Geometric Inequalities in Spaces of nonpositive curvature

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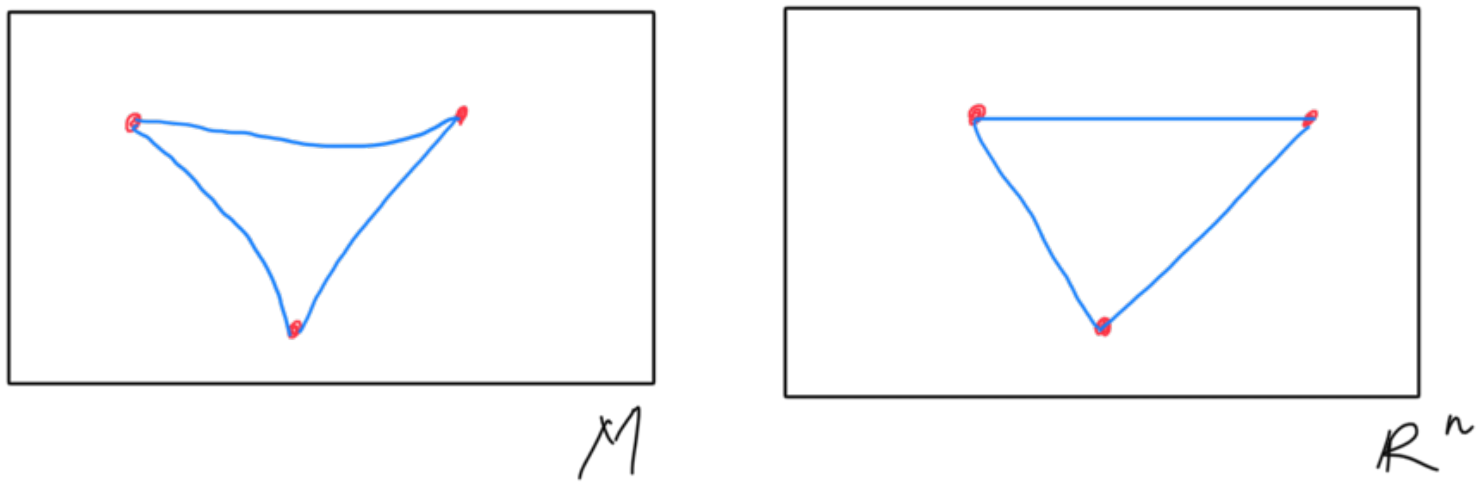
A Cartan–Hadamard space M is a complete simply connected Riem. mfld with nonpositive curvature.

Example: $\mathbb{R}^n$ & $\mathbb{H}^n$

Basic properties:

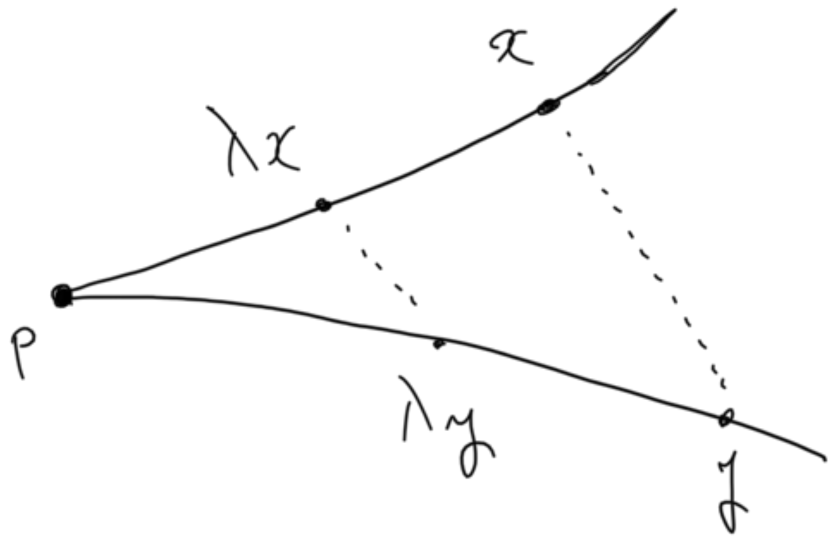
* M is diffeo. to $\mathbb{R}^n$
* Every pair of points can be joined by a unique geodesic
* CH-mflds are CAT(0) spaces.

geodesic metric spaces where tringles are "thinner" than those in $\mathbb{R}^n$.



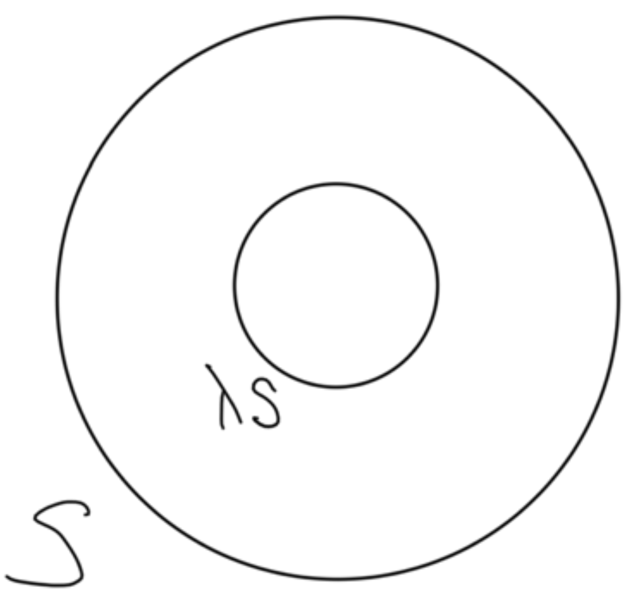
Another way to think of nonpositive curvature:

- Exp. map is <u>expansive</u>:



$$\boxed{d(\lambda x, \lambda y) \leq \lambda d(x,y)}$$

S,



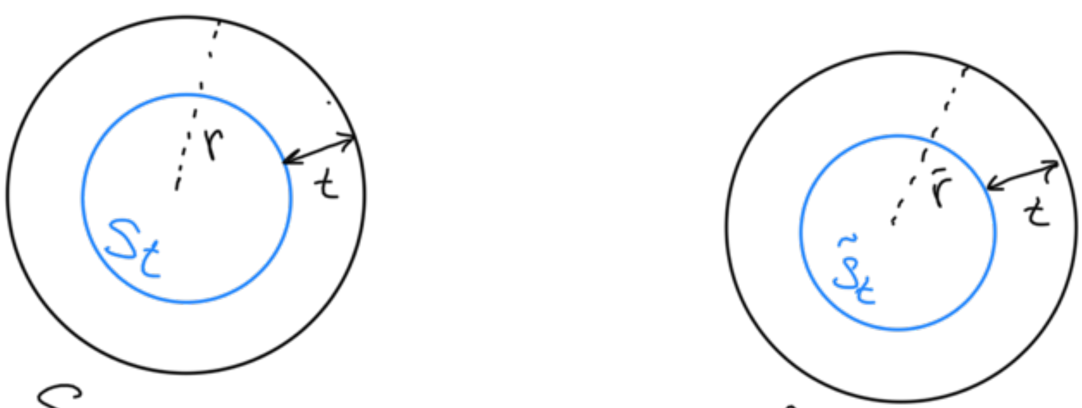
$$\boxed{|\lambda S| \leq \lambda^{n-1} |S|}$$

Balls in CH-mflds satisfy the Euclidean isop. inequality:

$S \subset M$ $\qquad$ $\tilde{S} \subset \mathbb{R}^n$

$$|S| = |\tilde{S}| \implies |B| \leq |\tilde{B}|$$

Proof



$S \subset M$ $\qquad$ $\tilde{S} \subset \mathbb{R}^n$

$$r \leq \tilde{r}$$

$$\Rightarrow \quad |S_t| = \left|\frac{r-t}{r} S\right|$$

$$\leq \left(\frac{r-t}{r}\right)^{n-1} |S|$$

$$\leq \left(\frac{\tilde{r}-t}{\tilde{r}}\right)^{n-1} |\tilde{S}|$$

$$= \left|\frac{\tilde{r}-t}{r} \tilde{S}\right|$$

$$= |\tilde{S}_t|$$

$$\Rightarrow \quad |B| = \int_0^r |S_t|\,dt \leq \int_0^r |\tilde{S}_t|\,dt \leq |\tilde{B}|$$

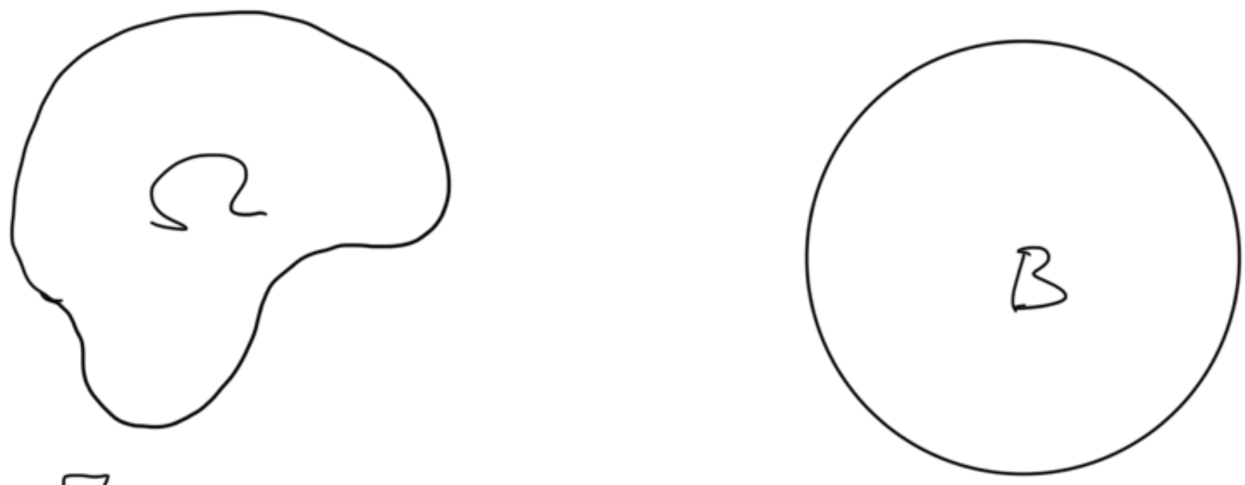
coarea formula

$\square$

Exercise: Why would not the above proof work for convex bodies?

CH-Conjecture (Aubin, Gromov, Burago-Zalgaller)

The Euclidean isop. ineq. holds in CH-mflds:



$\Gamma \subset M$ $\qquad\qquad$ $S \subset \mathbb{R}^n$

$$\boxed{|\Gamma| = |S| \quad \Rightarrow \quad |\Omega| \leq |B|}$$

CH- Conjecture

Known for:

- H^n (e.g. Steiner symmetrization)
- $n=2$ (Weil), $n=3$ (Kleiner), $n=4$ (Croke)
- Large volumes, when $K<0$ (Yau, Burago-Zalgaller)
- Small volumes (Johnson-Morgan)
- For every n there is a const C_n s.t.

$$\frac{|\Omega|^{n-1}}{|\Gamma|^n} \leq C_n$$

(Spruck-Hoffman)

Equivalent to Sobolev Ineq. (By the coarea formula)

$$\boxed{\left(\int_M f \ \right) \le \frac{1}{n|B^n|}\int_M |Df|}$$

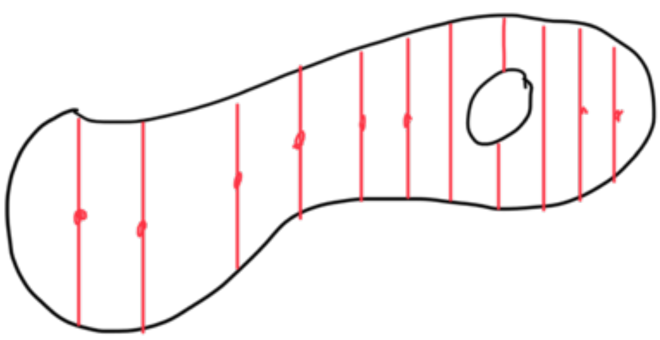
Open even in the convex case

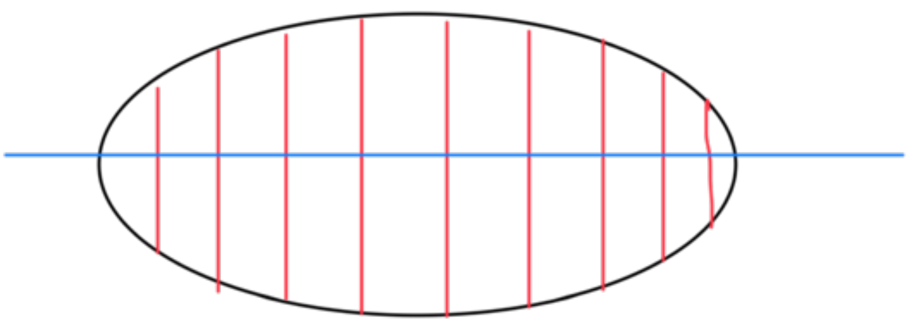
- Convexity is not so convenient in the absence of linear structure.

A couple of Proofs of the Classical Isop. Ineq. in $\mathbb{R}^n$

Steiner Symmetrization

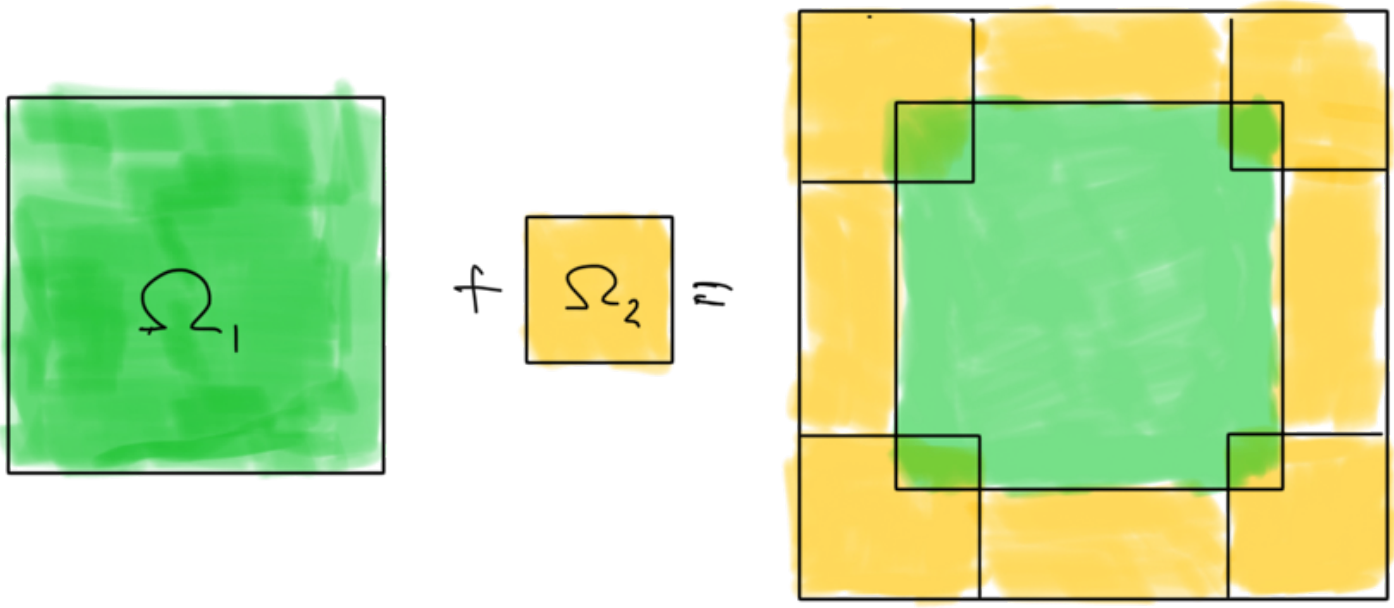
(requires existence of a minizer)





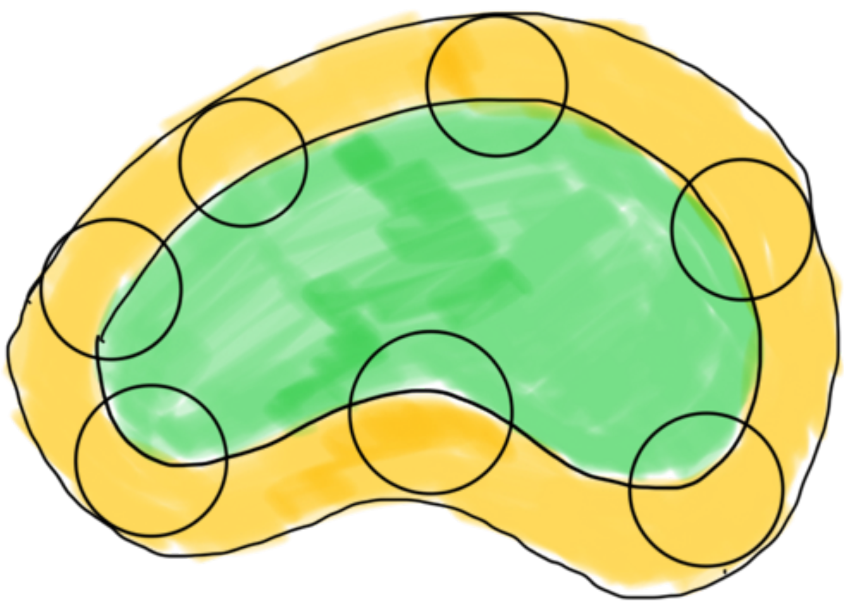
Brunn-Minkowski Ineq.

$$|\Omega_1+\Omega_2|^{\frac{1}{n}} \geq |\Omega_1|^{\frac{1}{n}} + |\Omega_2|^{\frac{1}{n}}$$



Proof is immediate for rectangles.
Follows for all regions by approximation.

$$\Omega_r := \Omega + r\underset{\text{unit ball}}{B} = \{x\in\mathbb{R}^n \mid \operatorname{dist}(\Omega,x)\leq r\}$$



$$\begin{aligned}
|\Omega_r| &= |\Omega + rB| \\
&\geq \left(|\Omega|^{1/n} + r|B|^{\frac{1}{n}}\right)^n \\
&\geq |\Omega| + nr|\Omega|^{\frac{n-1}{n}}|B|^{\frac{1}{n}}
\end{aligned}$$

$$\begin{aligned}
|\Gamma| &= \lim_{r\to 0} \frac{|\Omega_r|-|\Omega|}{r} \\
&\geq n|\Omega|^{\frac{n-1}{n}}|B|^{\frac{1}{n}}
\end{aligned}$$

Convexity in CH-mflds

- Similarities with $\mathbb{R}^n$:

  * distance function from a convex set is convex

    (a function $f: M \to \mathbb{R}$ is convex if its restriction to geodesics is convex)

  * Convex sets have dimension

    (their relative interior is a totally geodesic submfld, Cheeger-Ebin)

- Differences from $\mathbb{R}^n$

  * convex hull of 3 points may have interior!

* Signed distance function from the

boundary of a convex body might not be convex in the interior!

* Boundary of the convex hull of a set may not contain any geodesic segment!

Lytchak & Petrunin

(no Caratheodory thm)

(no simplices)

* Equivalent <u>notions of convexity</u> in $\mathbb{R}^n$ <u>diverge</u> in CH-mflds.

$$\boxed{h\text{-convex} \subsetneq d\text{-convex} \subsetneq \text{convex}}$$

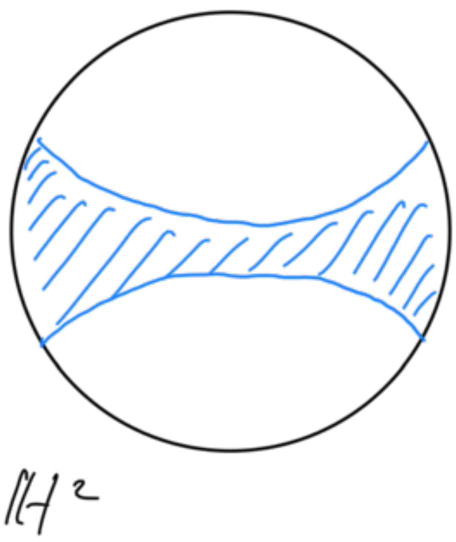
through each boundary point there passes a horosphere

dist function from boundary is convex inside the set.

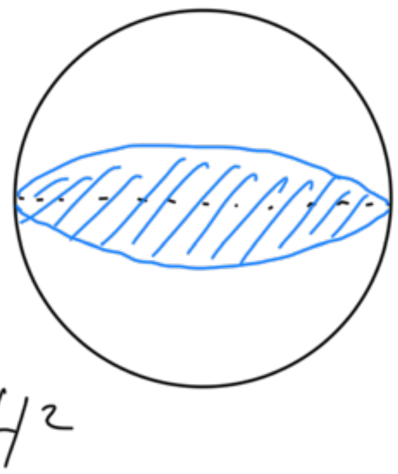
geod. between points are contained in the set.

★ <u>More strange things</u> will be mentioned later

<u>Example</u>



convex but not d-convex



tube about a geodesic: d-convex, but not h-convex.

<u>So how are we going to study the isop. problem in CH-mflds?</u>

- We can adopt a variational approach, in the sense of <u>Steiner</u>, which leads to integrals of generalized mean curvature (called quermassintegrals or mixed volumes in $\mathbb{R}^n$).

Then we can study to what extent various Alexander-Fenchel type inequalities hold.

Let us begin by reviewing Steiner Polynomial, Mean curvature integrals & Alexandrov-Fenchel Ineq in $\mathbb{R}^n$.

Steiner's Polynomial

How mean curvatures enter the picture

- If $\Gamma \subset \mathbb{R}^n$ is a smooth closed embedded hypersurface

$$\Gamma_t := \Gamma + t\nu$$

outward normal

is well-defined for small t

$$\boxed{|\Gamma_t| = |\Gamma| + C_1 M_1(\Gamma)t + \cdots + C_{n-1} M_{n-1}(\Gamma) t^{n-1}}$$

Steiner formula

total (first) mean curvature

total Gauss-Kronecker curvature

M_r : the rth mean curvature

C_r : constants depending only on n

$$\boxed{M_r(\Gamma) := \int_\Gamma \sigma_r(K)}$$

$$K = (K_1, \dots, K_{n-1})$$

principal curvatures of Γ

$$\sigma_r(K) = \sum_{i_1 < \cdots < i_r} K_{i_1} \cdots K_{i_r}$$

symmetric functions

elementary

$$J_0 := 1 \quad (\text{by convention})$$

So $\boxed{\mathcal{M}_0 = |\Gamma|}$

Proof [Steiner's polynomial]

$$f : \Gamma \longrightarrow \Gamma_t$$

$$f(x) := x + t\nu$$

$$df = I + t\, d\nu$$

the "shape operator"

κ_i are the eigen values of $d\nu$

$$|\Gamma_t| = \int_\Gamma \det(df)$$

$$= \int_\Gamma \det(I + t\, d\nu)$$

characteristic polynomial
of the shape operator

$\square$

Crofton's formula for total mean curv.

If Γ is convex, $\mathcal{M}_r(\Gamma)$ have a more geometric description:

$$\boxed{\mathcal{M}_r(\Gamma) := C_{n,r} \int_{V \in \mathrm{Grass}(n-1-r,\, n)} |\Pi_V(\Gamma)|}$$

Average size of projections of Γ into $(n-1-r)$-dim subspaces.

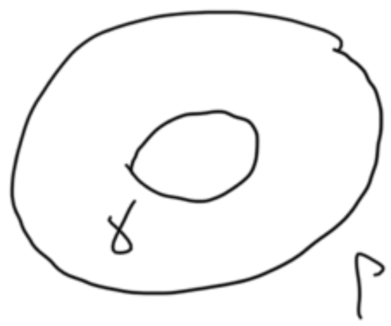
Ex: $\mathcal{M}_{n-2}$ is the <u>mean width</u>.

$$\boxed{\mathcal{M}_{-1}(\Gamma) := |\Omega|}$$

So if γ is nested inside Γ, then

$$\boxed{\mathcal{M}_r(\gamma) \leq \mathcal{M}_r(\Gamma)}$$

Monotonicity formula



Alexandrov-Fenchel Inequalities

$$\boxed{\frac{\mathcal{M}_k(\Gamma)^{n-k}}{\mathcal{M}_{k-1}(\Gamma)^{n-k-1}} \geq C_{n,k}\,\frac{\mathcal{M}_k(S^{n-1})^{n-k}}{\mathcal{M}_{k-1}(S^{n-1})^{n-k-1}}}$$

Examples

* $k=0$ (Isop. Ineq.)

$$\frac{|\Gamma|^n}{|\Omega|^{n-1}} \geq \frac{|S^{n-1}|^n}{(B^n)^{n-1}}$$

* $k=1$ (Minkowski's Ineq.)

$$\frac{M_1(\Gamma)^{n-1}}{|\Gamma|^{n-2}} \geq \frac{M_1(S^{n-1})^{n-1}}{|S^{n-1}|^{n-2}}$$

among convex hypersurfaces with the same area sphere (& only sphere) minimizes total (1st) mean curvature

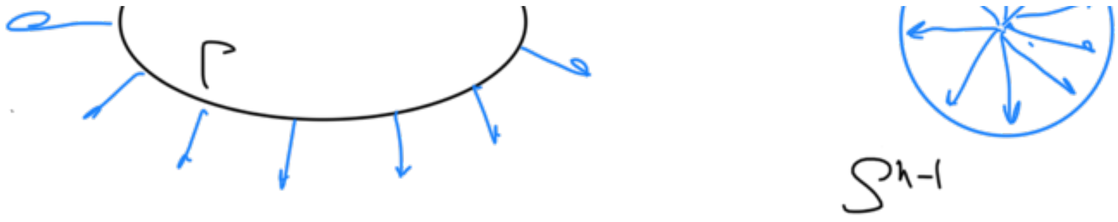
In particular, for $n=3$

$$\boxed{M_1(\Gamma) \geq \sqrt{16\pi|\Gamma|}}$$

* $k=n-1$ (Gauss Kronecker curv. Ineq.)

$$\boxed{M_{n-1}(\Gamma) \geq |S^{n-1}|}$$

the volume of the Gauss map



Onto CH-mflds

* <u>Steiner polynomial</u> holds in CH-mflds
  "   "
  with "=" replaced by $\geq$.

* <u>Alexandrov-Fenchel Ineq.</u> have been extended to H^n, via <u>harmonic curvature flow,</u> but not always in their sharpest form.
  (Andrews, Hu, Li, Wang, Xia, ...)

* Sharp Minkowski ineq. is not known even in H^3!

☆ Almost all fundamental questions are open in CH-mflds:

- Isop. Ineq. is open
(CH conjecture)

- Gauss-Kronecker ineq is open
(not even known that
$M_{n-1}(\Gamma) > \varepsilon$!)

- Minkowski ineq is open
(for $K \leq a < 0$)
(But for $K \leq 0$, solved recently)
in dim 3
G-S

☆ $$\text{GK-ineq} \xrightarrow{\text{Kleiner}} \text{Isop. Ineq.}$$

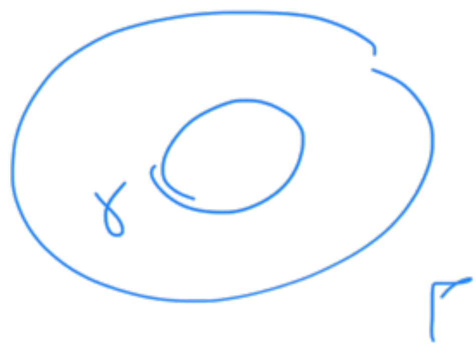
$$\text{Minkowski ineq.} \xrightarrow{\text{G-S}} \text{Isop. Ineq}$$

assuming d-convexity

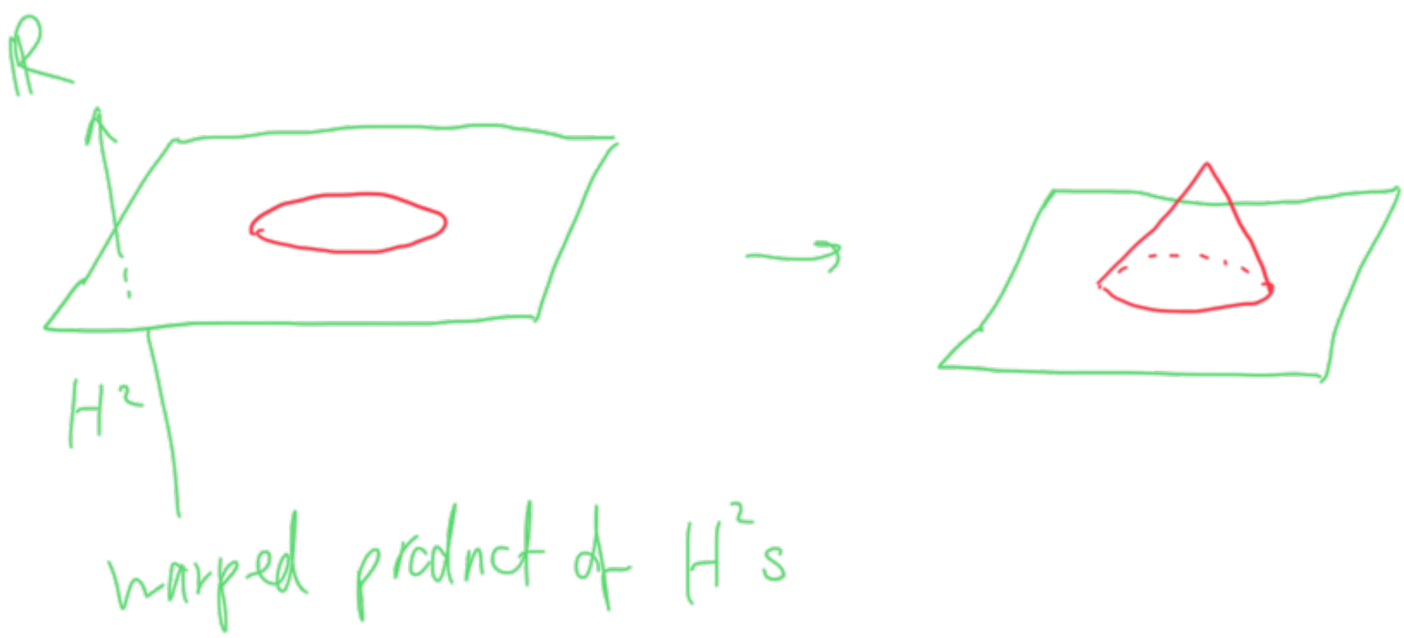
* **Warning**: More strange convexity phenomena in CH-mflds:

- Monotonicity fails for $\mathcal{M}_{n-1}$:

$$\mathcal{M}_{n-1}(\gamma) \not\leq \mathcal{M}_{n-1}(\Gamma)$$



Dekster's Example (JDG 1981)

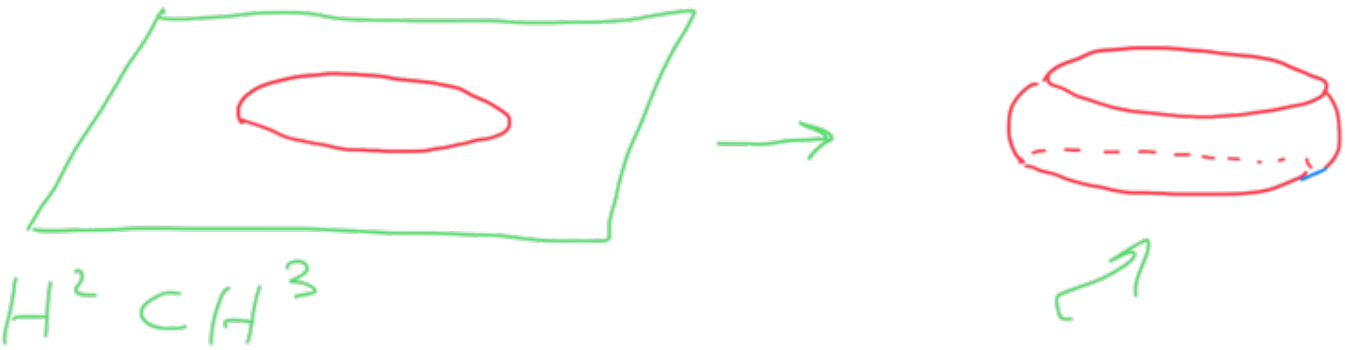


Nareira-Solanes Example

- $\mathcal{M}_1(\Gamma)$ in H^3 is not

minimized by spheres!

minimized by splines!



$H^2 \subset H^3$ "Pancake" in H^3

Minkowski $\longrightarrow$ Isop. Ineq

Suppose Minkowski ineq. holds for d-convex hypersurfaces $\Gamma \subset M$, i.e.

$$\mathcal{M}_1(\Gamma) \geq \mathcal{M}_1(S)$$

sphere in $\mathbb{R}^n$ with $|S| = |\Gamma|$

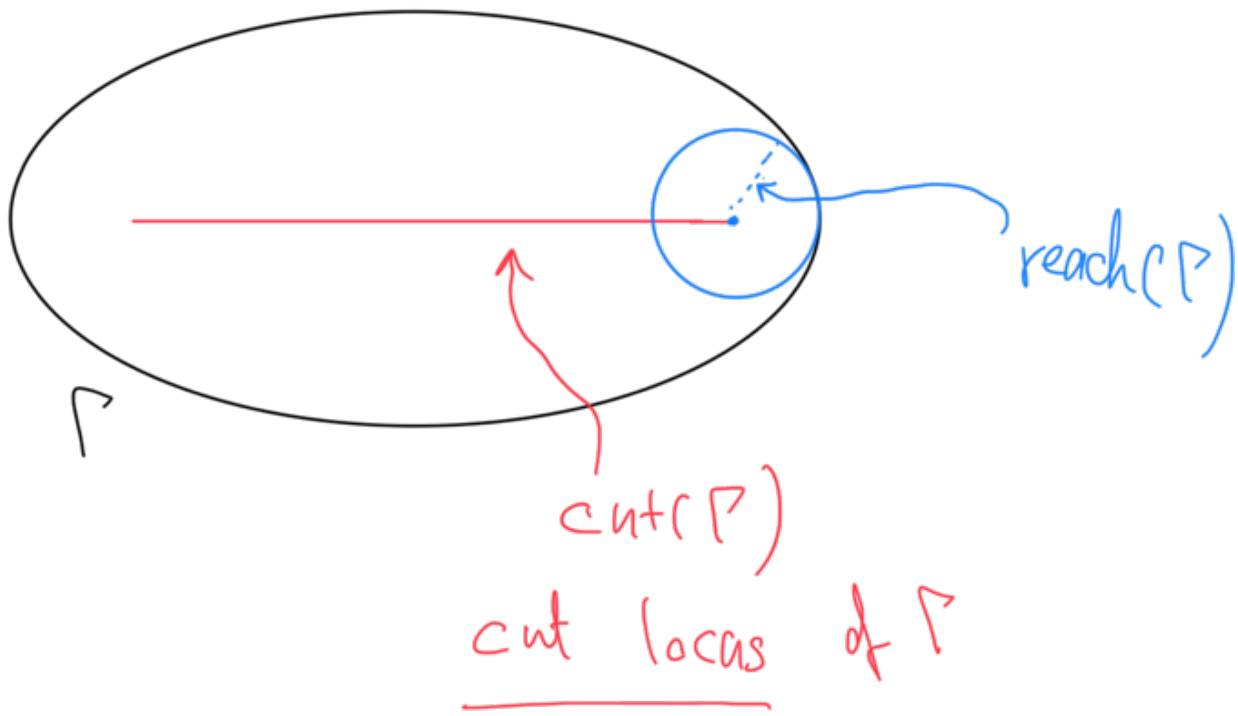
and "=" holds only if Γ bounds a Euclidean ball.

Then isop. ineq. holds in M for d-convex hypersurfaces.

Ingredients of the proof

Reach (in the sense of Federer)

Reach of a convex hypersurface $\Gamma \subset M$, is the sup radii of balls which roll freely inside Γ.



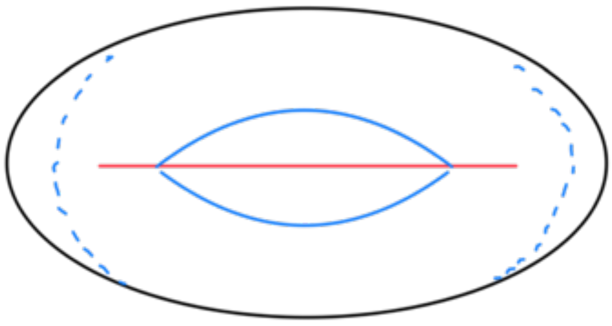
$$\boxed{\mathrm{reach}(\Gamma) = \mathrm{dist}(\Gamma, \mathrm{cut}(\Gamma))}$$

Inner and outer parallel hypersurfaces

Let Γ_{-t}, Γ_t be the inner and outer parallel hypersurfaces (level sets of d_Γ)

- $t \leqslant \text{reach}(\Gamma) \Rightarrow |(\Gamma_{-t})_t| = |\Gamma|$

- $t > \text{reach}(\Gamma) \Rightarrow |(\Gamma_{-t})_t| < |\Gamma|$



Coarea formula

$f : \Omega \to [a, b]$, Lipschitz

$|\nabla f| = 1$

$$|\Omega| = \int_a^b |f^{-1}(t)| \, dt$$

Proof:

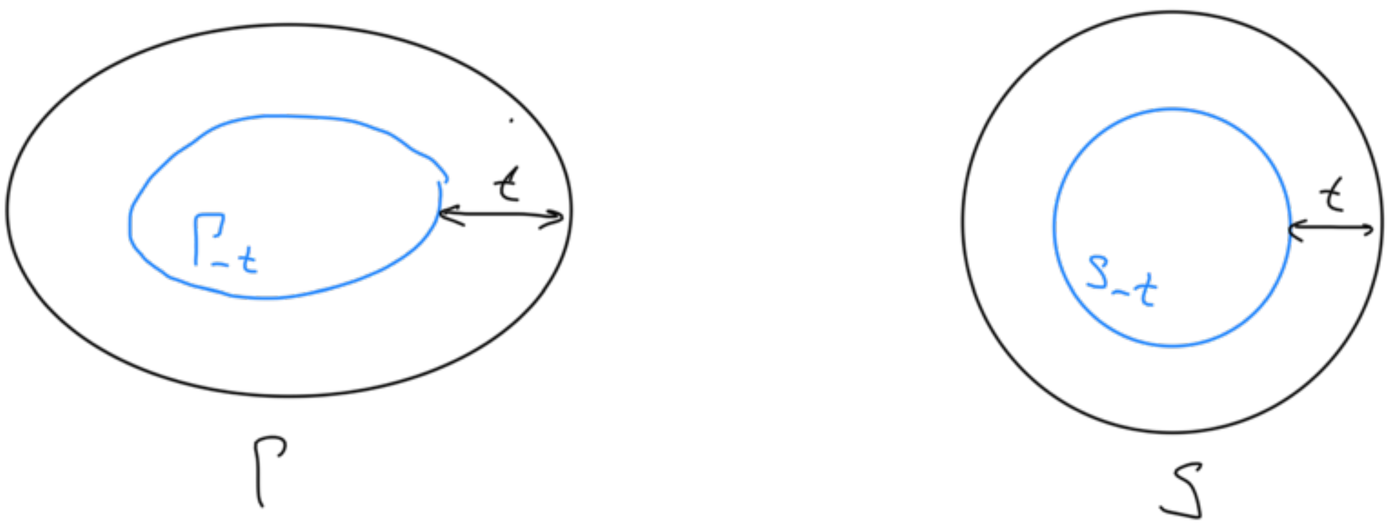
Let S be a sphere in $\mathbb{R}^n$ with $|S| = |\Gamma|$. Then

$$\text{inradius}(\Gamma) \leq \text{radius}(S)$$

We know that Isop. Ineq. holds for spheres. So suppose that Γ is not a sphere. Then

$$\mathcal{M}_1(\Gamma) > \mathcal{M}_1(S).$$

$$\Rightarrow \quad \underline{|\Gamma_{-t}| < |S_{-t}|, \text{ for small } (t)} \quad (\star)$$



If $|\Gamma_{-t}| \leq |S_{-t}|$, for all $t \in [0, \text{inradius}(\Gamma))$ we are done by the <u>coarea formula</u>.

Suppose then that

$$\boxed{|\Gamma_{-t_0}| > |S_{-t_0}|} \quad \text{for some } t_0$$

Let

$$\bar{s} := \sup\{ s \mid \ |(\Gamma_{-t_0})_t| \geq |S_{-t_0+t}|,$$

for all $t \in [0, S]$ }

Then

$$\boxed{|(\Gamma_{-t_0})_{\bar{s}}| = |S_{-t_0+\bar{s}}|} \quad (\ast\ast)$$

$$\Rightarrow \quad \mathcal{M}_1((\Gamma_{-t_0})_{\bar{s}}) > \mathcal{M}_1(S_{-t_0+\bar{s}})$$

$\Rightarrow$ $\underline{\bar{s} = t_0}$ (otherwise we can push higher)

There are now two cases to consider:

1. $t_0 > \text{reach}(\Gamma)$
2. $t_0 \leq \text{reach}(\Gamma)$

Case 1

$$|(\Gamma_{-t_0})_{\bar{s}}| < |\Gamma_{-t_0+s}| = |\Gamma| = |S| = |S_{-t_0+\bar{s}}|$$

violates $(\ast\ast)$

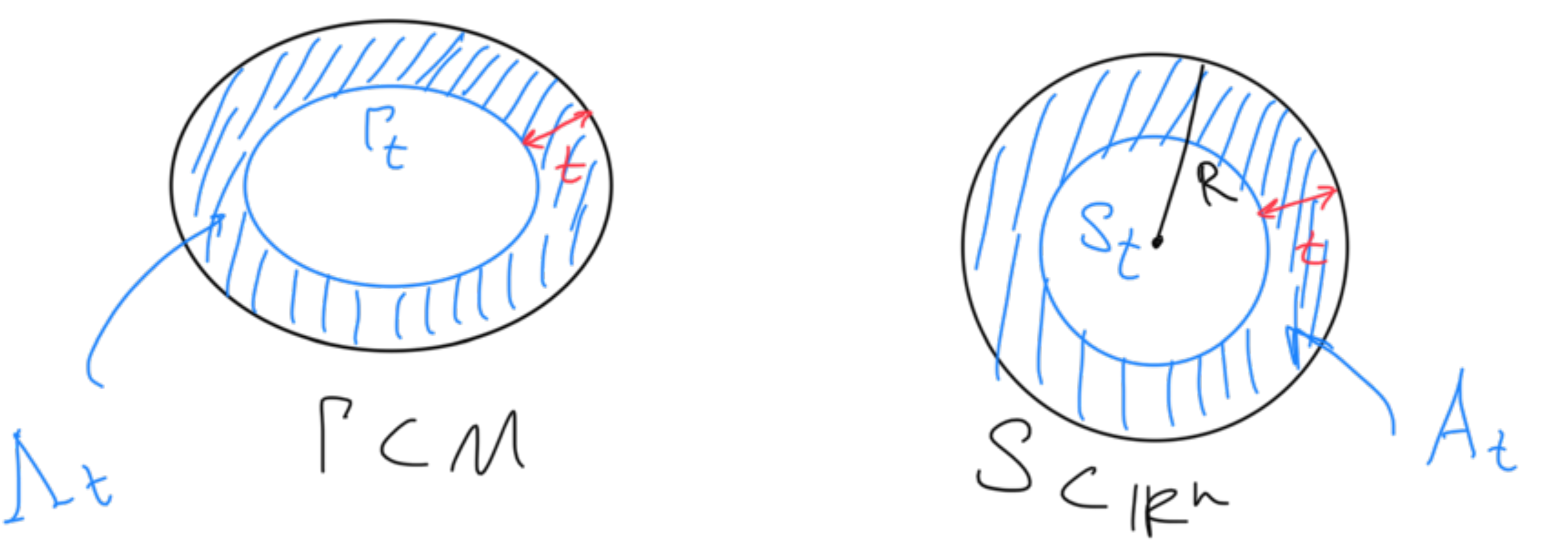
Case 2

$$|(\Gamma_{-t_0})_s| = |\Gamma_{-t_0+s}|, \quad \forall s \in [0,t_0]$$

violates (*)

Afterword

We have actually proved something stronger than the isop. ineq:



$$|\Gamma| = |S|$$

- Λ_t : Annular domain between Γ & Γ_t
- A_t : " " " S & S_t

$$\boxed{|\Lambda_t| \leq |A_t|}$$

for all $t \in [0, \mathrm{inrad}(\Omega)]$

In particular,

$$\boxed{|\Omega| \leqslant |B_R \setminus B_{R-\mathrm{inrad}(\Omega)}|}$$

Bonnesen style
isop. Ineq.

Questions

- Can one prove this annular isop. ineq. <u>without</u> Mink. Ineq?

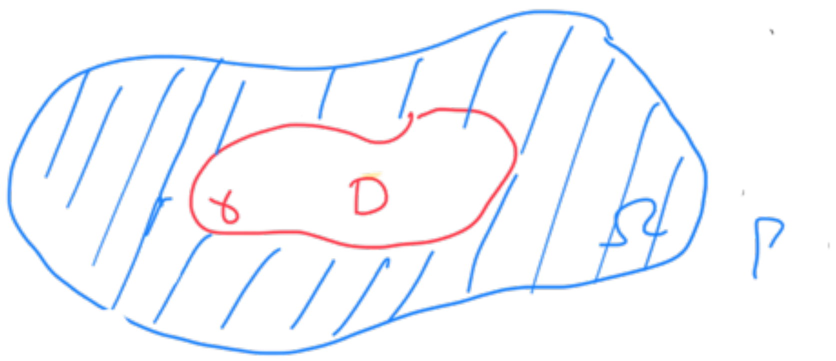
- Even in H^n, or $\mathbb{R}^n$?

- Does it just follow from the main isop. ineq.?

- In dimensions 3, & 4 we already have the isop. ineq. in CH-mflds. So can we get the Mink. ineq. out of that?

The Comparison Formula

How to compare $M_r(\Gamma)$ & $M_r(\gamma)$ in CH-mflds.



$u : M \to \mathbb{R}$, $C^{1,1}$

$\nabla u \neq 0$ on $\Omega - D$

$u =$ const on Γ & γ.

$K^u = (K_1^u, \dots, K_{n-1}^u)$, principal curvatures of level sets of u

$E_1, \dots, E_{n-1}$, principal directions

$$\boxed{\begin{aligned} M_r(\Gamma) - M_r(\gamma) &= (r+1)\int_{\Omega - D} \sigma_{r+1}(K^u) \\ &+ \int_{\Omega - D} \Big(-\sum K_{i_1}^u \cdots K_{i_{r-1}}^u K_{i_r n} \\ &\qquad + \frac{1}{|\nabla u|} \sum K_{i_1}^u \cdots K_{i_{r-2}}^u |\nabla u|_{i_{r-1}} R_{i_r i_{r-1} i_r n} \Big) \end{aligned}}$$

The "good term"

The "bad term"

$|Du|_i := \nabla_{E_i}(|Du|)$

$R_{ijkl} := R(E_i, E_j, E_k, E_l)$ (Riemann tensor)

$K_{ij} := R_{ijij}$ (Sectional curvature)

Note: For $r=0$,

$$\boxed{|\Gamma| - |\gamma| = 2\int_{\Omega - D} \sigma_1(K^u)}$$

this is a well-known formula which follows from stokes thm,

because $\boxed{\sigma_1(K^u) = \operatorname{div}\left(\frac{Du}{|Du|}\right)}$

Proof of the Comparison Formula

Proof 1 : Divergence of Newton Operators

developed by Reilly

$$P(\lambda) := \det(\lambda I - \nabla^2 u)$$

Hessian of u

$T_r^u :=$ Truncation of $P(\nabla^2 u)$ by removing terms of order higher than r.

Newton operator (Reilly)

$$\Rightarrow \quad \sigma_r(K^u) = \frac{\langle T_r(\nabla u), \nabla u \rangle}{|\nabla u|^{r+2}}$$

$$\boxed{\begin{aligned} &\operatorname{div}\left(T_{r-1}^u\left(\frac{\nabla u}{|\nabla u|^r}\right)\right) \\ &= \left\langle \operatorname{div}(T_{r-1}^u), \frac{\nabla u}{|\nabla u|^r} \right\rangle + r\,\frac{\langle T_r^u(\nabla u), \nabla u \rangle}{|\nabla u|^{r+2}} \end{aligned}}$$

Divergence identity for Newton operators.

Integrating this formula yields the

comparison formula via Stokes

Proof 2: Chern's formulas

$$\sigma_r(K) = \Phi_r(E_1, \dots, E_{n-1})$$

Chern-type forms

$$\boxed{\Phi_r = \sum \underbrace{\varepsilon(i_1 \dots i_{n-1})}_{\pm 1}\, w_n^{i_1} \wedge \dots \wedge w_n^{i_r} \wedge \theta^{i_{r+1}} \wedge \dots \wedge \theta^{i_{n-1}}}$$

the sum ranges over $1 \le i_1, \dots, i_{n-1} \le n-1$

with $i_1 < \dots < i_r$ & $i_{r+1} < \dots < i_{n-1}$

$$\theta^i(E_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

dual one-forms to E_i

$$w_j^i(\cdot) := \langle \nabla_{(\cdot)} E_i, E_j \rangle$$

connection one forms

$w^i(E\ )\ \ k$

$$\boxed{\omega^i_n(E_k) = K_i} \quad , \quad k \neq n$$

$$\mathcal{M}_r(\Gamma) - \mathcal{M}_r(\gamma) = \int_{\Gamma - \gamma} \Phi_r = \int_{\Omega - D} d\Phi_r$$

Stokes thm

$$\boxed{\begin{aligned} d\Phi_r &= (-1)^{n-1}(r+1)\,\Phi_{r+1} \wedge \theta^n \\ &+ (-1)^{r-1} \sum \varepsilon(i_1 \dots i_{n-1})\, \omega^{i_1}_n \wedge \dots \wedge \omega^{i_{r-1}}_n \wedge \Omega^{i_r}_n \wedge \theta^{i_{r+1}} \wedge \dots \wedge \theta^{i_{n-1}} \end{aligned}}$$

$$\Omega^i_j(\cdot, \star) := -\langle R(\cdot, \star) E_i, E_j \rangle$$

curvature 2-forms

$$\omega^i_n(E_n) = \frac{\langle \nabla_{E_n}(\nabla u), E_i \rangle}{|\nabla u|} = \frac{u_{ni}}{|\nabla u|}$$

Applications of the Comparison Formula

The "bad term" (involving mixed curvatures) drops out when:

- $r = 1$
- $K_M = \text{const}$
- γ, Γ are parallel.

leading to several new inequalities

- Let M be a CH-mfld & Γ, γ $C^{1,1}$ convex hypersurfaces, then we have the following applications:

★ Monotonicity for total mean curvature:

$$M_1(\Gamma) - M_1(\gamma) = 2\int_{\Omega - D} \sigma_2(\kappa^u) - (n-1)\int_{\Omega - D} Ric\left(\frac{Du}{|Du|}\right)$$

$$\geq 0$$

$\Longrightarrow$ $K \leq a$

$$\boxed{M_1(\Gamma) > -(n-1)a|\Omega|}$$

Sharp for $n=3$.

(generalizes Gallego-Solanes in $\mathbb{H}^3$)

★ Monotonicity for parallel Surfaces

If Γ, γ are parallel (and γ is still smooth, i.e., within reach(Γ))

then $|\partial_n|_\Gamma = 0$. So

$$M_r(\Gamma) - M_r(\gamma) \geq$$

$$(r+1)\int_{\Omega - D} \sigma_{r+1}(\kappa^n) - a(n-r)\int_{\Omega - D} \sigma_{r-1}(\kappa^n)$$

$$\geq 0$$

(generalizes Schroeder-Strake for $r = n-1$)

★ Monotonicity for const curvature:

$$M_r(\Gamma) - M_r(\gamma) = (r+1)\int_{\Omega - D} \sigma_{r+1}(\kappa^n) - a(n-r)\int_{\Omega - D} \sigma_{r-1}(\kappa^n)$$

$$\geq 0$$

(established earlier by Solanes,

follows from Crofton formula for Quermass-integrals)

* Rigidity thm for curvature

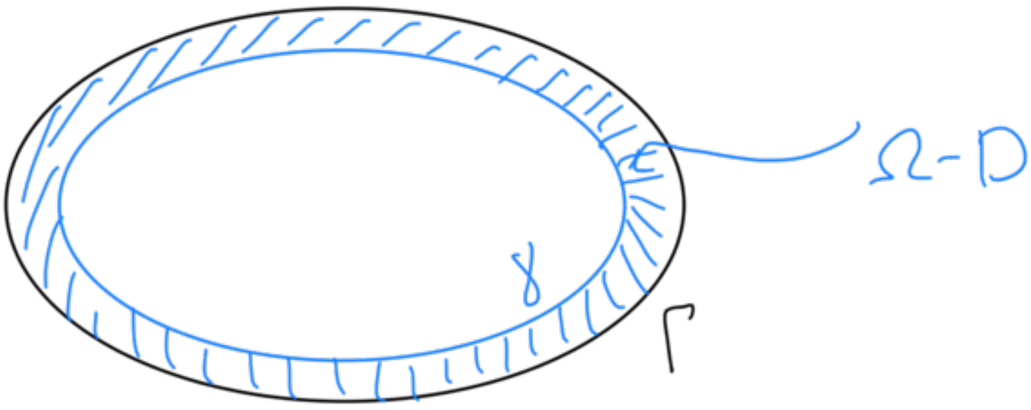
Thm: Let Γ be st. convex in CH-mfld M^3 with $K \leq a \leq 0$. Suppose $K \equiv a$ on Γ. Then $K \equiv a$ on Ω.

(Refines Gromov, Green-Wu, Ziller, Shroeder-Strake, Seshadri, for $a=0$)

- Generalized very recently by G-Petrunin to hypersurfaces with semi-def. 2nd fund. form in all dimensions.

Proof: After replacing Γ by another convex surface in Ω may assume that $K < a$ at some point in every ngbhd of Γ in Ω.

Let γ be an inner parallel surface.



Choose γ so close to Γ s.t. $K<a$ at some point of γ.

By the comparison formula

$$M_2(\Gamma) - M_2(\gamma) \geq -a\int_{\Omega-D} \sigma_1 = -a(|\Gamma| - |\gamma|)$$

$$\sigma_2 = K_P - K_M$$

Gauss' Eq.

$$\Rightarrow M_2(\Gamma) = \int_\Gamma \sigma_2$$

$$= \int K_P - \int K_M$$

Gauss-Bonnet thm

$$= 4\pi - \int K_M$$

$$= 4\pi - a|\Gamma|$$

So

$$4\pi - \cancel{a|\Gamma|} - M_2(\gamma) \geq -\cancel{a|\Gamma|} + a|\gamma|$$

$$\cancel{\mathcal{M}_2(\gamma')} - \mathcal{M}_2(\gamma) \geq \cancel{-a|\gamma'|} + a|\gamma|$$

$$\mathcal{M}_2(\gamma) \leq 4\pi - a|\gamma|$$

But

$$\mathcal{M}_2(\gamma) = 4\pi - \int_\gamma K \geq 4\pi - a|\gamma|$$

So

$$\mathcal{M}_2(\gamma) = 4\pi - a|\gamma|$$

$$\Rightarrow \quad K \equiv a \quad \text{on } \gamma$$

Contradiction.

Minkowski inequality in CH-3 mfld

$$\boxed{\mathcal{M}_1(\Gamma) \geq \sqrt{16\pi|\Gamma| - 2a|\Gamma|^2}}$$

- Sharpest ineq. known in CH 3-mflds, even $\mathbb{H}^3$
- Sharp for $a=0$

- Gallego-Solanes Examples in $\mathbb{H}^3$ satisfy

$$\mathcal{M}_1(\Gamma) \geq \sqrt{16\pi|\Gamma| + \frac{\pi^2}{4}|\Gamma|^2}$$

2.47

So if the sharp Minkowski inequality is of the form

$$\mathcal{M}_1(\Gamma) \geq \sqrt{16\pi|\Gamma| - \lambda a|\Gamma|^2}$$

then

$$2 \leq \lambda \leq 2.47$$

Proof: uses harmonic mean curvature flow

the only flow known to deform convex hypersurfaces to a point in CH- mflds.

$$X : \Gamma \times [0,T) \longrightarrow M, \quad X_t(\cdot) := X(\cdot,t)$$

$$X_t'(p) = -F_t(p)\nu_t(p), \quad X_0(p) = p$$

$$F_t := \left(\frac{1}{\kappa_1^t} + \cdots + \frac{1}{\kappa_{n-1}^t} \right)^{-1}$$

$$= \frac{\sigma_{n-1}^t}{\sigma_{n-2}^t}$$

$$\varphi(t) := M_1(\Gamma_t)^2 - 16\pi|\Gamma_t| + 2a|\Gamma_t|^2$$

We compute φ' as follows:

By the comparison & coarea formula:

$$M_1'(\Gamma_t) = -2\int_\Gamma (\sigma_2^t - Ric(\nu_t))\, F_t\, d\mu_t$$

$$\leq -2\int_\Gamma \frac{(\sigma_2^t)^2}{\sigma_1^t}\, d\mu + 2a\int \frac{\sigma_2^t}{\sigma_1^t}\, d\mu_t$$

$$|\Gamma_t|' = -M_2(\Gamma_t)$$

$$\varphi'(t) = 2M_1(\Gamma_t)\, M_1'(\Gamma_t) - 16\pi|\Gamma_t|' + 4a|\Gamma_t||\Gamma_t|'$$

$$\leq 4M_2(\Gamma_t)\big(-M_2(\Gamma_t) + 4\pi$$

$- a |\Gamma_t|)$

$$\sigma_2^t(p) = K_{\Gamma_t}(p) - K_M(T_p\Gamma_t)$$

Gauss' Equation

$$\int_{\Gamma_t} K_\Gamma = 4\pi$$

Gauss - Bonnet

$$\Rightarrow \mathcal{M}_2(\Gamma_t) \geq 4\pi - a|\Gamma_t|$$

$$\Rightarrow \varphi' \leq 0$$

But $\displaystyle\lim_{t \to a} \varphi(t) = 0$

$$\therefore \varphi(0) \geq 0 \qquad \square$$

Note: If Γ is h-convex,

we can show that

$$M(\Gamma) \geq \sqrt{16\pi|\Gamma| - \frac{7}{2}a|\Gamma|^2}$$

But it is not known if HMCF preserves h-convexity.