_t\Big) d\mu_t \\ &\leq -2\int_\Gamma (G_t - a)\frac{G_t}{H_t} d\mu_t \\ &\leq -2\int_\Gamma \frac{(G_t)^2}{H_t}\, d\mu_t. \end{aligned} \tag{6}$$

So, by Cauchy-Schwarz inequality,

$$\mathcal{M}_t \mathcal{M}_t' \leq -2\mathcal{M}_t \int_\Gamma \frac{(G_t)^2}{H_t} d\mu_t \leq -2\mathcal{G}_t^2, \tag{7}$$

where $\mathcal{G}_t = \mathcal{G}(\Gamma_t) := \int_\Gamma G_t d\mu_t$ is the *total Gauss-Kronecker curvature* of Γ_t. Let H be the function on $\Omega \setminus \{o\}$ given by $H(X_t(p)) := H_t(p)$. Also define u on $\Omega \setminus \{o\}$ by $u(X_t(p)) = t$, which yields that $|\nabla u(X_t)| = 1/F_t$. Then $H = \mathrm{div}(\nabla u/|\nabla u|)$, and Stokes' theorem together with the coarea formula yields that

$$|\Gamma_t| - |\Gamma_{t+h}| = \int_{\Omega_t \setminus \Omega_{t+h}} H = \int_t^{t+h} \left(\int_\Gamma H_s F_s\, d\mu_s\right) ds = \int_t^{t+h} \mathcal{G}_s ds,$$

where Ω_t is the convex domain bounded by Γ_t. Thus

$$|\Gamma_t|' = -\mathcal{G}_t.$$

It follows that

$$\phi'(t) = 2\mathcal{M}_t \mathcal{M}_t' - 16\pi|\Gamma_t|' + 4a|\Gamma_t||\Gamma_t|' \leq -4\mathcal{G}_t\Big(\mathcal{G}_t - 4\pi + a|\Gamma_t|\Big) \leq 0, \tag{8}$$

where the last inequality is due to Gauss' equation and Gauss-Bonnet theorem. Indeed, by Gauss' equation, for all $p \in \Gamma_t$,

$$G_t(p) = K_{\Gamma_t}(p) - K_M(T_p\Gamma_t), \tag{9}$$

where K_{Γ_t} is the sectional curvature of Γ_t, and $K_M(T_p\Gamma_t)$ is the sectional curvature of M with respect to the tangent plane $T_p\Gamma_t \subset T_pM$. So, by Gauss-Bonnet theorem,

$$\mathcal{G}_t = 4\pi - \int_{p\in\Gamma_t} K_M(T_p\Gamma_t) \geq 4\pi - a|\Gamma_t|. \tag{10}$$

Hence $\phi' \leq 0$ as claimed. But since Γ_t is convex and collapses to a point, $|\Gamma_t| \to 0$, which yields that

$$\lim_{t\to T} \phi(t) = \lim_{t\to T} \mathcal{M}_t^2 \geq 0.$$

Thus $\phi(0) \geq 0$, which yields the desired inequality (2).

If equality holds in (2), then $\phi(0) = 0$ which yields $\phi(t) \equiv 0$, since $\phi(t) \geq 0$ and $\phi'(t) \leq 0$. Then $\phi'(t) \equiv 0$. So equalities hold in (8), which yields $\mathcal{M}_t\mathcal{M}_t' = -2\mathcal{G}_t^2$. Consequently, the inequalities in (7) become equalities. This forces $G_t/H_t = \lambda(t)$, by the equality case in Cauchy-Schwarz inequality. So Γ_t are parallel to Γ, which means that all points of Γ have constant distance from o. Hence Γ is a (geodesic) sphere. Finally, equalities in (7) force equalities in (6). This forces $\mathrm{Ric}(\nu_t) \equiv 0$, which in turn yields that the sectional curvatures with respect to planes containing ν_t must vanish, since they are nonpositive. Consequently all sectional curvatures of M vanish in the (geodesic) ball bounded by Γ, by [15, Lem. 5.4], which completes the proof. □

Note 2.1. Andrews and Wei [3] showed that harmonic mean curvature flow preserves h-convexity of hypersurfaces in hyperbolic space. If this property holds in any Cartan-Hadamard space, then the proof of Theorem 1.1 may be refined to establish in that space the stronger inequality

$$\mathcal{M}(\Gamma) \geq \sqrt{16\pi|\Gamma| - \frac{7}{2}\,a|\Gamma|^2}, \tag{11}$$

when Γ is h-convex and $a < 0$. To establish this claim, we rescale the metric of M so that $a = -1$, for convenience. Then, similar to the proof of Theorem 1.1, we set

$$\phi(t) := \mathcal{M}_t^2 - 16\pi|\Gamma_t| - \frac{7}{2}|\Gamma_t|^2,$$

and compute that

$$\phi'(t) = 2\mathcal{M}_t\mathcal{M}_t' + \big(16\pi + 7|\Gamma_t|\big)\mathcal{G}_t.$$

Next recall that by (6)

$$\begin{aligned}
\mathcal{M}_t\mathcal{M}_t' &\leq -2\int H_t \int \frac{(G_t)^2 + G_t}{H_t} \\
&= -2\int H_t \int \frac{(G_t + 1/2)^2 - 1/4}{H_t} \\
&\leq -2\left(\int \left(G_t + \frac{1}{2}\right)\right)^2 + \frac{1}{2}\int H_t \int \frac{1}{H_t},
\end{aligned}$$

where all integrals take place over Γ with respect to $d\mu_t$. If Γ_t is h-convex, then its principal curvatures are ≥ 1. Indeed the principal curvatures of horospheres in M are bounded below by 1, since principal curvatures of spheres of radius ρ in M are $\geq \coth(\rho)$ [26, p. 184]. It follows that

$$2 \leq H_t \leq 2G_t, \tag{12}$$

which in turn yields

$$\int H_t \int \frac{1}{H_t} \leq \int 2G_t \int \frac{1}{2} = |\Gamma_t|\mathcal{G}_t. \tag{13}$$

Furthermore, since by (10) $\mathcal{G}_t \geq 4\pi + |\Gamma|$,

$$\left(\int \left(G_t + \frac{1}{2}\right)\right)^2 = \mathcal{G}_t^2 + \mathcal{G}_t|\Gamma_t| + \frac{1}{4}|\Gamma_t|^2 \geq \big(4\pi + 2|\Gamma_t|\big)\mathcal{G}_t.$$

So we have

$$\mathcal{M}_t\mathcal{M}_t' \leq -2\big(4\pi + 2|\Gamma_t|\big)\mathcal{G}_t + \frac{1}{2}|\Gamma_t|\mathcal{G}_t \leq -\left(8\pi + \frac{7}{2}|\Gamma_t|\right)\mathcal{G}_t,$$

which in turn yields

$$\phi'(t) \leq -2\left(8\pi + \frac{7}{2}|\Gamma_t|\right) + \big(16\pi + 7|\Gamma_t|\big)\mathcal{G}_t = 0.$$

Hence, since $\lim_{t\to T}\phi(t) \geq 0$, it follows that $\phi(0) \geq 0$, which establishes the desired inequality (11). Note that (12) was the only place where h-convexity was used in the above argument, which was for the sole purpose of establishing (13). Thus (11) holds whenever there exists a function $\lambda\colon [0,T) \to \mathbf{R}$ such that $\lambda(t) \leq H_t \leq \lambda(t)G_t$.

3. General Convex Surfaces

Here we employ an approximation argument to extend (2), which we established for smooth strictly convex surfaces in the last section, to all convex surfaces in a Cartan-Hadamard 3-manifold. This involves some basic facts about convex sets and their distance functions in Riemannian geometry which can be found in [15, Sec. 2 & 3] plus recent comparison results for total mean curvature obtained in [16].

A subset of a Cartan-Hadamard manifold M is *convex* if it contains the geodesic segment connecting every pair of its points. A *convex hypersurface* $\Gamma \subset M$ is the boundary of a compact convex set with interior points. Let $d_\Gamma\colon M \to \mathbf{R}$ be the *distance function* of Γ, and Ω be the domain bounded by Γ. The *signed* distance function of Γ is defined by setting $\widehat{d_\Gamma} := d_\Gamma$ on $M \setminus \Omega$ and $\widehat{d_\Gamma} := -d_\Gamma$ on Ω. The level sets

$$\Gamma_t := \big(\widehat{d_\Gamma}\big)^{-1}(t)$$

are called *parallel hypersurfaces* of Γ. Unless noted otherwise, we assume that $t \geq 0$ and call Γ_t the *outer parallel* hypersurface, while Γ_{-t} will be called the *inner parallel* hypersurface of Γ. A fact which will be used frequently below is that Γ_t are $\mathcal{C}^{1,1}$ and convex for $t > 0$ [15, Sec. 2 & 3]. In particular, for $t > 0$, Γ_t is twice differentiable almost everywhere and so its total mean curvature $\mathcal{M}(\Gamma_t)$ is well defined and positive.

Lemma 3.1. *For any $\mathcal{C}^{1,1}$ convex hypersurface Γ in a Cartan-Hadamard manifold, $t \mapsto \mathcal{M}(\Gamma_t)$ is a continuous nondecreasing function for $t \geq 0$.*

Proof. Let Ω_t denote the domain bounded by Γ_t. By [16, (12)], for $0 \leq t_1 \leq t_2$,

$$\mathcal{M}(\Gamma_{t_2}) - \mathcal{M}(\Gamma_{t_1}) = \int_{\Omega_{t_2} \setminus \Omega_{t_1}} \big(2\sigma_2(\kappa) - \mathrm{Ric}(\nabla \widehat{d}_\Gamma)\big), \tag{14}$$

where $\kappa = (\kappa_1, \ldots, \kappa_{n-1})$ refers to the principal curvatures of parallel hypersurfaces of Γ and σ_2 is the second symmetric elementary function. Since Γ is convex, $\sigma_2(\kappa) \geq 0$, and by assumption the Ricci curvature of M is nonpositive. Thus $t \mapsto \mathcal{M}(\Gamma_t)$ is nondecreasing. The above expression also yields the continuity of $t \mapsto \mathcal{M}(\Gamma_t)$, since the integrand depends only on Γ and M. So the integral vanishes as $t_1 \to t_2$, or $t_2 \to t_1$. □

Now for any convex hypersurface Γ in a Cartan-Hadamard manifold, which may not be $\mathcal{C}^{1,1}$, we set

$$\mathcal{M}(\Gamma) := \lim_{t \to 0^+} \mathcal{M}(\Gamma_t).$$

Since $\mathcal{M}(\Gamma_t) \geq 0$, and by Lemma 3.1, $\mathcal{M}(\Gamma_t)$ does not increase as $t \to 0^+$, the above limit exists. Furthermore, continuity of $t \mapsto \mathcal{M}(\Gamma_t)$ ensures that, in case Γ is $\mathcal{C}^{1,1}$, the above definition coincides with the regular definition of $\mathcal{M}(\Gamma)$ as the integral of mean curvature. Now that $\mathcal{M}(\Gamma)$ is well-defined for all convex hypersurface, we may state the main result of this section:

Theorem 3.2. *Minkowski's inequality (2) holds for all convex surfaces Γ in a Cartan-Hadamard 3-manifold M with curvature $K \leq a \leq 0$.*

To establish this theorem we need the following facts:

Lemma 3.3. *Smooth strictly convex hypersurfaces are dense in the space of $\mathcal{C}^k$ convex hypersurfaces of a Cartan-Hadamard manifold with respect to $\mathcal{C}^k$ topology, for $k \geq 0$.*

Proof. Let Γ be a convex hypersurface in a Cartan-Hadamard manifold M, and $u\colon M \to \mathbf{R}$ be the distance function from the domain Ω bounded by Γ. Let x_0 be a point in the interior of Ω, and ρ be the distance function from x_0. Then, for $\varepsilon > 0$, $u^\varepsilon(x) := u(x) + \varepsilon\rho^2(x)$ is a strictly convex function in the sense of Greene and Wu [20]. Consequently, the Greene-Wu convolution u^ε_λ yields a family of smooth strictly convex functions converging to u^ε with respect to $\mathcal{C}^k$ norm over any compact set, as $\lambda \to 0$ [21, Thm. 2 & Lem. 3.3]; see [15, p. 21–22]. In particular, for any given integer $i > 0$, we may choose ε and λ so small that a level set Γ^i of u^ε_λ lies within a neighborhood of Γ of radius $1/i$. Then Γ_i converges to Γ with respect to $\mathcal{C}^k$ topology, which completes the proof. □

We say that a set is *nested inside* a convex hypersurface Γ provided that it lies in the convex domain bounded by Γ. The following monotonicity property is a quick consequence of an analogous result in [16] for $\mathcal{C}^{1,1}$ surfaces:

Lemma 3.4. *Let γ, Γ be a pair of of convex hypersurfaces in a Cartan-Hadamard manifold. Suppose that γ is nested inside Γ. Then $\mathcal{M}(\gamma) \leq \mathcal{M}(\Gamma)$.*

Proof. For every $t > 0$, γ_t and Γ_t are $\mathcal{C}^{1,1}$ convex hypersurfaces, with γ_t nested inside Γ_t. Thus $\mathcal{M}(\gamma_t) \leq \mathcal{M}(\Gamma_t)$ by [16, Cor. 4.1]. Letting $t \to 0$ completes the proof. □

The next observation follows from the fact that the nearest point projection into a convex set is distance nonincreasing in Cartan-Hadamard manifolds [5, Prop. 2.4(4)].

Lemma 3.5. *Let γ, Γ be a pair of convex hypersurfaces in a Cartan-Hadamard manifold, with γ nested inside Γ. Then $|\gamma| \leq |\Gamma|$, with equality only if $\gamma = \Gamma$.*

Now were are ready to establish the main result of this section:

Proof of Theorem 3.2. By Lemma 3.3, there exists a family $\Gamma^i \subset M$ of smooth strictly convex hypersurfaces which converge to Γ with respect to $\mathcal{C}^0$ topology. After replacing each Γ^i by an outer parallel hypersurface, we may assume that $\Gamma^i \subset M \setminus \Omega$, where Ω is the domain bounded by Γ. By Theorem 1.1, $\mathcal{M}(\Gamma^i)$ satisfy (2). Thus it suffices to check that $|\Gamma^i| \to |\Gamma|$ and $\mathcal{M}(\Gamma^i) \to \mathcal{M}(\Gamma)$. For every $\varepsilon > 0$, there exists an integer N such that Γ^i lies in the region bounded by Γ and Γ_ε for $i \geq N$. Thus

$$|\Gamma| \leq |\Gamma^i| \leq |\Gamma_\varepsilon|, \quad \text{and} \quad \mathcal{M}(\Gamma) \leq \mathcal{M}(\Gamma^i) \leq \mathcal{M}(\Gamma_\varepsilon),$$

by Lemmas 3.5 and 3.4. As $\varepsilon \to 0$, $|\Gamma_\varepsilon| \to |\Gamma|$, and by Lemma 3.1, $\mathcal{M}(\Gamma_\varepsilon) \to \mathcal{M}(\Gamma)$ as well, which completes the proof. □

4. Isoperimetric Inequality

Here we prove Theorem 1.2, using the generalized Minkowski's inequality (2) derived in the last section, and a Steiner type formula which we will establish below. To this end we need to define the total Gauss-Kronecker curvature of a general convex hypersurface Γ in a Cartan-Hadamard manifold. Similar to our treatment for total mean curvature in the last section, we set

$$\mathcal{G}(\Gamma) := \lim_{t \to 0^+} \mathcal{G}(\Gamma_t), \tag{15}$$

where recall that Γ_t denote the outer parallel hypersurfaces of Γ. By [16, Cor. 4.4], $\mathcal{G}(\Gamma_t)$ does not increase as $t \to 0^+$. Thus, since $\mathcal{G}(\Gamma_t) \geq 0$, the above limit exists. Let us also record that:

Lemma 4.1. *The total Gauss-Kronecker curvature $\mathcal{G}(\Gamma)$ is continuous in the space of $\mathcal{C}^{1,1}$ convex surfaces Γ in a Cartan-Hadamard 3-manifold M with respect to $\mathcal{C}^1$ topology.*

Proof. Let Γ^i be a sequence of $\mathcal{C}^{1,1}$ convex surfaces in M converging to Γ with respect to $\mathcal{C}^1$ topology. Then Gauss' equation (9) together with Gauss-Bonnet theorem yields

$$\mathcal{G}(\Gamma^i) = 4\pi - \int_{p\in\Gamma^i} K_M(T_p\Gamma^i) \quad \longrightarrow \quad 4\pi - \int_{p\in\Gamma} K_M(T_p\Gamma) = \mathcal{G}(\Gamma),$$

as desired. $\square$

Now we can establish the following Steiner type formula for general convex surfaces, using tube formulas of Gray [19] together with an approximation argument.

Lemma 4.2. *Let Γ be a convex surface in a Cartan-Hadamard 3-manifold. Then, for any $t \geq 0$,*

$$|\Gamma_t| \geq |\Gamma| + \mathcal{M}(\Gamma)t + \mathcal{G}(\Gamma)t^2. \tag{16}$$

Proof. If Γ is smooth, then (16) holds by Steiner's formula in spaces of nonpositive curvature [19, Thm. 10.31(ii)]. The general case follows by approximation. If for every $\varepsilon > 0$ we can show

$$|\Gamma_{\varepsilon+t}| \geq |\Gamma_\varepsilon| + \mathcal{M}(\Gamma_\varepsilon)t + \mathcal{G}(\Gamma_\varepsilon)t^2,$$

then (16) follows by letting $\varepsilon \to 0$. Hence we may assume that Γ is $\mathcal{C}^{1,1}$, after replacing Γ with Γ_ε. Then, by Lemma 3.3, there exists a family of smooth convex surfaces $\Gamma^i \subset M$ such that $\Gamma^i \to \Gamma$ with respect to $\mathcal{C}^1$ topology. As described in the proof of Theorem 3.2, we may assume that Γ^i lie outside the domain bounded by Γ, which yields $\mathcal{M}(\Gamma^i) \to \mathcal{M}(\Gamma)$ via Lemma 3.4. Furthermore, $\mathcal{G}(\Gamma^i) \to \mathcal{G}(\Gamma)$ as well, by Lemma 4.1. Finally $|\Gamma^i| \to |\Gamma|$ and $|(\Gamma^i)_t| \to |\Gamma_t|$ by Lemma 3.5, as shown in the proof of Theorem 3.2. Thus, as Γ^i satisfy (16), Γ does as well. $\square$

The next fact is well-known when Γ is smooth. For the sake of completeness, we quickly check that it holds under minimal regularity:

Lemma 4.3. *Let Γ be a closed oriented $\mathcal{C}^{1,1}$ hypersurface embedded in a Riemannian manifold M, and Γ_t be the parallel hypersurfaces of Γ for $-\varepsilon < t < \varepsilon$. Then*

$$\mathcal{M}(\Gamma) = \frac{d}{dt}\Big|_{t=0} |\Gamma_t|.$$

Proof. Since Γ is $\mathcal{C}^{1,1}$, the signed distance function $\widehat{d}_\Gamma$ of Γ is $\mathcal{C}^{1,1}$ on an open neighborhood U of Γ in M [15, Lem. 2.6]. Thus $H = \operatorname{div}(\nabla \widehat{d}_\Gamma)$ almost everywhere on U, where

H is the mean curvature of parallel hypersurfaces of Γ in U. Consequently, by Stokes' theorem and the coarea formula, for $t \geq 0$,

$$|\Gamma_t| - |\Gamma| = \int_{\Lambda_t} \operatorname{div}(\nabla \widehat{d}_\Gamma) = \int_{\Lambda_t} H = \int_0^t \mathcal{M}(\Gamma_s) ds,$$

where Λ_t is the domain bounded between Γ and Γ_t. Furthermore, by (14), $\mathcal{M}(\Gamma_s)$ is continuous for $0 \leq s \leq t$ (formula (14) holds for all pairs of parallel closed $\mathcal{C}^{1,1}$ hyeprsurfaces in Riemannian manifolds [16, Thm. 3.1]). Thus, by the mean value theorem for integrals, the right derivative of $|\Gamma_t|$ at $t = 0$ is equal to $\mathcal{M}(\Gamma)$. Similarly, the left derivative at $t = 0$ is equal to $\mathcal{M}(\Gamma)$, which completes the proof. □

The last observation we need follows quickly from Lemma 3.5 and the fact that the exponential map is distance nonreducing in Cartan-Hadamard manifolds:

Lemma 4.4. *Let Γ be a convex hypersurface in a Cartan-Hadamard n-manifold bounding a domain Ω with inradius r, and S be a sphere in $\mathbf{R}^n$ with radius R. Suppose that $|\Gamma| = |S|$. Then $r \leq R$, with equality only if Ω is isometric to a ball in $\mathbf{R}^n$.*

Now we are ready to establish the main result of this section:

Proof of Theorem 1.2. Let $S \subset \mathbf{R}^3$ be a sphere with $|S| = |\Gamma|$ and radius R. By Lemma 4.4, $R \geq r := \operatorname{inrad}(\Omega)$. Let Γ_{-t}, S_{-t} denote respectively the inner parallel surfaces of Γ, S at distance $t \in (0, r)$, as defined in the last section. It is enough to show that $|\Gamma_{-t}| \leq |S_{-t}|$, for then by the coarea formula

$$|\Omega| = \int_0^r |\Gamma_{-t}| dt \leq \int_0^r |S_{-t}| dt = |\Lambda|, \tag{17}$$

where Λ is the annular region between S and S_{-r}. The application of the coarea formula here is warranted since the distance function $\widehat{d}_\Gamma$ is Lipschitz and $|\nabla \widehat{d}_\Gamma| = 1$ almost everywhere [15, Sec. 2]. Furthermore $|\Lambda|$ is the desired upper bound, since $R = \sqrt{|\Gamma|/(4\pi)}$. Now suppose, towards a contradiction, that

$$|\Gamma_{-t_0}| > |S_{-t_0}|$$

for some $t_0 \in (0, r)$. Since Γ is d-convex, Γ_{-t} are convex. So, by Lemmas 3.5 and 4.2,

$$|\Gamma| \geq |(\Gamma_{-t_0})_{t_0}| \geq |\Gamma_{-t_0}| + \mathcal{M}(\Gamma_{-t_0}) t_0 + \mathcal{G}(\Gamma_{-t_0}) t_0^2.$$

There exists $s_0 > 0$ such that $|\Gamma_{-t_0}| = |S_{-t_0+s_0}|$. By Theorem 3.2,

$$\mathcal{M}(\Gamma_{-t_0}) \geq \mathcal{M}(S_{-t_0+s_0}) > \mathcal{M}(S_{-t_0}).$$

Furthermore, recall that by Gauss' equation and Gauss-Bonnet theorem (10), $\mathcal{G}((\Gamma_{-t_0})_s) \geq 4\pi = \mathcal{G}(S_{-t_0})$, for all $s > 0$. So

$$\mathcal{G}(\Gamma_{-t_0}) \geq \mathcal{G}(S_{-t_0}),$$

by definition (15). Hence we obtain

$$|\Gamma| > |S_{-t_0}| + \mathcal{M}(S_{-t_0})t_0 + \mathcal{G}(S_{-t_0})t_0^2 = |(S_{-t_0})_{t_0}| = |S| = |\Gamma|,$$

which is the desired contradiction. Finally, suppose that equality holds in (4). Then equality holds in (17). Consequently $|\Gamma_{-t}| = |S_{-t}|$, since we just showed that $|\Gamma_{-t}| \leq |S_{-t}|$. It follows, via Lemma 4.3, that

$$\mathcal{M}(\Gamma) = \frac{d}{dt}|\Gamma_t|\Big|_{t=0} = -\frac{d}{dt}|\Gamma_{-t}|\Big|_{t=0} = -\frac{d}{dt}|S_{-t}|\Big|_{t=0} = \mathcal{M}(S).$$

So, by Theorem 1.1, Γ must bound a Euclidean ball. □

5. Higher Dimensions

Let Γ be a convex hypersurface in a Cartan-Hadamard n-manifold M bounding a domain Ω, and S be a sphere in $\mathbf{R}^n$ with $|S| = |\Gamma|$ bounding a ball B. The analogue of Minkowski inequality (1) in M is that

$$\mathcal{M}(\Gamma) \geq \mathcal{M}(S), \tag{18}$$

with equality only if Ω is isometric to B. The analogue of the isoperimetric inequality (4) is that, if Γ is d-convex, $r := \mathrm{inrad}(\Omega)$, and R is the radius of B, then

$$|\Omega| \leq |B \setminus B_{R-r}|, \tag{19}$$

where B_ρ stands for a ball of radius ρ in $\mathbf{R}^n$ with the same center as B, and equality holds only if Ω is isometric to B. By Lemma 4.4, $r \leq R$, thus B_{R-r} is well-defined.

Theorem 5.1. *Let M be a Cartan-Hadamard n-manifold. Suppose that the Minkowski type inequality (18) holds for all $\mathcal{C}^{1,1}$ convex hypersurfaces $\Gamma \subset M$ with equality only if the domain Ω bounded by Γ is isometric to a ball in $\mathbf{R}^n$. Then the isoperimetric inequality (19) also holds in M for all domains Ω with $\mathcal{C}^{1,1}$ d-convex boundary Γ, and equality holds only if Ω is isometric to a ball in $\mathbf{R}^n$.*

The proof of the above theorem uses the notion of reach in the sense of Federer [12,43], see [15, Sec. 2]. The *reach* of a convex hypersurface $\Gamma \subset M$, bounding a domain Ω, is the supremum value of ρ such that through each point of Γ there passes a ball of radius ρ contained in Ω. It is well-known that $\mathrm{reach}(\Gamma) > 0$ if and only if Γ is $\mathcal{C}^{1,1}$ [15, Lem. 2.6]. Lemma 3.5 quickly yields:

Lemma 5.2. *Let Γ be a d-convex hypersurface in a Cartan-Hadamard manifold bounding a domain Ω, and $t \in [0, \mathrm{inrad}(\Omega))$. Then $|(\Gamma_{-t})_t| \leq |\Gamma|$ with equality if and only if $t \leq \mathrm{reach}(\Gamma)$.*

Now we are ready to establish the main result of this section:

Proof of Theorem 5.1. If Ω is isometric to B, then equality holds in (19) and there is nothing left to prove. So suppose that Ω is not isometric to B. Then, by the rigidity assumption for Minkowski's inequality (18), $\mathcal{M}(\Gamma) > \mathcal{M}(S)$. Consequently, since Γ is $\mathcal{C}^{1,1}$, there exists $\varepsilon > 0$ such that

$$|\Gamma_{-t}| < |S_{-t}|, \qquad \text{for} \qquad t \in (0, \varepsilon], \tag{20}$$

by Lemma 4.3. Furthermore note that, by Lemma 4.4,

$$r < R. \tag{21}$$

So S_{-t} is well-defined for $t \in (0, r)$. If $|\Gamma_{-t}| \leq |S_{-t}|$ for all $t \in [\varepsilon, r)$, then by the coarea formula

$$|\Omega| = \int_0^r |\Gamma_{-t}| \, dt < \int_0^r |S_{-t}| \, dt = |B \setminus B_{R-r}|,$$

and we are done. So suppose, towards a contradiction, that $|\Gamma_{-t_0}| > |S_{-t_0}|$, for some $t_0 \in [\varepsilon, r)$. Let

$$\overline{s} := \sup \left\{ s \leq t_0 \;\middle|\; (\Gamma_{-t_0})_t| \geq |S_{-t_0+t}| \;\text{ for all }\; t \in [0, s] \right\}.$$

Then $\overline{s} > 0$, and

$$|(\Gamma_{-t_0})_{\overline{s}}| = |S_{-t_0+\overline{s}}|. \tag{22}$$

Note that $(\Gamma_{-t_0})_{\overline{s}}$ cannot bound a Euclidean ball, for otherwise the radius of $(\Gamma_{-t_0})_{\overline{s}}$ would be equal to that of $S_{-t_0+\overline{s}}$, i.e., $r - t_0 + s = R - t_0 + s$, or $r = R$, which would violate (21). Consequently, by the rigidity assumption for (18),

$$\mathcal{M}(\Gamma_{-t_0})_{\overline{s}} > \mathcal{M}(S_{-t_0+\overline{s}}). \tag{23}$$

But since $\overline{s} > 0$, $(\Gamma_{-t_0})_{\overline{s}}$ is $\mathcal{C}^{1,1}$. So (22) and (23) yield that $|(\Gamma_{-t_0})_{\overline{s}+\delta}| > |S_{-t_0+\overline{s}+\delta}|$ for δ small, via Lemma 4.3. Hence

$$\overline{s} = t_0,$$

by the definition of $\overline{s}$. There are now two possibilities: either $t_0 > \mathrm{reach}(\Gamma)$, or $t_0 \leq \mathrm{reach}(\Gamma)$. If $t_0 > \mathrm{reach}(\Gamma)$, then by Lemma 5.2,

$$|(\Gamma_{t_0})_{\overline{s}}| < |\Gamma_{-t_0+\overline{s}}| = |\Gamma| = |S| = |S_{-t_0+\overline{s}}|,$$

which is not possible by (22). If, on the other hand, $t_0 \leq \mathrm{reach}(\Gamma)$, then, again by Lemma 5.2 and the definition of $\overline{s}$,

$$|\Gamma_{-t_0+t}| = |(\Gamma_{-t_0})_t| \geq |S_{-t_0+t}|,$$

for all $t \in [0, t_0]$, which violates (20), for t close to t_0. So we arrive at the desired contradiction. $\square$

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