Tangent Cones and Regularity of Real Hypersurfaces

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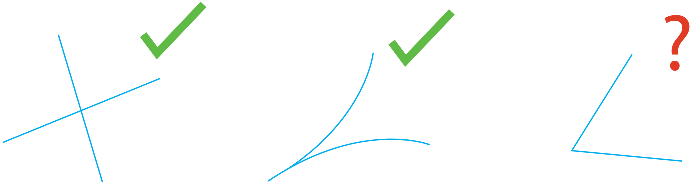
The following results were obtained in joint work with Ralph Howard.

Can a planar real algebraic curve have a corner?

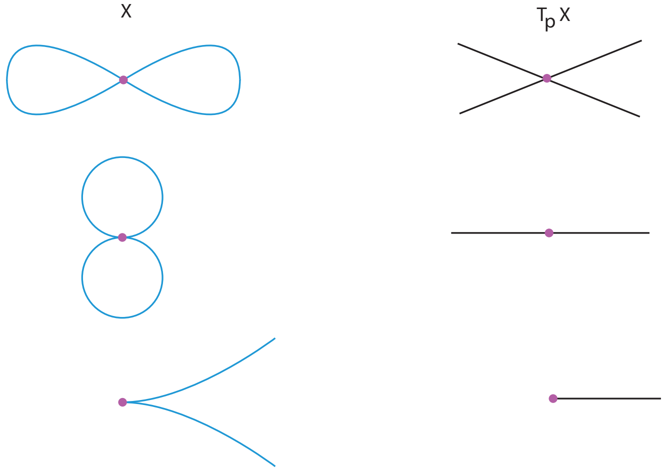


Crossings Cusps Corners

Definition (Whitney)

The tangent cone T_pX of a set $X \subset \mathbf{R}^n$ at a point $p \in X$ consists of the limits of all (secant) *rays* which originate from p and pass through a sequence of points $p_i \in X \smallsetminus \{p\}$ which converges to p.

Examples



Useful fact:

Let

$$X_{p,\lambda} := \lambda(X - p) + p$$

be the *homothetic expansion* of X by the factor λ centered at p. Then

$$T_pX = \limsup_{\lambda \to \infty} X_{p,\lambda}.$$

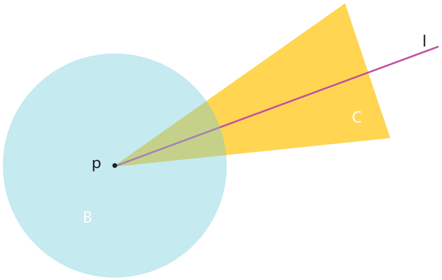
So, T_pX is the *"outer limit"* of the blowups of X centered at p.

This means that for every point $x \in T_pX$ there exists a subsequence X_{p,λ_i} which eventually intersects every neighborhood of x.

Another useful fact:

A ray ℓ belongs to T_pM if and and only if for any cone $C_\delta(\ell)$ around ℓ and ball $B_r(p)$ centered at p,

$$C_\delta(\ell) \cap B_r(p) \cap X \neq \emptyset$$



Characterization of $\mathcal{C}^1$ hypersurfaces

Theorem (1)

Let $X \subset \mathbf{R}^n$ be a locally closed set. Suppose that T_pX is flat (i.e. a hyperplane) for each $p \in X$, and depends continuously on p.

Then X is a union of C^1 hypersurfaces.

Further, if the multiplicity of each T_pX is at most $m < 3/2$, then X is a hypersurface.

Basic idea for the proof:

Show that locally X is a multi-sheeted graph:

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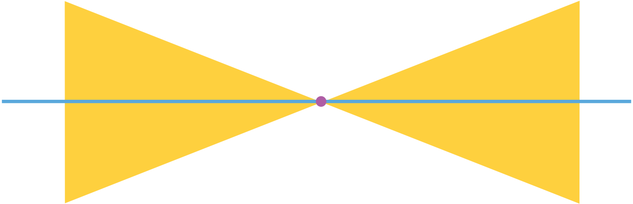

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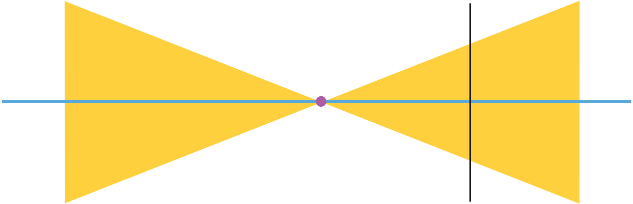

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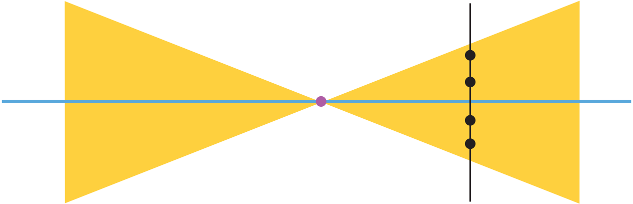

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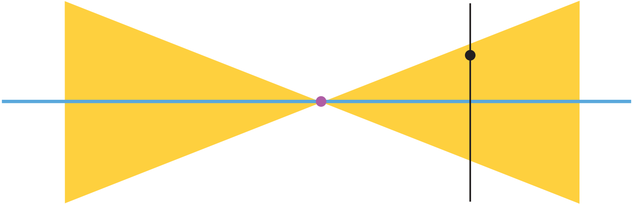

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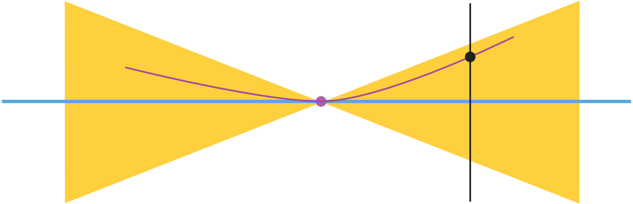
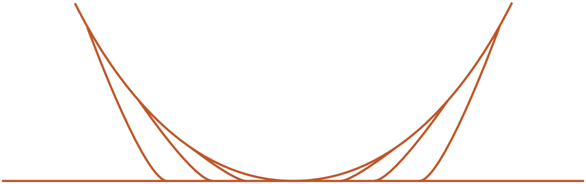


The assumption on multiplicity is necessary:



Definition

The (lower) *multiplicity* of T_pX with respect to some measure μ on $\mathbf{R}^n$ is defined as

$$m_\mu(T_pX) := \liminf_{\lambda\to\infty} \frac{\mu\Big(X_{p,\lambda} \cap B^n(p,r)\Big)}{\mu\Big(T_pX \cap B^n(p,r)\Big)},$$

for some $r > 0$.

If the Hausdorff dimension of T_pX is an integer d, then we define the multiplicity with respect to the *Hausdorff measure* $\mathcal{H}^d$ as

$$m(T_pX) := m_{\mathcal{H}^d}(T_pX).$$

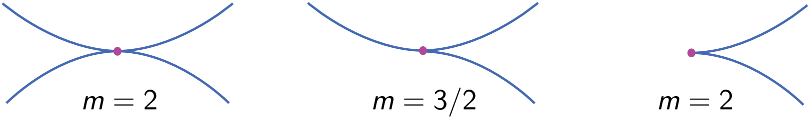


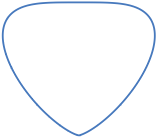
Figure: Examples of multiplicity

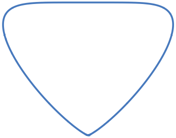
Example

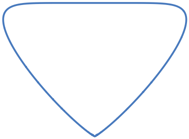
For any given $\alpha > 0$, there is a convex real algebraic hypersurface which is not $\mathcal{C}^{1,\alpha}$:

$$(1-y)y^{2n-1} = x^{2n}$$

for $n = 2, 3, 4, \ldots$. These curves are $\mathcal{C}^1$, and are $\mathcal{C}^\infty$ everywhere except at the origin o. But they are not $\mathcal{C}^{1,\alpha}$, for $\alpha > 1/(2n-1)$, in any neighborhood of o.





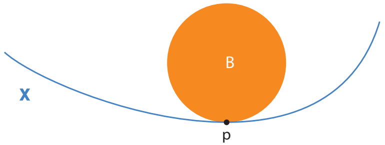


So the $\mathcal{C}^1$ conclusion in the last theorem was optimal

To have more regularity we need more conditions:

Note also that the continuity of $p \mapsto T_pX$ is not a pointwise condition which may be easily checked.

Both issues may be remedied by assuming that the set X has *positive support*:



This means that through each point p of X there passes a ball of *uniform radius* whose interior is disjoint from X. If two such balls with disjoint interiors pass through p then we say X has *double positive support*.

Example

All convex hypersurfaces (i.e., the boundaries of convex sets with interior points) have positive support.

More generally, the boundary of any set with *positive reach* (as defined by Federer) has positive support.

Theorem (2)

Let $X \subset \mathbf{R}^n$ be a locally closed set with flat tangent cones and positive support.

Suppose that either X is a hypersurface, or the multiplicity of each T_pX is at most $m < 3/2$.

Then X is a $\mathcal{C}^1$ hypersurface.

Furthermore, if X has double positive support, then it is $\mathcal{C}^{1,1}$.

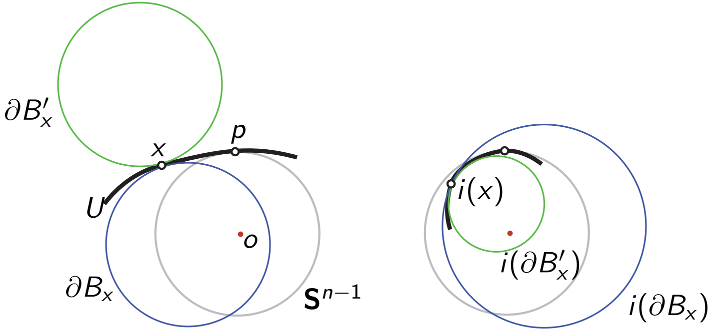


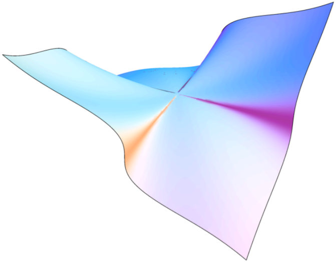
Figure: The inversion trick

Example

There are real algebraic hypersurfaces with flat tangent cones which are supported by balls at each point but are not $\mathcal{C}^1$:

$$z^3 = x^5 y + x y^5$$

This surface is $\mathcal{C}^\infty$ in the complement of o, and has a support ball at o, but its tangent planes along the x and y axis are vertical.



So the assumption on the uniformity of the radii of the support balls in the last theorem was necessary.

On the other hand, the assumption that T_pX be flat may be relaxed when X is real analytic:

Theorem (3)

Let $X \subset \mathbf{R}^n$ be a real analytic hypersurface with positive support. If T_pX is a hypersurface for all p in X, then X is $\mathcal{C}^1$.

In particular, convex real analytic hypersurfaces are $\mathcal{C}^1$.

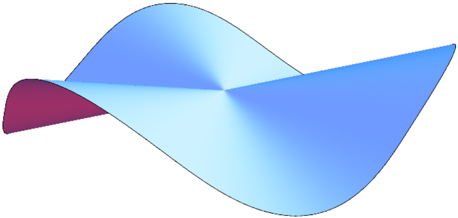
Note: By a "hypersurface" $X \subset \mathbf{R}^n$ we always mean a set which is locally homeomorphic to $\mathbf{R}^{n-1}$.

Example

There are real algebraic hypersurfaces whose tangent cones are hypersurfaces but are not hyperplanes, such as the Fermat cubic

$$x^3 + y^3 = z^3.$$

All points of this surface, except the origin, are regular and therefore the tangent cones are flat there. But the tangent cone at the origin is the surface itself, since the surface is invariant under homotheties.



So how do we prove the last theorem:

Theorem (3)

Let $X \subset \mathbf{R}^n$ be a real analytic hypersurface with positive support. If T_pX is a hypersurface for all p in X, then X is $\mathcal{C}^1$.

It suffices to show that T_pX is symmetric with respect to p.

Then T_pX has to be flat due to the existence of a support ball at p.

So we just need to show that: *if the tangent cone of a real analytic hypersurfaces is a hypersurface, then it is symmetric*.

To this end, we need to understand the relation between 3 notions of tangent cones:

- T_pX, the *tangent cone* we have already defined
- $\widetilde{T}_pX$, the *symmetric tangent cone*
- The *algebraic tangent cone* defined for analytic sets

The symmetric tangent cone is the limit of all *secant lines* (as opposed to secant rays) which pass through p.

Thus

$$\widetilde{T}_pX = T_pX \cup (T_pX)^*.$$

where $(T_pX)^*$ is the reflection of X with respect to p.

Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be an analytic function, with $f(o) = 0$. By Taylor's theorem

$$f(x) = h_f(x) + r_f(x)$$

where $h_f(x)$ is a nonzero homogenous polynomial of degree m, i.e.,

$$h_f(\lambda x) = \lambda^m h_f(x),$$

for every $\lambda \in \mathbf{R}$, and $r_f\colon \mathbf{R}^n \to \mathbf{R}$ is a continuous function which satisfies

$$\lim_{x \to 0} |x|^{-m} r_f(x) = 0.$$

Then the algebraic tangent cone of $X = Z(f) := f^{-1}(0)$ at o is defined as

$$Z(h_f).$$

$$\widetilde{T}_o Z(f) \subset Z(h_f).$$

Proof.

Suppose $v \in \widetilde{T}_o Z(f)$. Then, there are points $x_i \in Z(f) \setminus \{o\}$, and numbers $\lambda_i \in \mathbf{R}$ such that $\lambda_i x_i \to v$. Since $x_i \in Z(f)$,

$$0 = f(x_i) = h_f(x_i) + r_f(x_i)$$

which yields that

$$0 = |x_i|^{-m}\big(h_f(x_i) + r_f(x_i)\big) = h_f\left(|x_i|^{-1} x_i\right) + |x_i|^{-m} r_f(x_i).$$

Consequently

$$0 = \lim_{x_i \to o} h_f\left(|x_i|^{-1} x_i\right) + 0 = h_f\left(\lim_{x_i \to o} |x_i|^{-1} x_i\right).$$

But,

$$\lim_{x_i \to o} |x_i|^{-1} x_i = \lim_{x_i \to o} |\lambda_i x_i|^{-1} |\lambda_i| x_i = \pm |v|^{-1} v.$$

So $0 = h_f\left(|v|^{-1} v\right) = |v|^{-m} h_f(v)$; therefore, $v \in Z(h_f)$. $\square$

On the other hand, in general,

$$Z(h_f) \not\subset \widetilde{T}_o Z(f).$$

Consider for instance

$$f(x,y) = x(y^2 + x^4).$$

Then $Z(f)$ is just the y-axis, while $h_f(x,y) = xy^2$, so $Z(h_f)$ is both the x-axis and the y-axis.

But, if we let $\widetilde{Z}(h_f) \subset Z(h_f)$ be the set of points p where h_f changes sign at p. Then

$$\widetilde{Z}(h_f) \subset \widetilde{T}_o Z(f).$$

Proof

Suppose $v \in \widetilde{Z}(h_f)$ but $v \notin \widetilde{T}_o Z(f)$. Then there is open ngbhd U of v in $\mathbf{R}^n$ and an open ball B centered at o such that $f \neq 0$ on $cone(U) \cap B \setminus \{o\}$. Set

$$f_\lambda(x) := \lambda^m f(\lambda^{-1} x).$$

Then $f_\lambda \neq 0$ on $cone(U) \cap B \setminus \{o\}$ for $\lambda \geqslant 1$. But

$$f_\lambda(x) = \lambda^m h_f(\lambda^{-1} x) + \lambda^m r_f(\lambda^{-1} x) = h_f(x) + \lambda^m r_f(\lambda^{-1} x),$$

which yields that

$$\lim_{\lambda \to \infty} f_\lambda(x) = h_f(x).$$

So, for large λ, f_λ changes sign on U, which implies that $f_\lambda = 0$ at some point of U—a contradiction.

Proposition

Let $U \subset \mathbf{R}^n$ be an open neighborhood of o and $f : U \to \mathbf{R}^n$ be a $\mathcal{C}^{k \geqslant 1}$ function with $f(o) = 0$ which does not vanish to order k at o. Suppose that $Z(f)$ is homeomorphic to $\mathbf{R}^{n-1}$, f changes sign on $Z(f)$, and $T_o Z(f)$ is also a hypersurface. Then

$$\widetilde{T}_o Z(f) = \widetilde{Z}(h_f) = T_o Z(f).$$

In particular, $T_o Z(f)$ is symmetric with respect to o, i.e.,

$$T_o Z(f) = -T_o Z(f).$$

Lemma

Let $X = Z(f) \subset \mathbf{R}^n$ be a analytic hypersurface. Then for every point $p \in X$ there exists an open neighborhood U of p in $\mathbf{R}^n$, and an analytic function $g\colon U \to \mathbf{R}$ such that $Z(g) = X \cap U$, and g changes sign on $X \cap U$.

Proof

Let $p = o$, and $\mathcal{C}^\omega_o$ denote the ring of germs of analytic functions at o.

$\mathcal{C}^\omega_o$ is a Noetherian and is a unique factorization domain.

So f is the product of finitely may irreducible factors f_i in $\mathcal{C}^\omega_o$, and it follows, by Łojasiewicz's structure theorem for real analytic varieties, that

$$\dim\big(Z(g)\big) = n - 1.$$

Now let $(g) \subset \mathcal{C}_o^\omega$ be the ideal generated by g, i.e., the collection of all germs ϕg where $\phi \in \mathcal{C}_o^\omega$.

Further let $I(Z(g)) \subset \mathcal{C}_o^\omega$ be the ideal of germs in $\mathcal{C}_o^\omega$ which vanish on $Z(g)$. Then, by the *real nullstellensatz*:

$$(g) = I\big(Z(g)\big).$$

So the gradient of g cannot vanish identically on $Z(g)$; because otherwise, $\partial g/\partial x_i \in I(Z(g)) = (g)$ which yields that

$$\frac{\partial g}{\partial x_i} = \phi_i g$$

for some $\phi_i \in \mathcal{C}_o^\omega$. Consequently, by the product rule, all partial derivatives of g of any order must vanish at o, which is not possible since g is real analytic.

So we may assume that g has a regular point in $Z(g) \subset X \cap U$.

Then g must assume different signs on U, and therefore $U \setminus Z(g)$ must be disconnected.

This implies that $Z(g) = X \cap U$ via Jordan-Brouwer separation theorem. $\square$

Application

Theorem (4)

Let $X \subset \mathbf{R}^n$ be a real algebraic convex hypersurface homeomorphic to $\mathbf{R}^{n-1}$. Then X is an entire graph.

Proof

$$P(x_1, \dots, x_n) := \left(\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}, \frac{1}{x_n}\right)$$

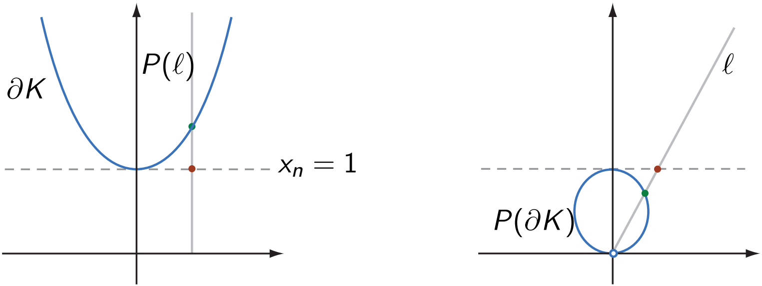
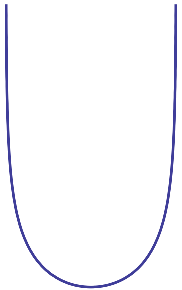


Figure: The projective transformation trick

Example

The last theorem does not hold in the real analytic category!

$$x^2 + e^{-y} = 1$$

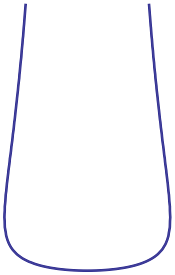


So there is real geometric difference between the categories of real algebraic and real analytic convex hypersurfaces.

Example

There are (nonconvex) real algebraic hypersurfaces homeomorphic to $\mathbf{R}^{n-1}$ which are not entire graphs:

$$y(1 - x^2 y) = 1$$



So the convexity assumption in the last theorem is essential as well.

Finally let us return to the original question which motivated our results (Why don't real algebraic curves have corners?).

One answer already follows from the general results we discussed.

A more specific answer also follows from Newton-Puiseux fractional power series ....

Lemma (Newton-Puiseux)

Let $\Gamma \subset \mathbf{R}^2$ *be a real analytic curve and* $p \in \Gamma$ *be a nonisolated point.*

Then there is an open neighborhood U *of* p *in* $\mathbf{R}^2$ *such that* $\Gamma \cap U = \cup_{i=1}^{k} \Gamma_i$ *where each "branch"* Γ_i *is homeomorphic to* $\mathbf{R}$ *via a real analytic (injective) parametrization* $\gamma_i \colon (-1, 1) \to \Gamma_i$*.*

Lemma

Let $\gamma\colon(-1,1)\to\mathbf{R}^2$ *be a nonconstant real analytic map. Then* γ *has continuously turning tangent lines.*

Proof.

Let $a=0$. Suppose $\|\gamma'(0)\|=0$. Then, by analyticity of γ', there is an integer $m>0$ and an analytic map $\xi\colon(-\epsilon,\epsilon)\to\mathbf{R}^2$ with $\|\xi(0)\|\neq 0$ such that $\gamma'(t)=t^m\xi(t)$. Thus

$$T(t):=\frac{\gamma'(t)}{\|\gamma'(t)\|}=\frac{t^m\xi(t)}{\|t^m\xi(t)\|}=\left(\frac{t}{|t|}\right)^m\frac{\xi(t)}{\|\xi(t)\|},$$

which in turn yields:

$$\lim_{t\to 0^+}T(t)=1^m\frac{\xi(0)}{\|\xi(0)\|}=(-1)^{2m}\frac{\xi(0)}{\|\xi(0)\|}=(-1)^m\lim_{t\to 0^-}T(t).$$

$\square$

Corollary

Each branch of a real analytic curve $\Gamma \subset \mathbf{R}^2$ at a nonisolated point $p \in \Gamma$ is either $\mathcal{C}^1$ near p or has a cusp at p. $\square$

Thanks!