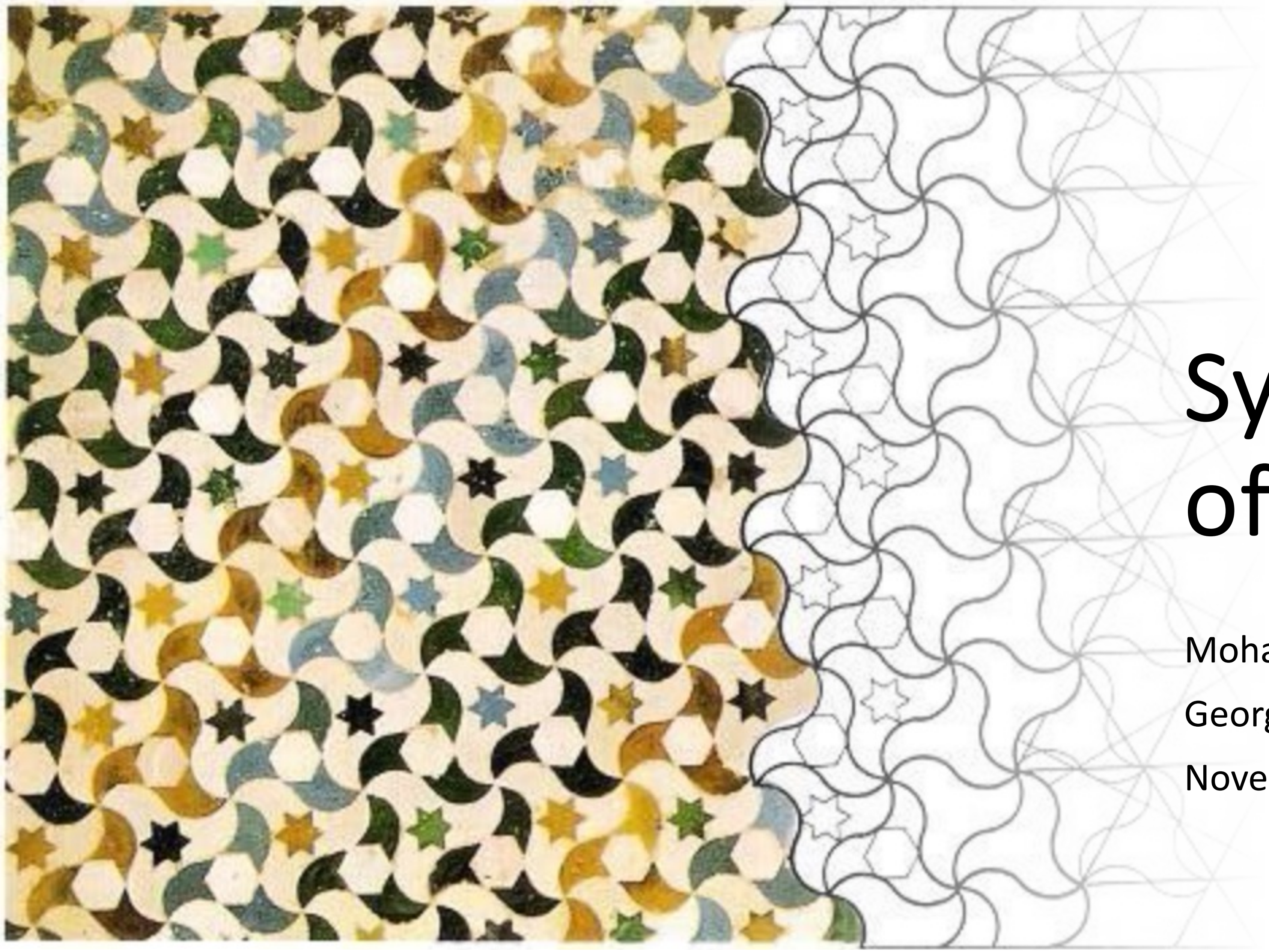

Symmetries of Tilings

Mohammad Ghomi

Georgia Tech

November 2023

This talk is a brief introduction to the mathematical theory of planar tilings, with many historical examples.

Specifically, we discuss classification of periodic tilings based on their symmetry groups.

The most complex symmetry patterns are found in Andalusia (Moorish Spain), Persia (modern day Iran), Morocco, Syria, and the rest of the Islamic world.

A ***tiling*** or ***tesselation*** is a covering of the plane by a number of nonoverlapping geometric shapes.

Roman:

A ***tiling*** or ***tessellation*** is a covering of the plane by a number of nonoverlapping geometric shapes.

Roman:

Westminster Abbey, London, England (Cosmati Pavement, 13th century)

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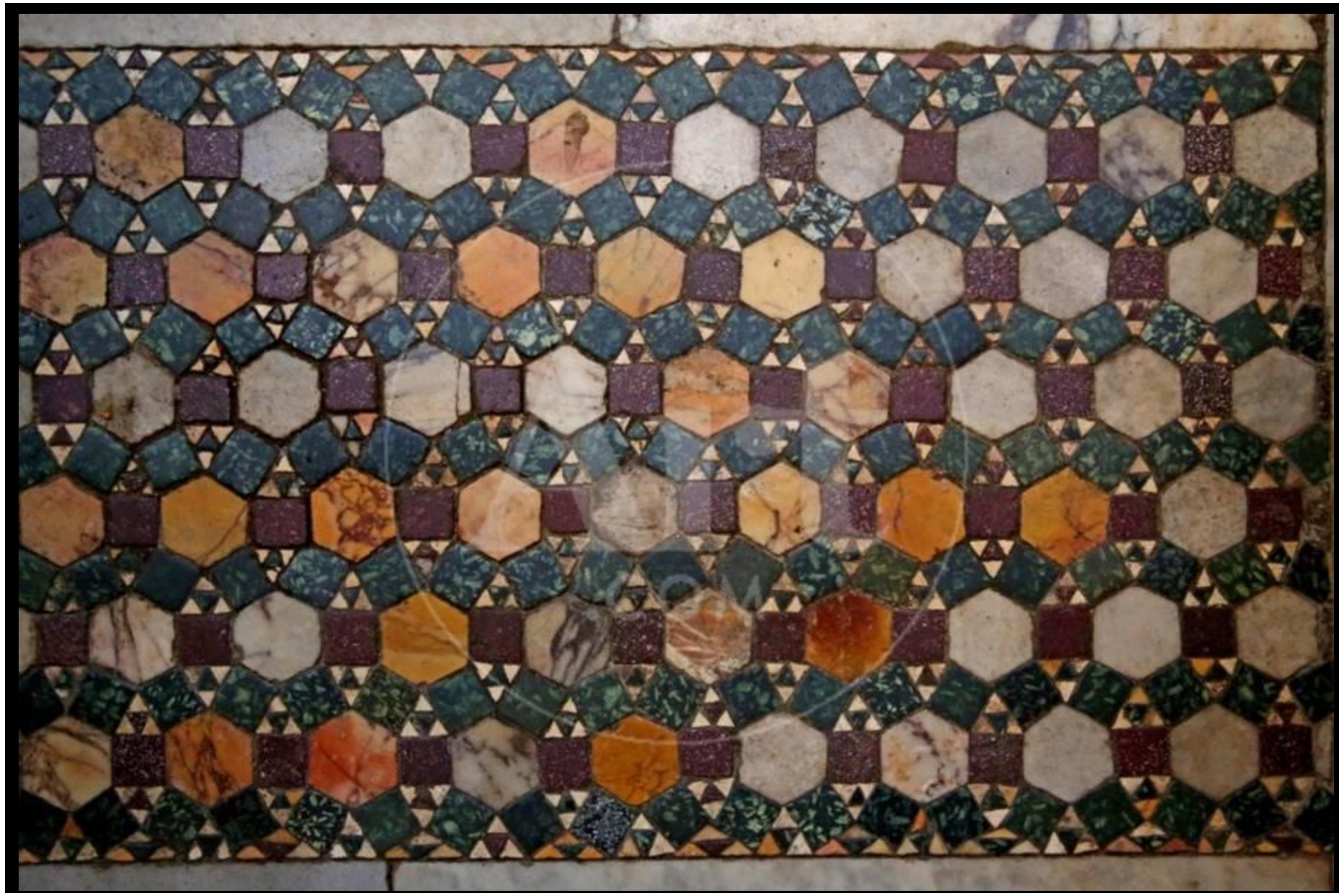
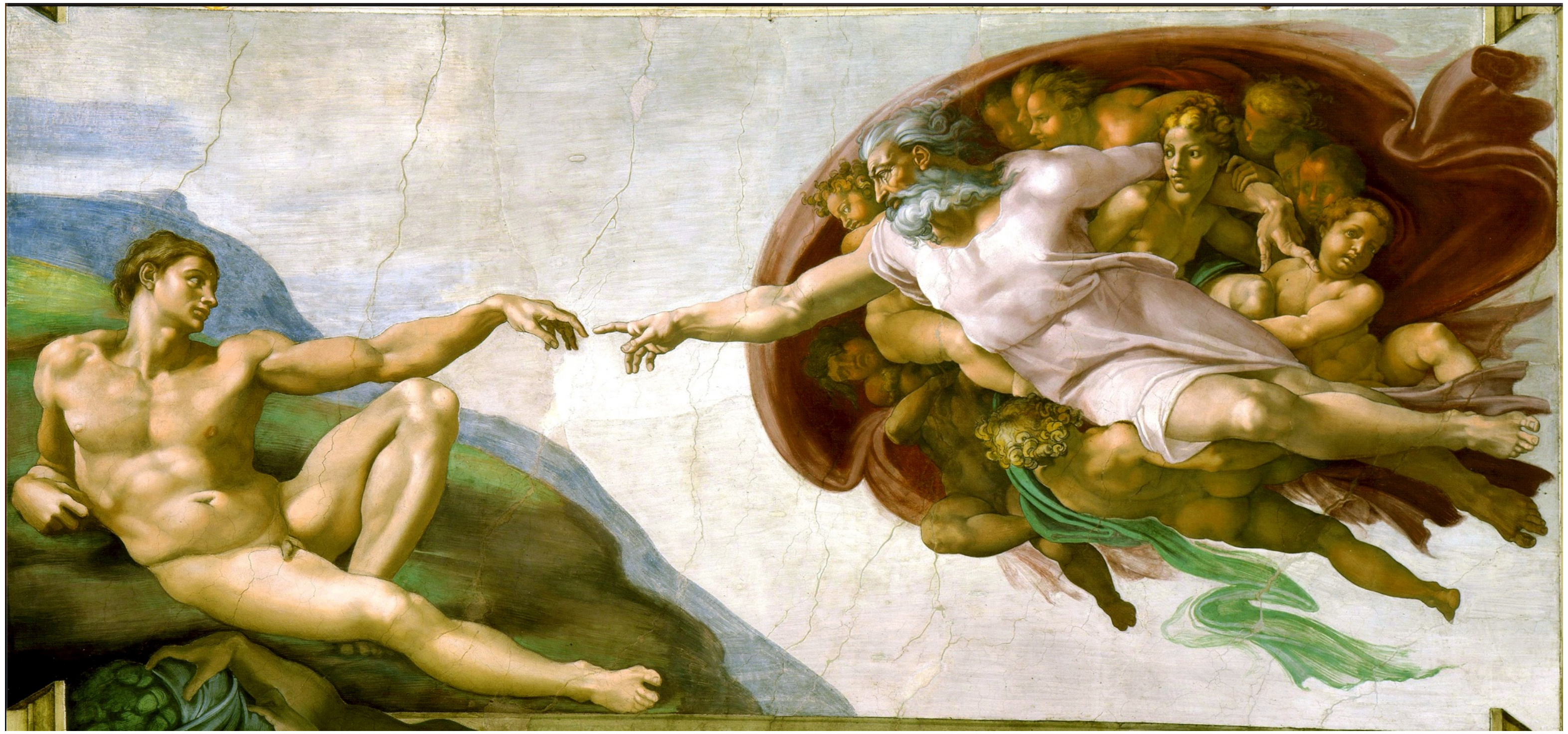


Westminster Abbey, London, England (Cosmati Pavement, 13th century)

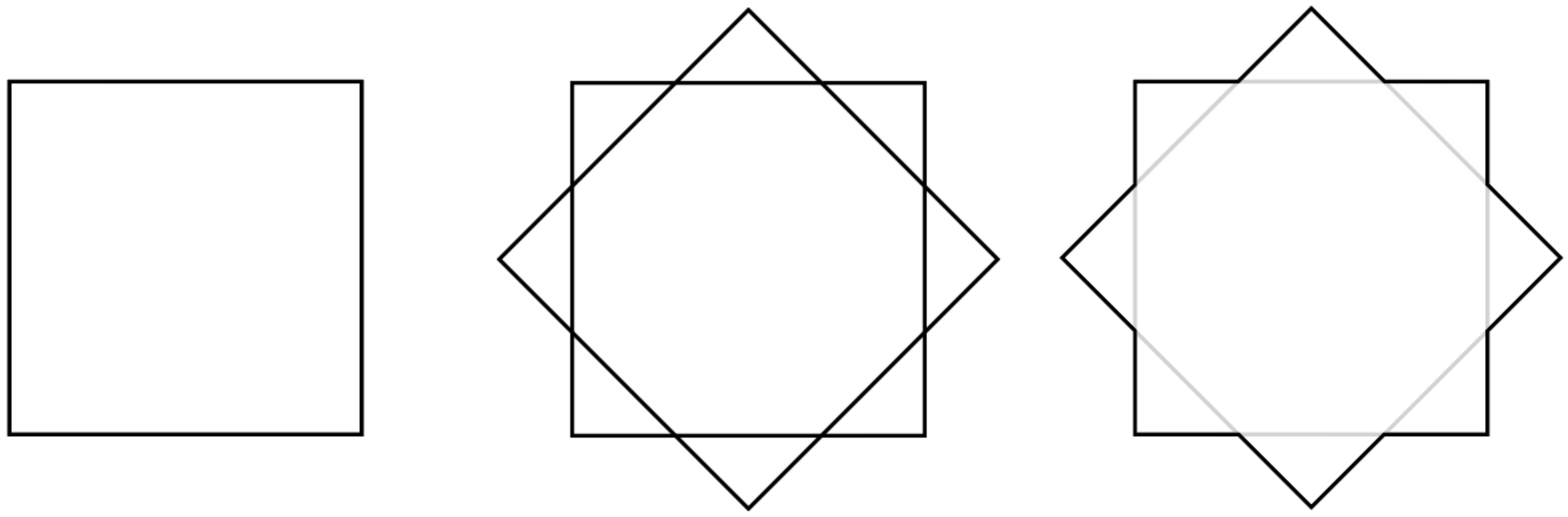
The examples of tilings in the Greco-Roman architecture, and later the Christian world, were relatively simple in terms of their symmetry structure.



In Spain and Persia, much richer patterns were developed, perhaps due to the **ban on depiction of human figures in Islam**.

Persian:
(Kashan,
Iran)

A common octagonal motif in Islamic tiling.



Persian:

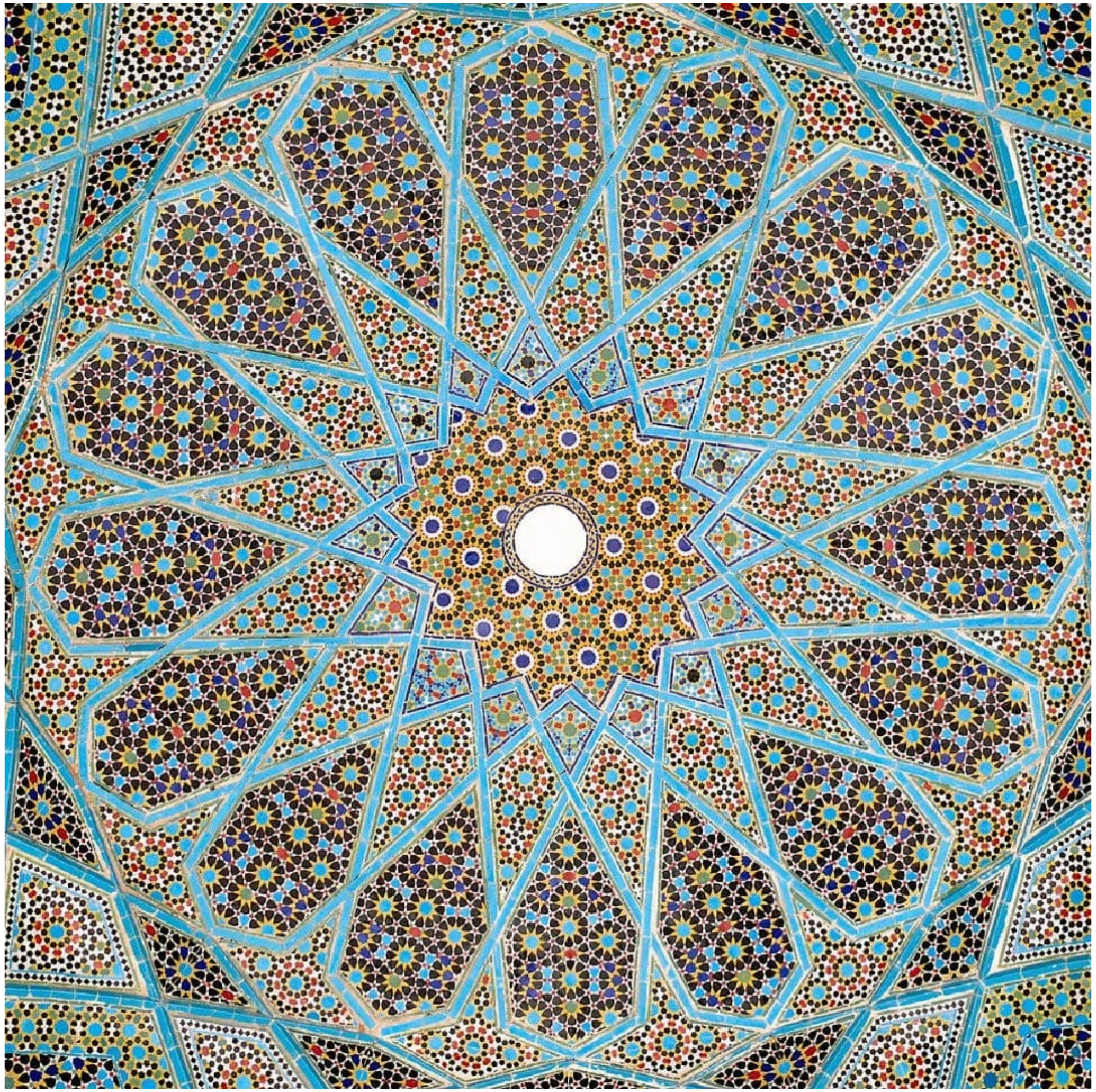
(Nasir-ol-Molk

Mosque,

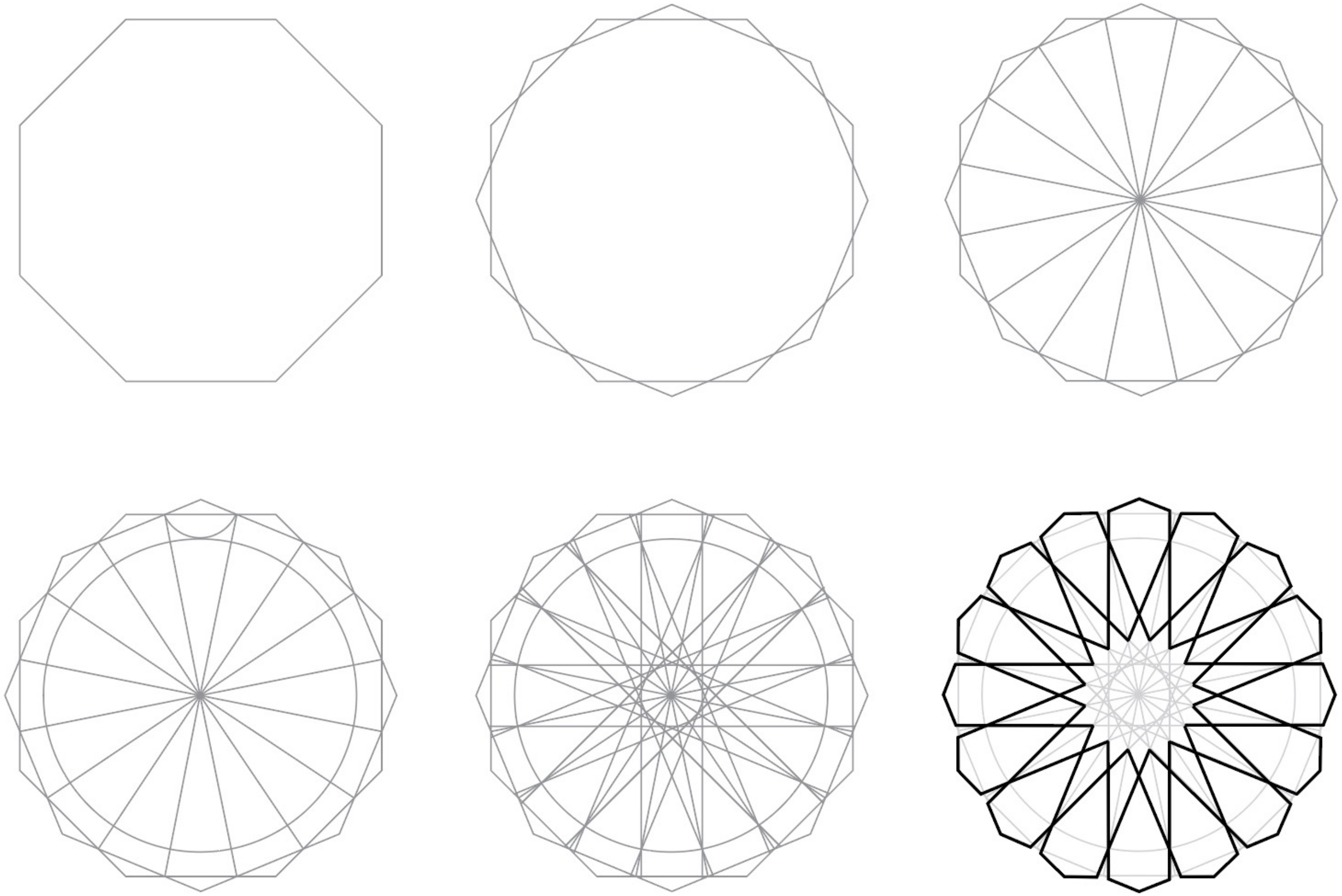
Shiraz, Iran).

Persian:
(Mausoleum of Hafez
In Shiraz, Iran).

A 16-fold rosette
in the ***Girih*** pattern

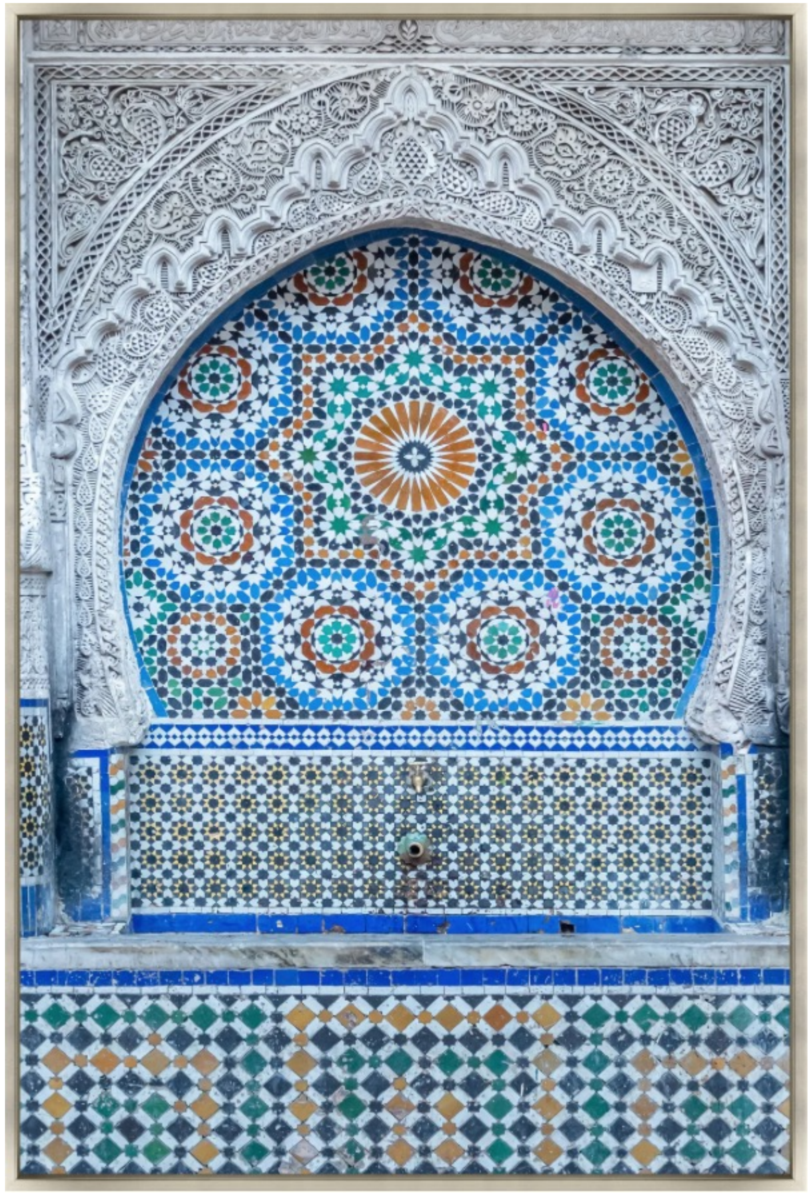


Construction of a 16-fold rosette



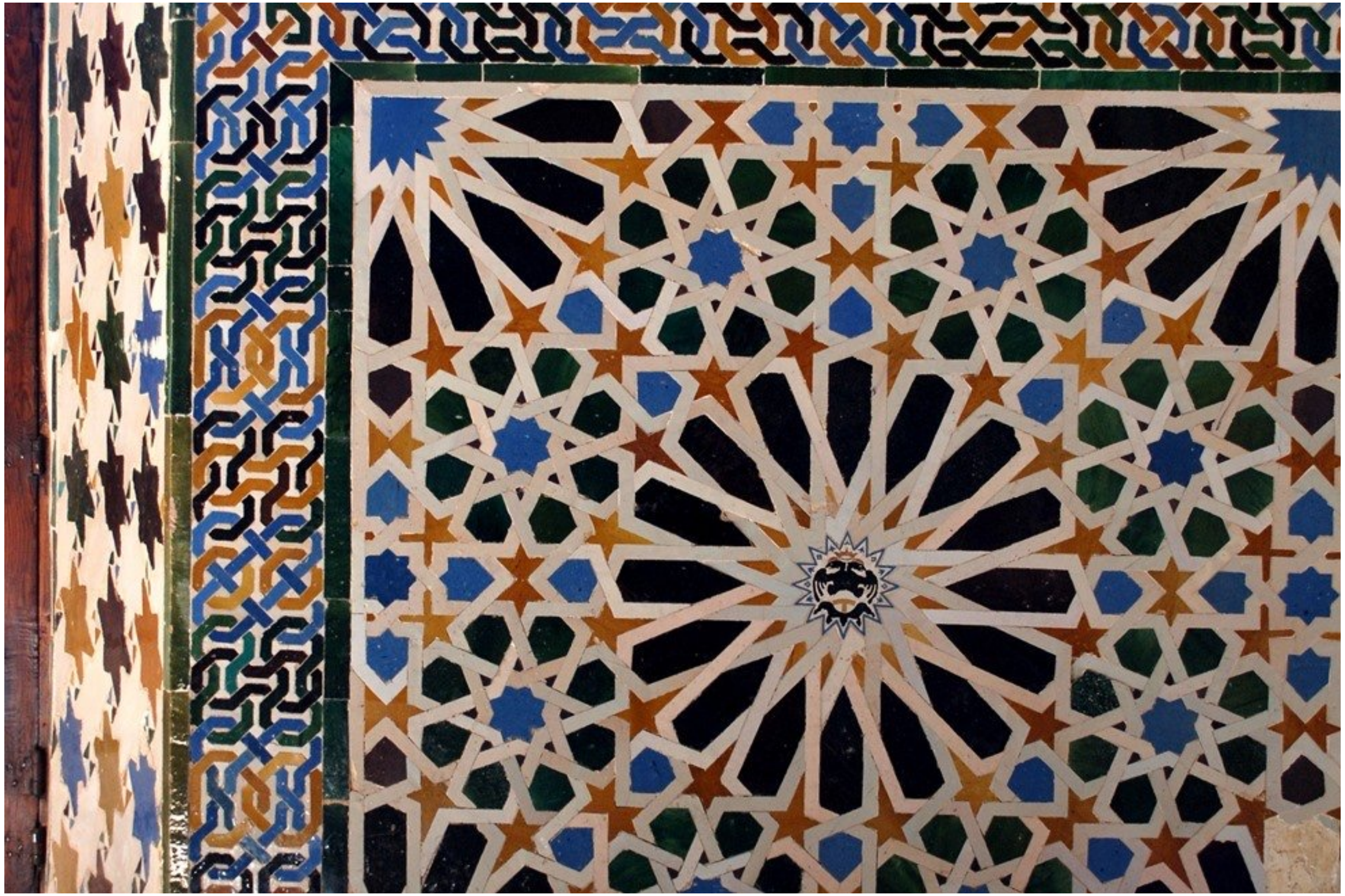
A Moroccan

Tiling with the octagonal motif

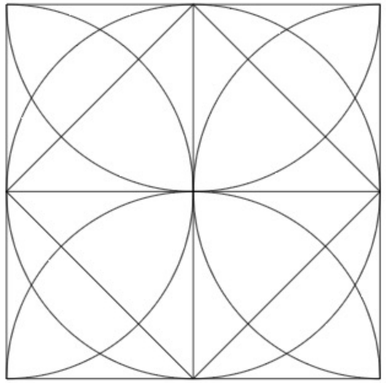


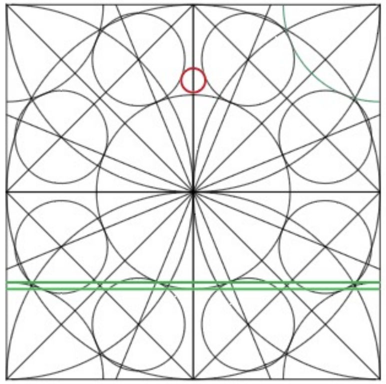
Another Moroccan example
with the 12-fold rosette
in the window,
and hexagonal surround tiles.

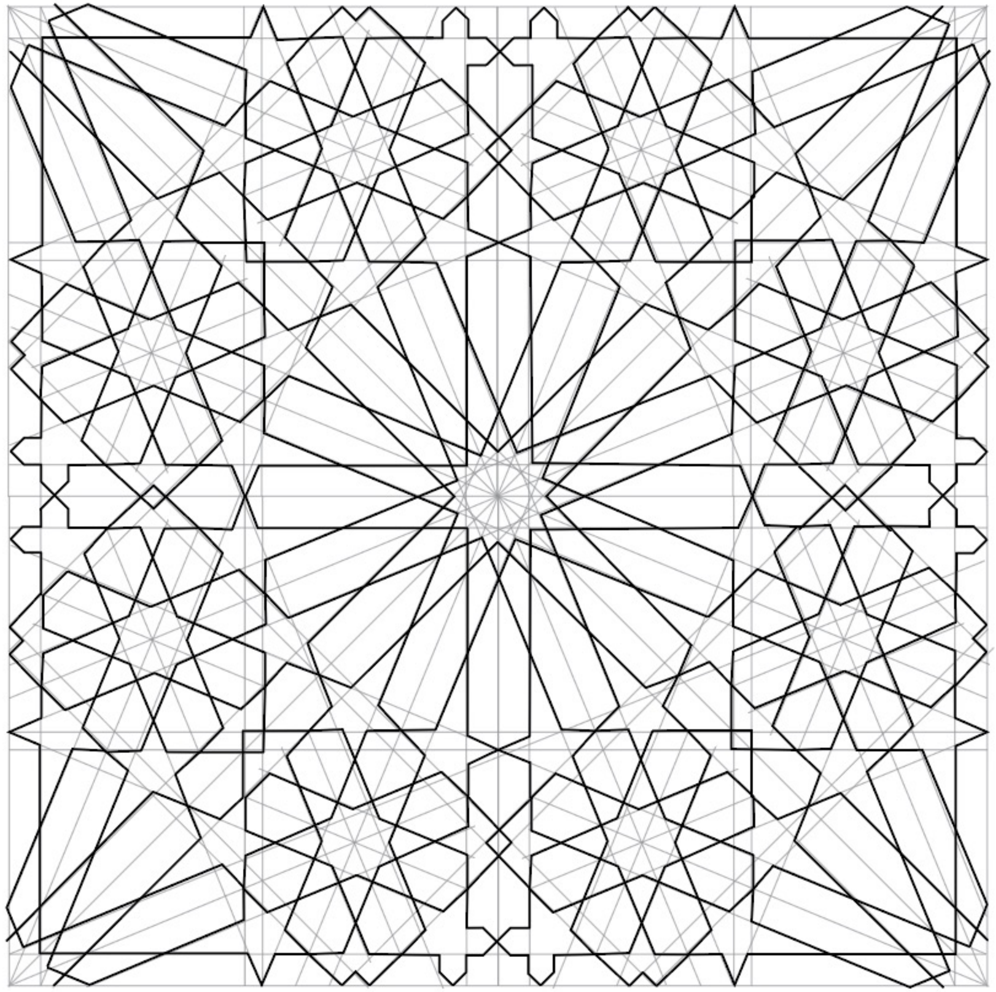
The 12-fold rosette appears in a well-known pattern in Alhambra, Spain in combination with an 8-fold rosette

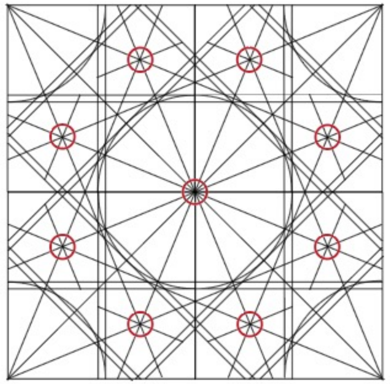


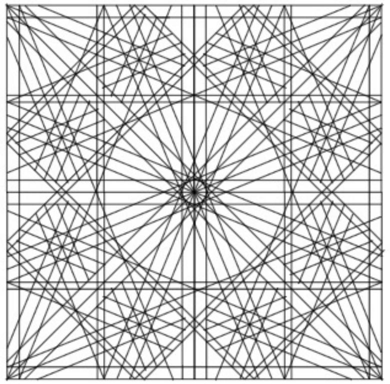
Details for generating the Alhambra pattern



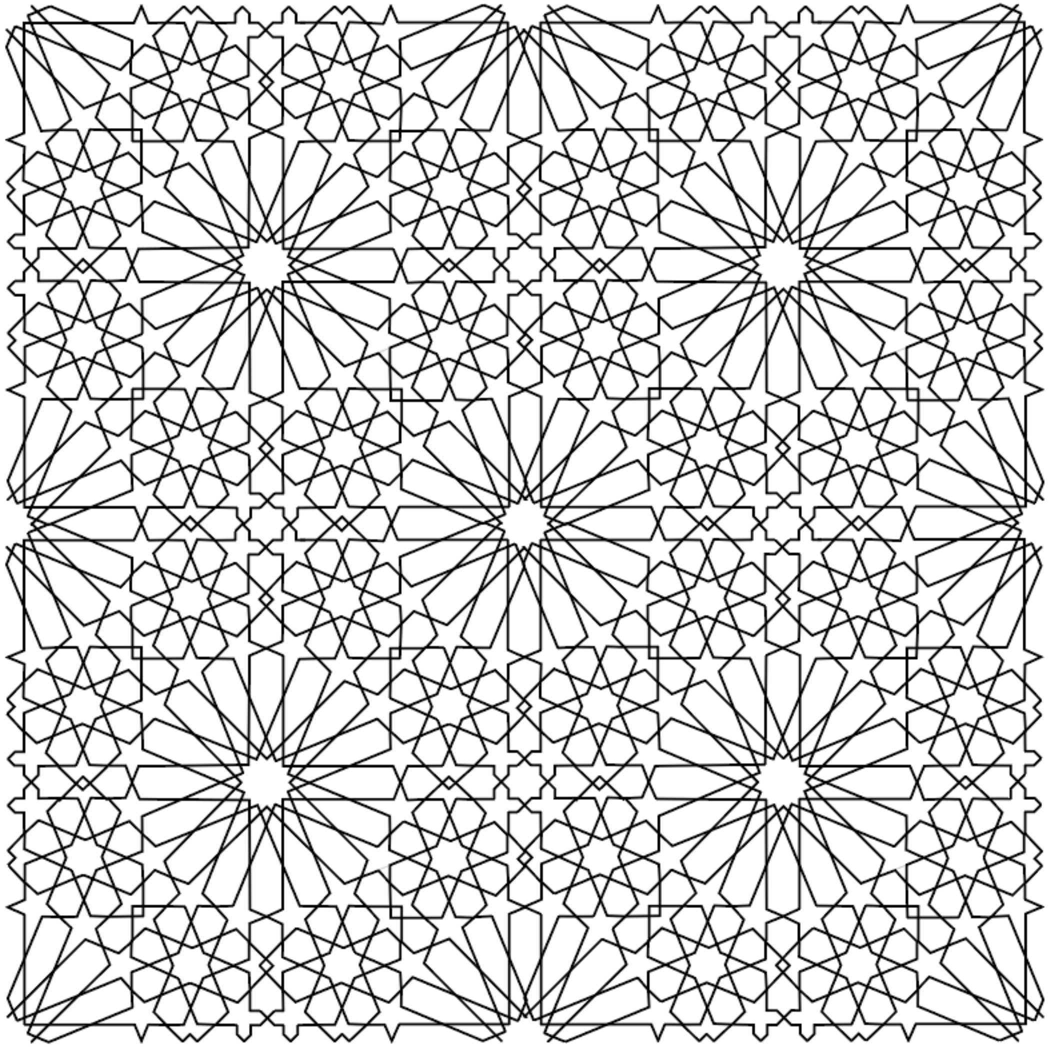








The last panel now tiles the space

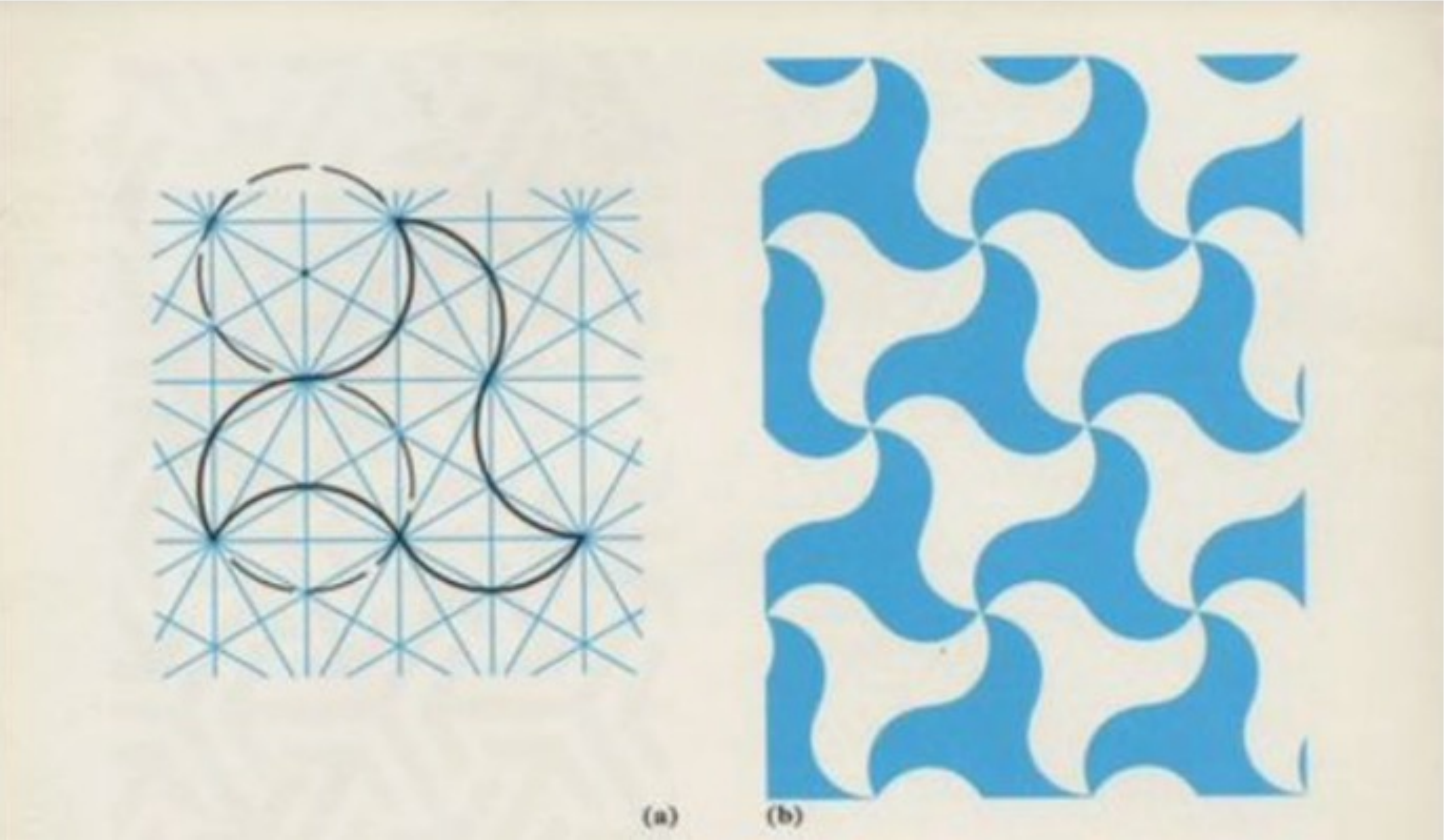


Another example from Alhambra with the octagonal motif.

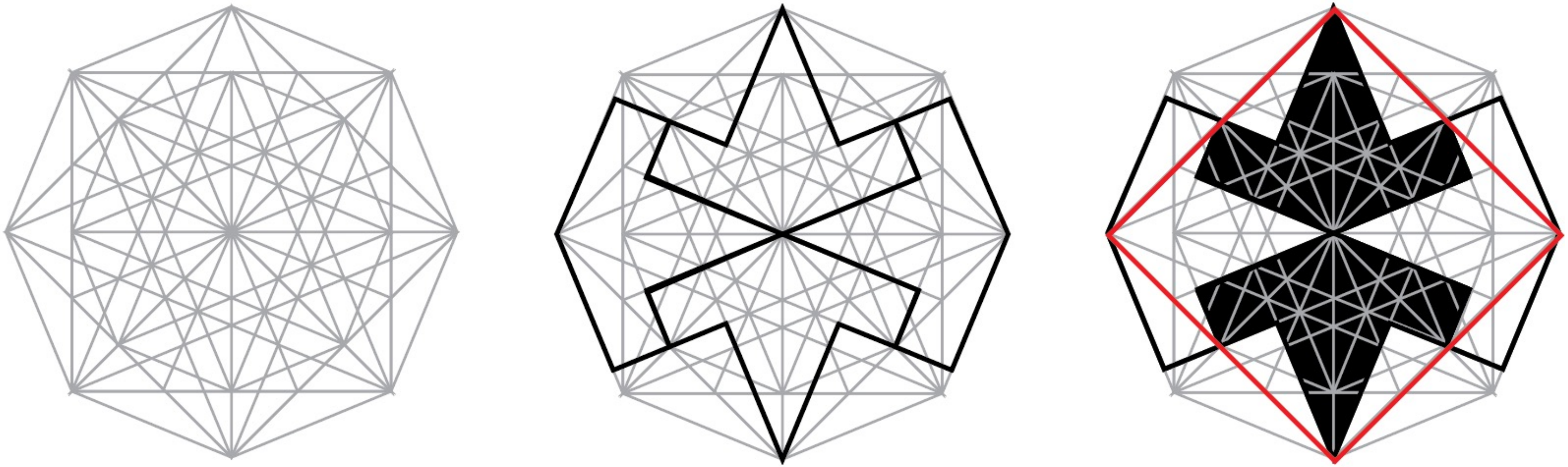
More examples from Alhambra (some look quite contemporary)

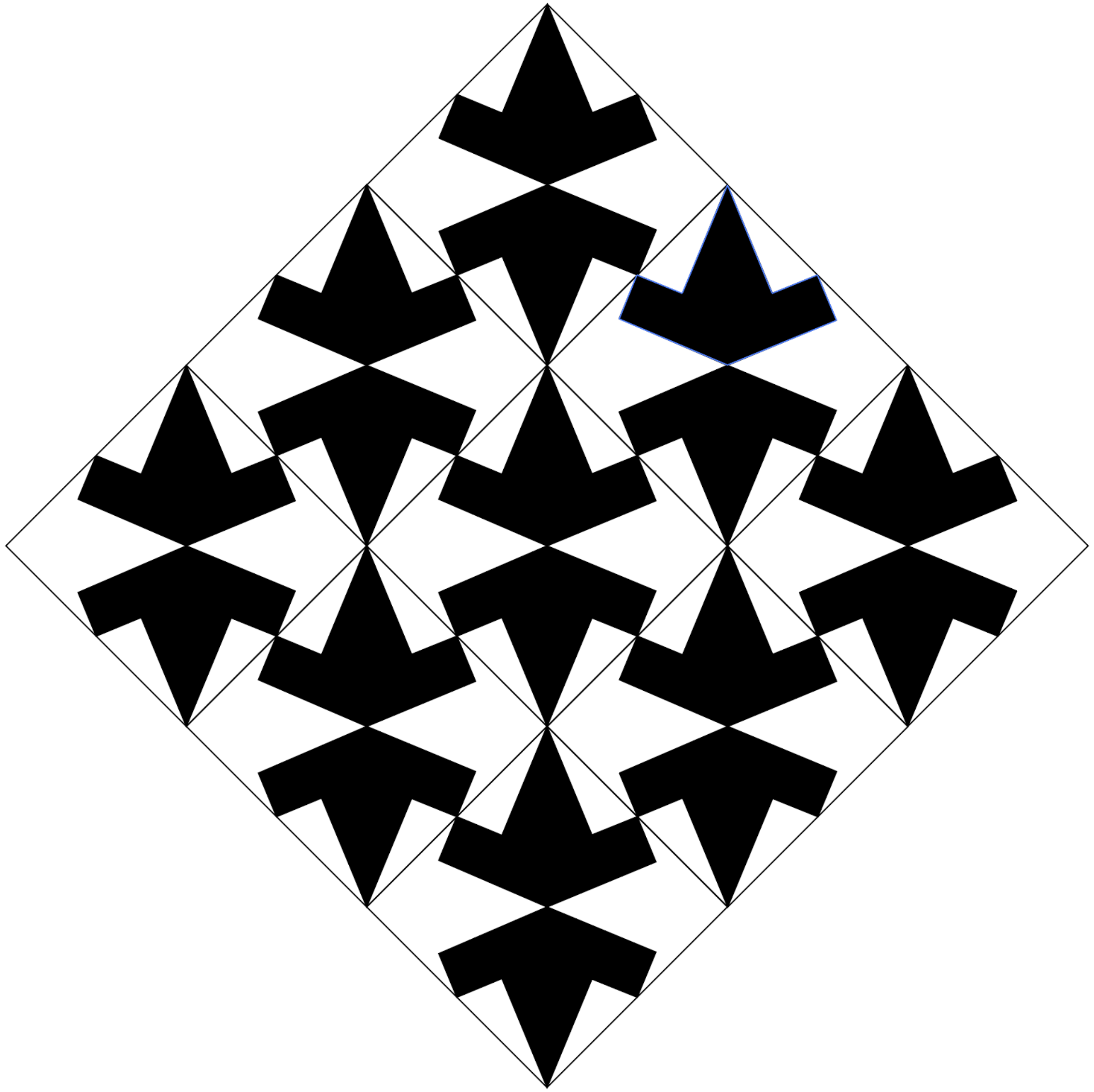
More examples from Alhambra

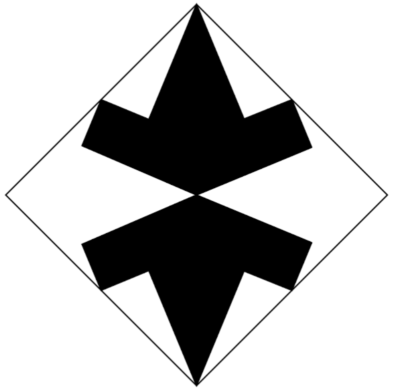
More examples from Alhambra



More examples from Alhambra (Art Deco?)







Spain (Seville):

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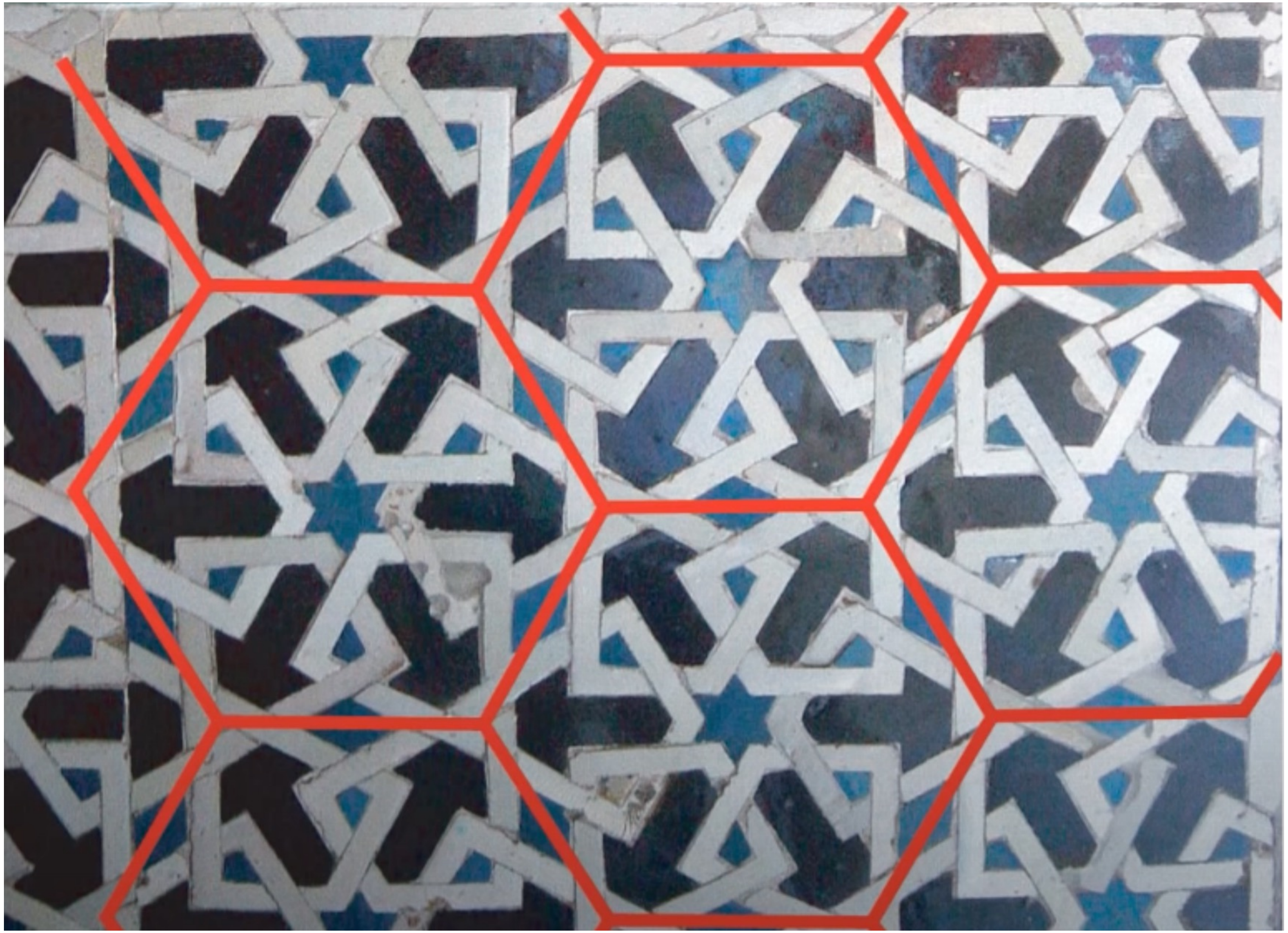
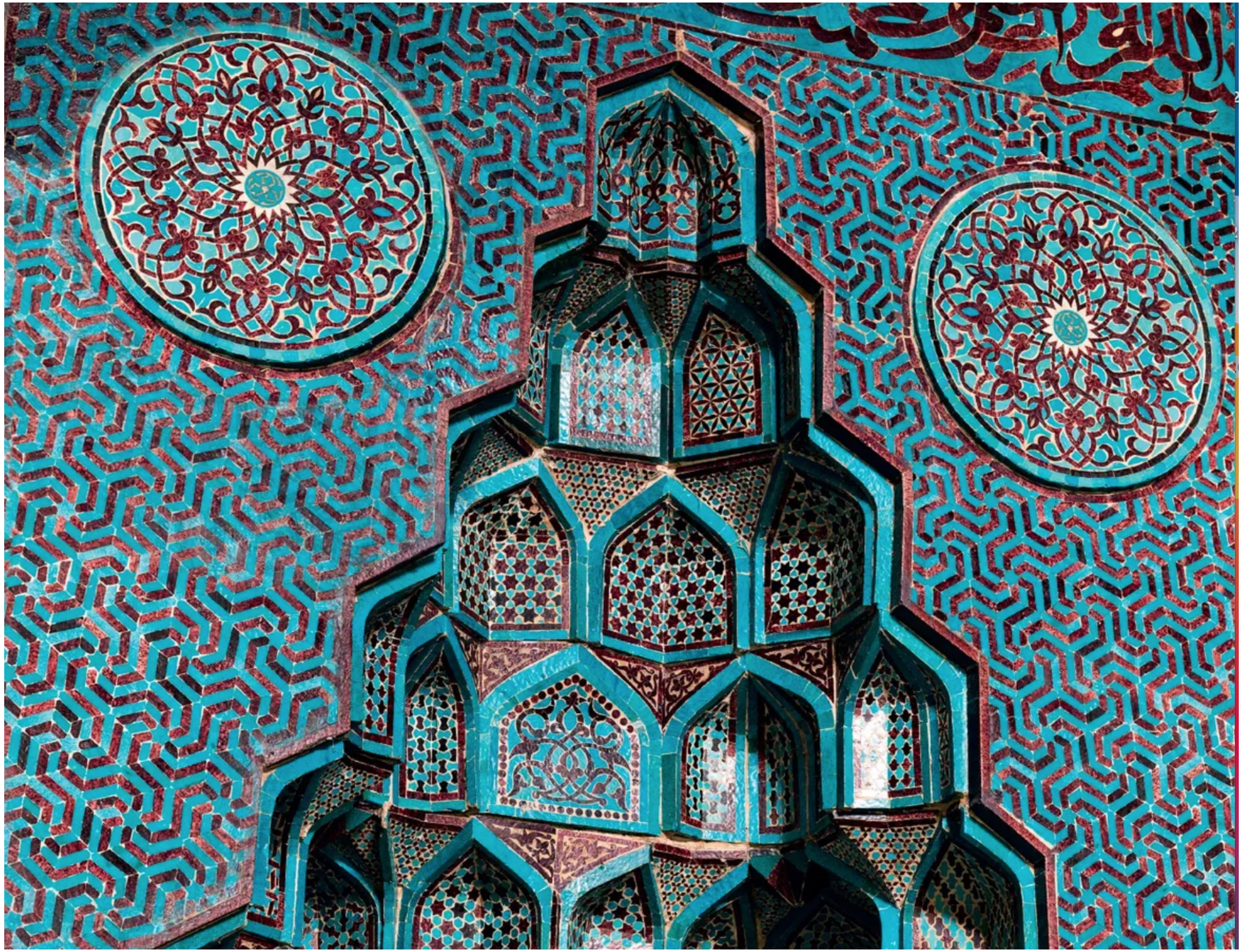
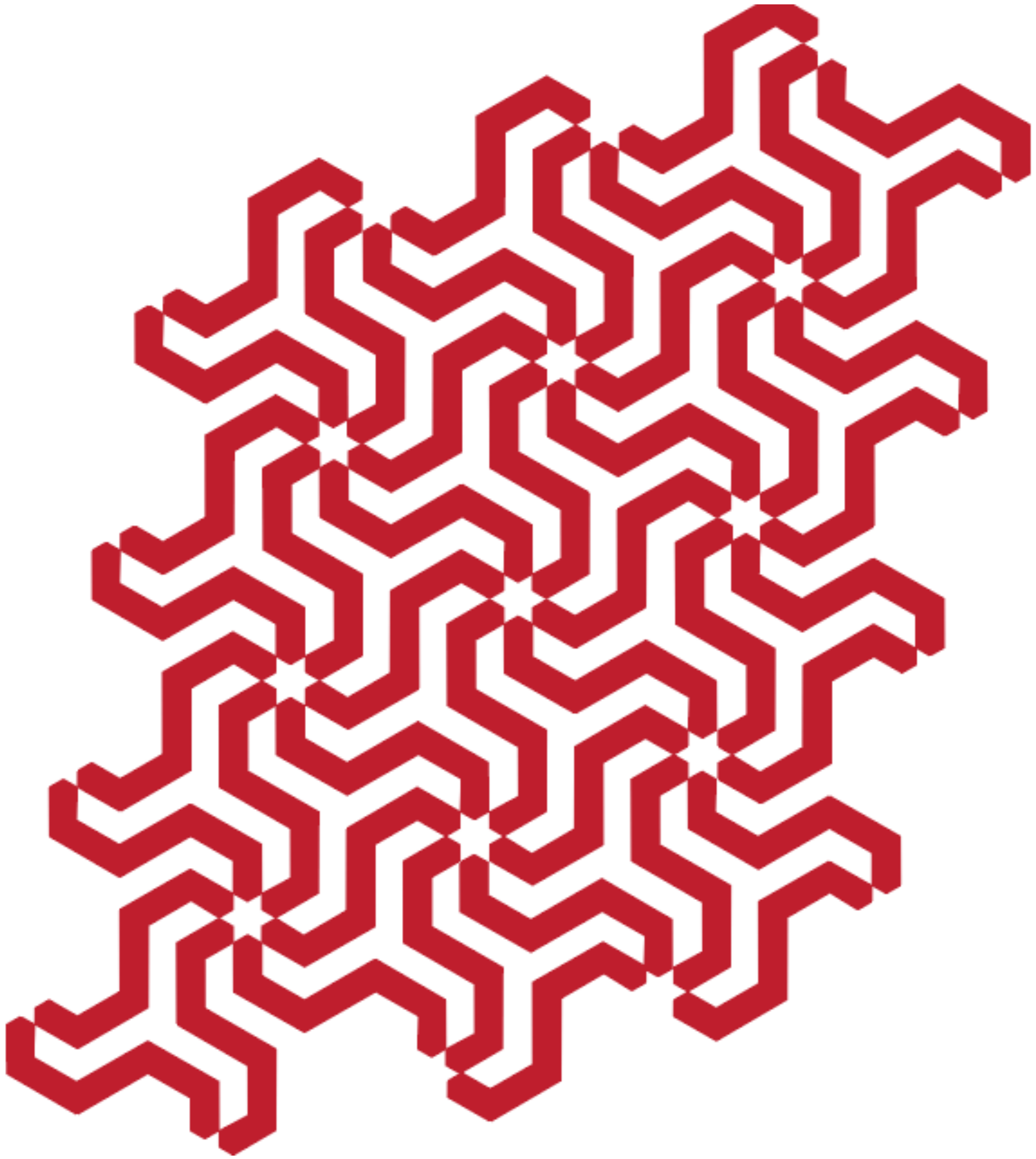


Jame Mosque, Yazd, Iran



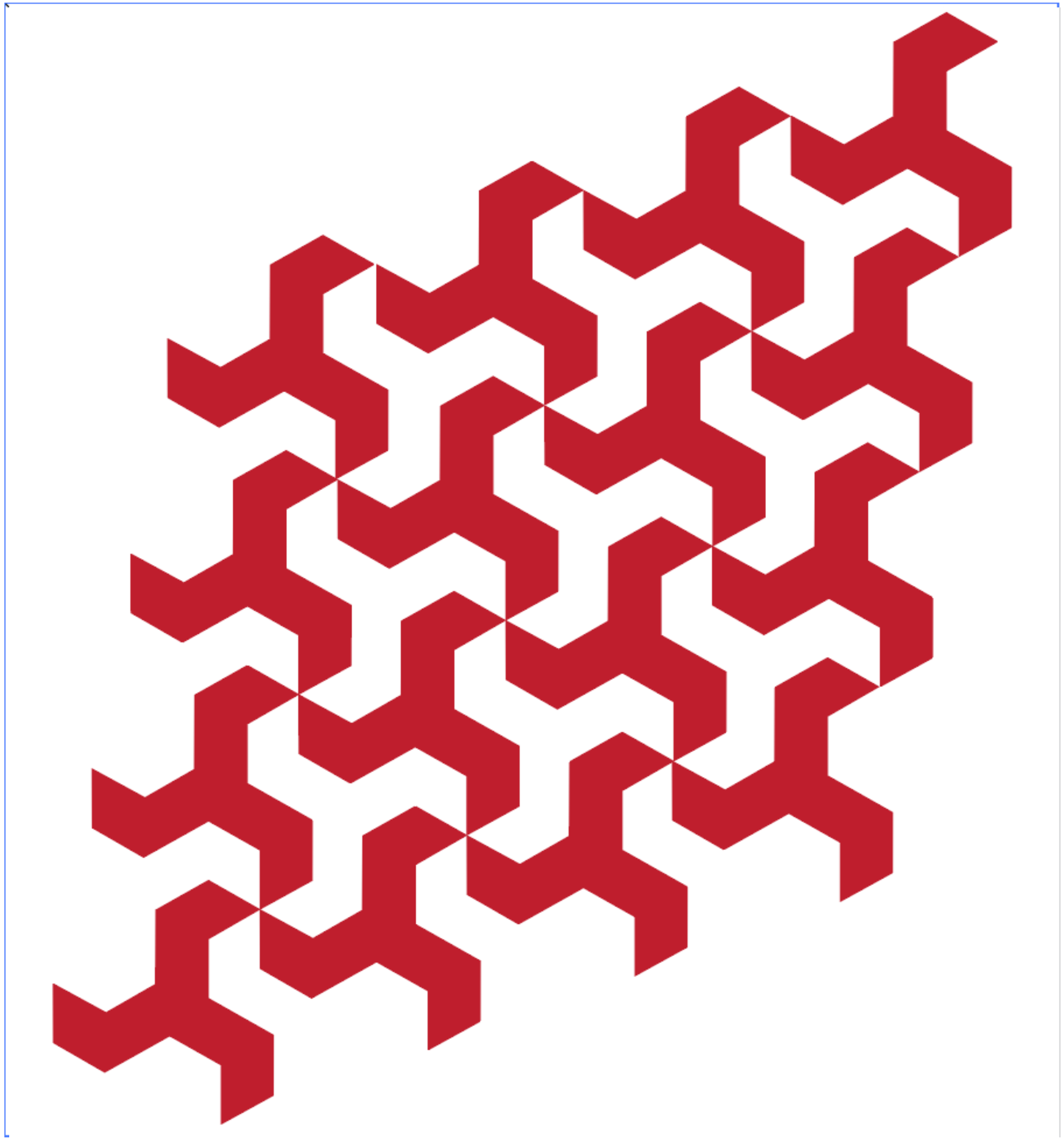
Jame Mosque,
Yazd, Iran

(Similar to one of
Alhambra patterns)



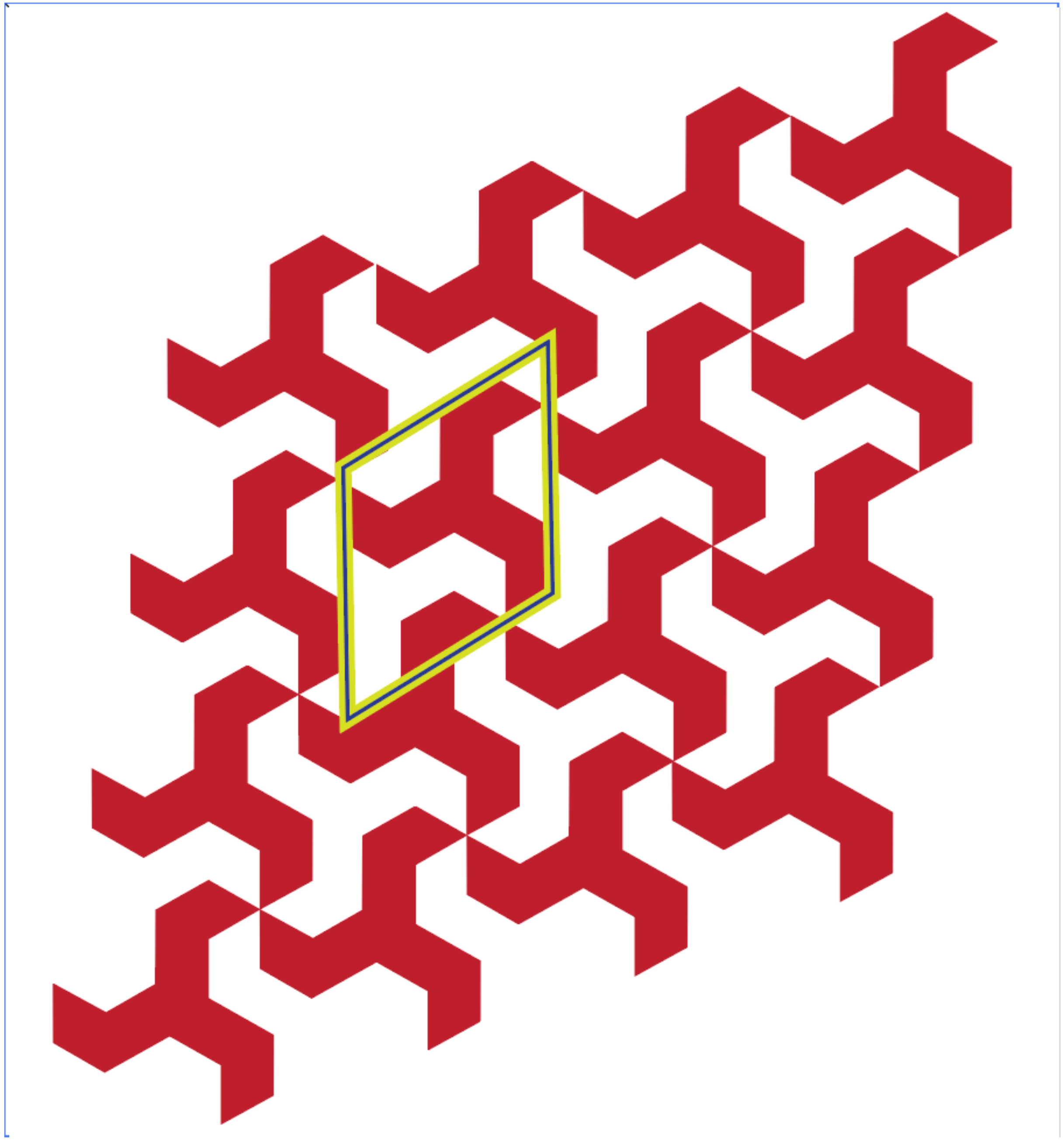
Jame Mosque,
Yazd, Iran

(Similar to one of Alhambra patterns)

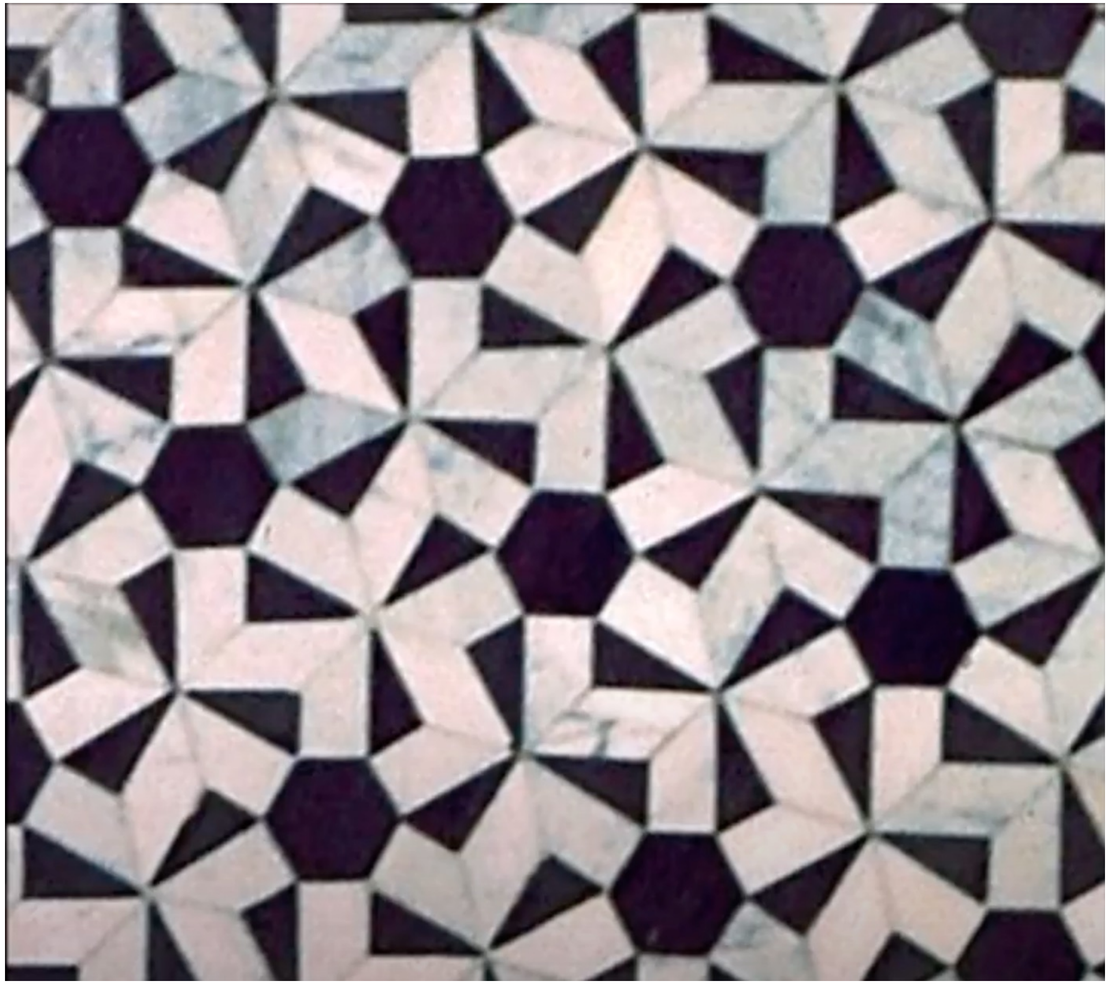


Jame Mosque,
Yazd, Iran

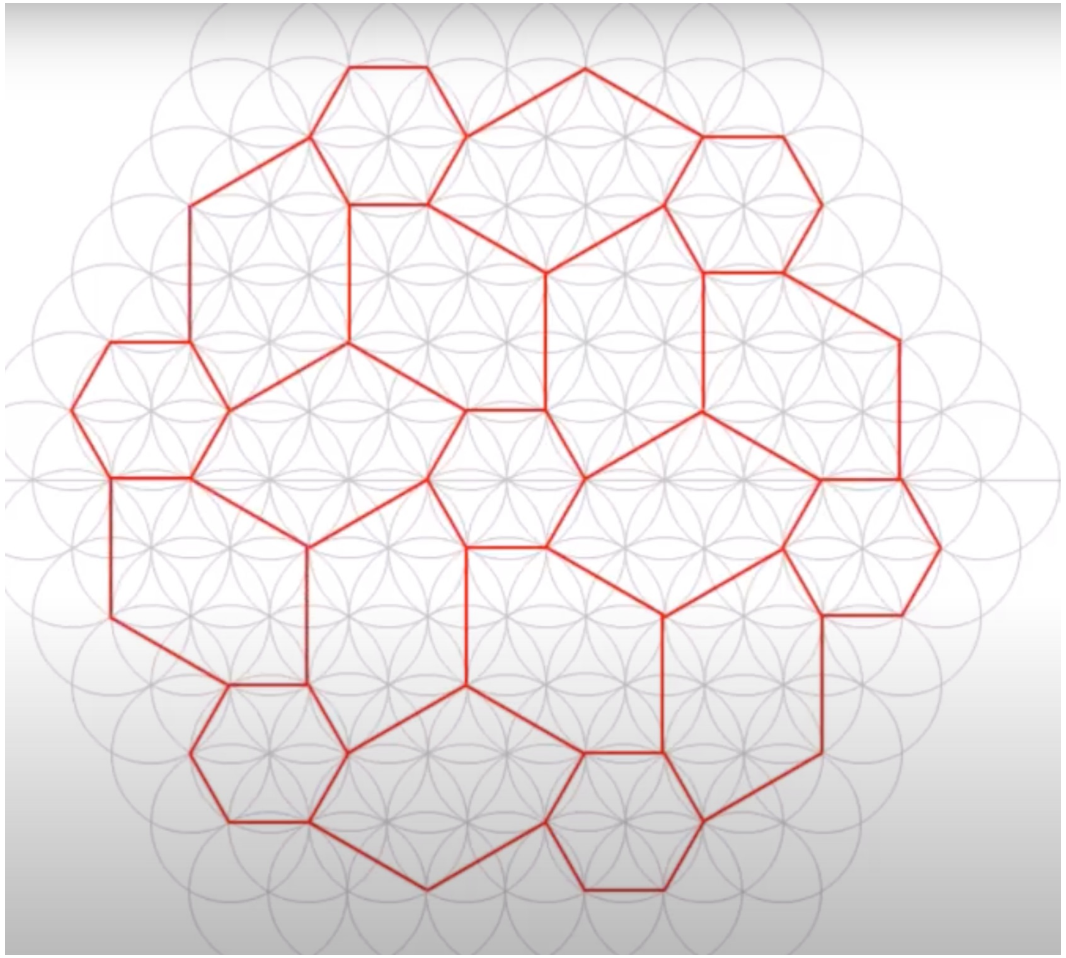
Can be tiled using a single rhombus!



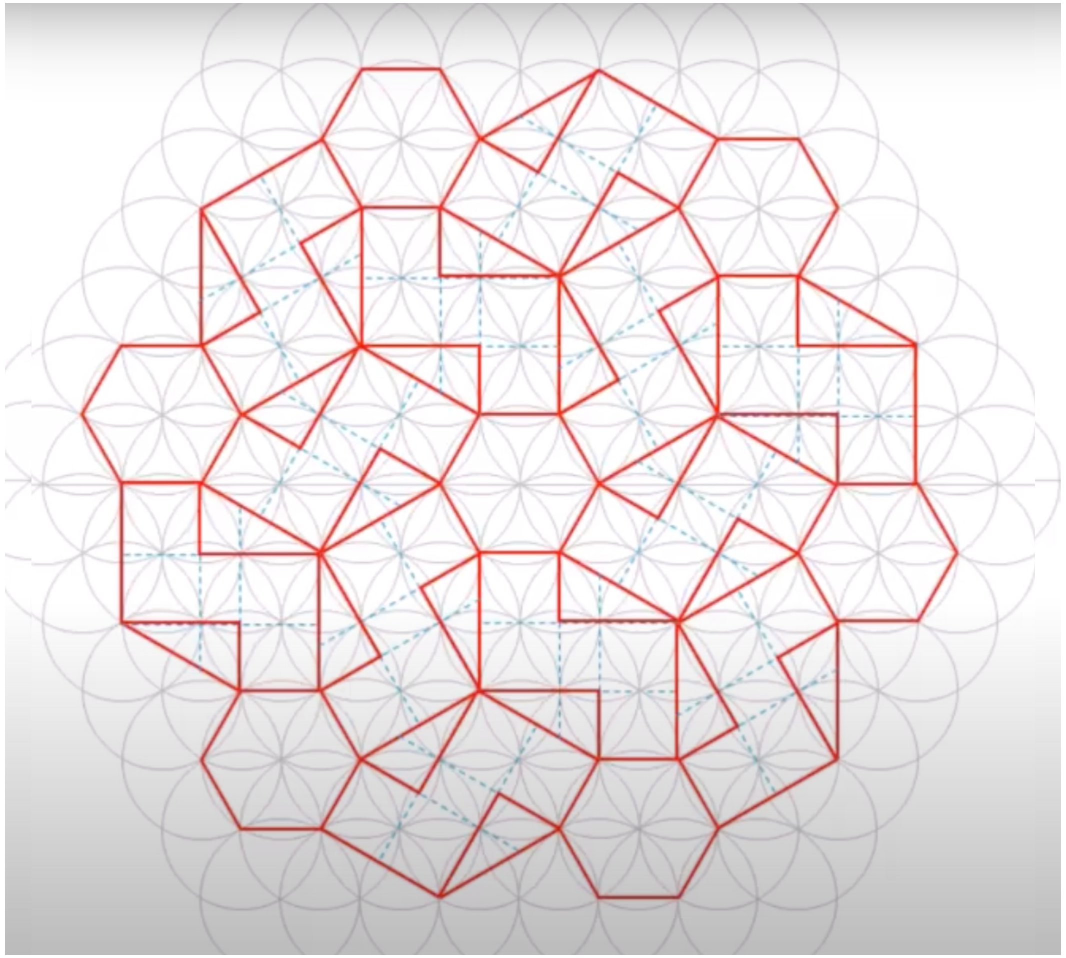
Grand mosque of Damascus



Grand mosque of Damascus

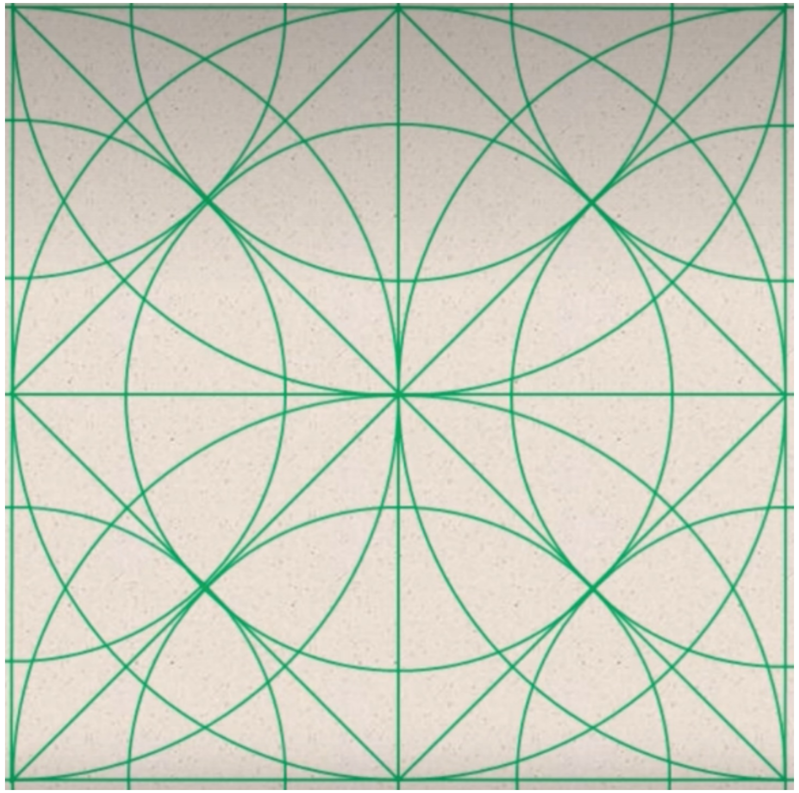


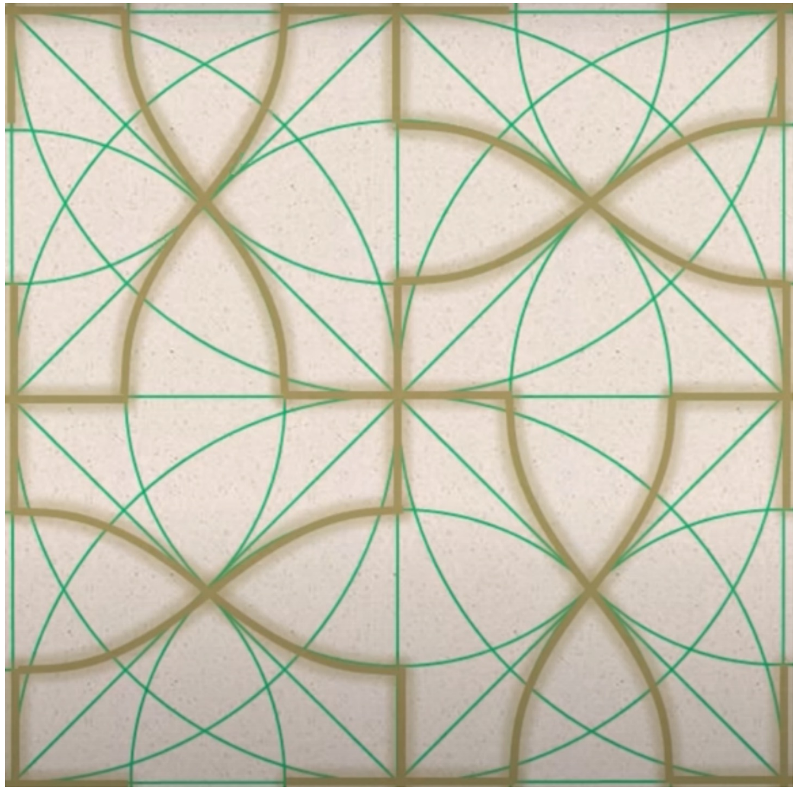
Grand mosque of Damascus



Alhambra

Alhambra





How can we understand or classify all these patterns?

The key is ***symmetry***.

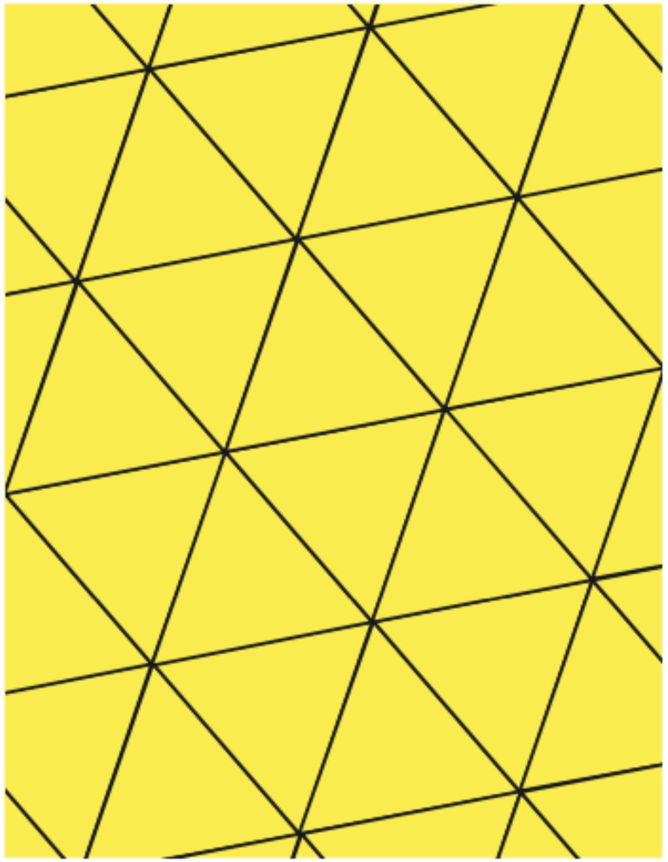
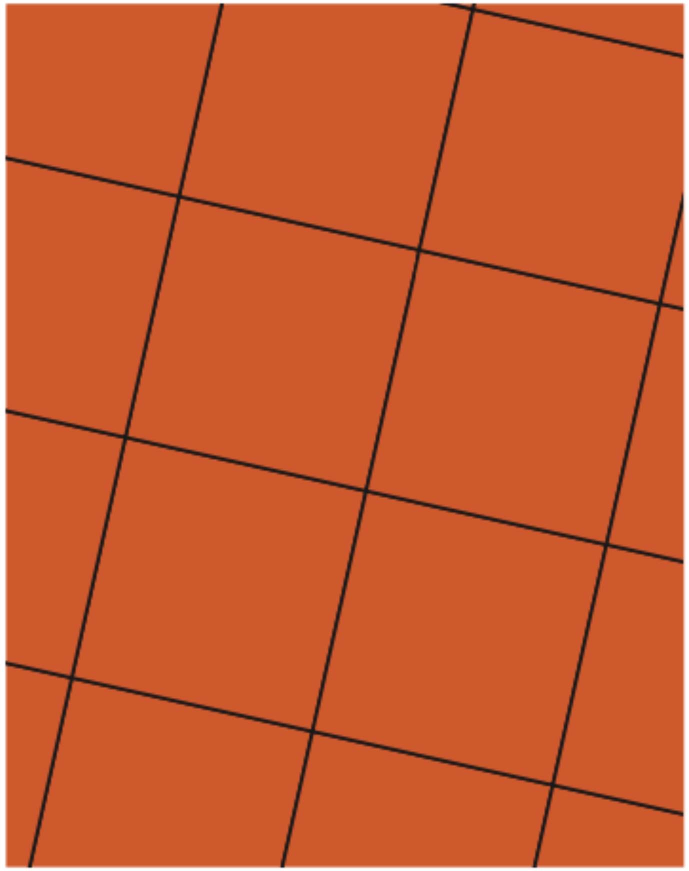
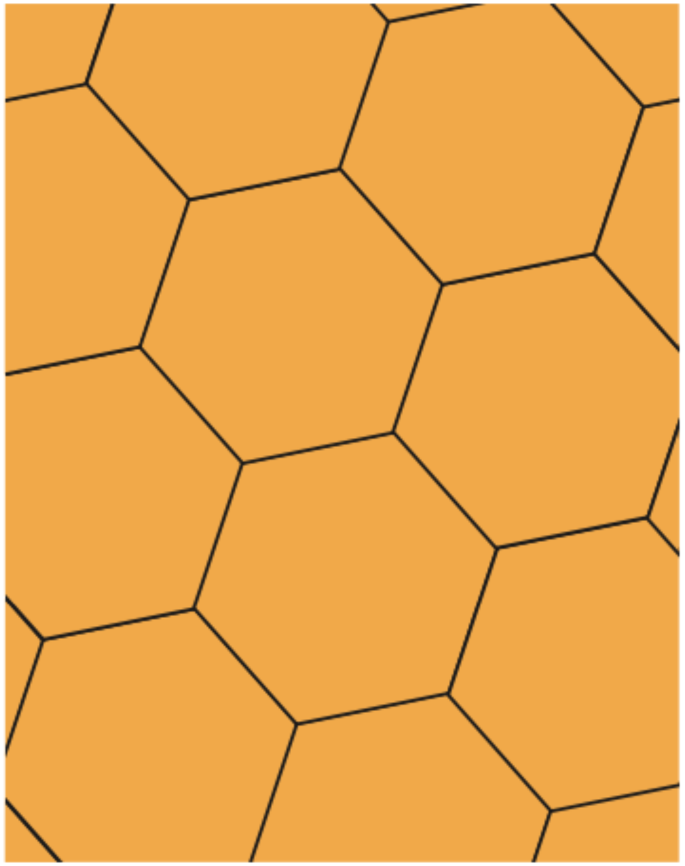
There are 17 possible symmetry patterns that a periodic tiling can have.

In traditional Western art and architecture, only a handful of these patterns were explored.

In the palace of Alhambra alone, at least 13 of these patterns are present under one roof (by some counts all can be found there).

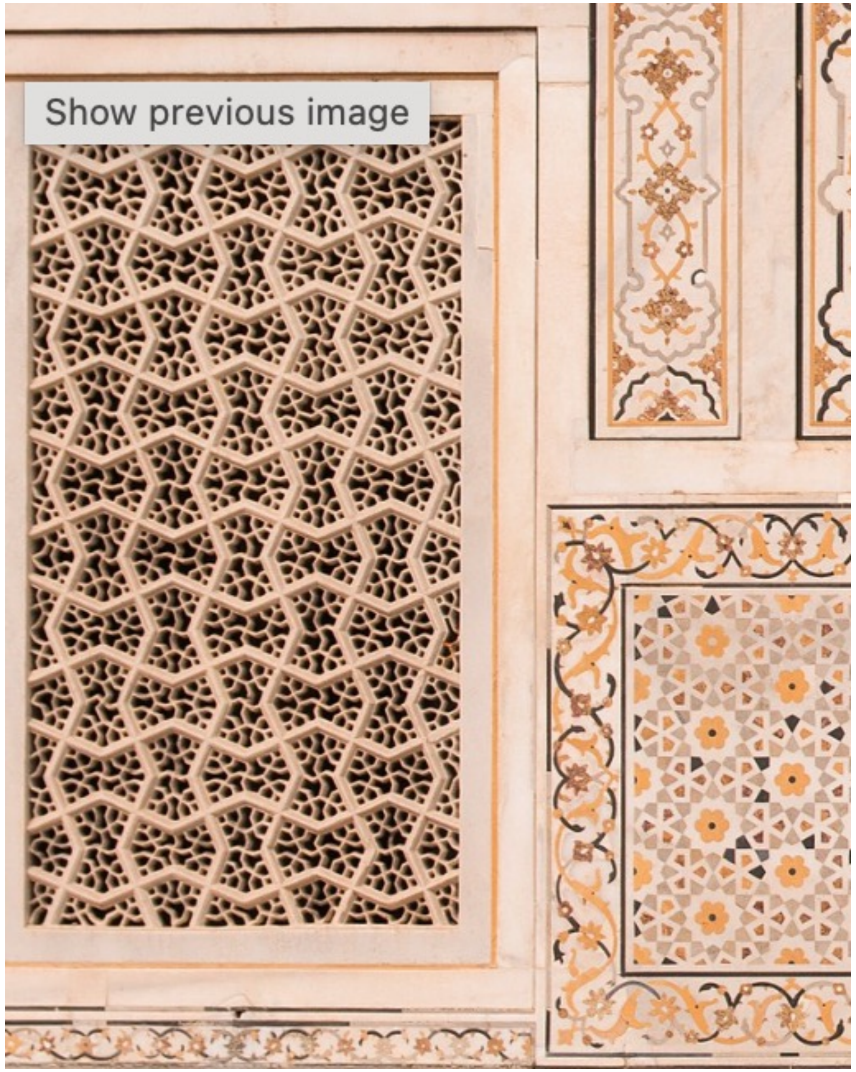
A ***tile*** is any shape which can cover the plane without overlaps or gaps.

There are only three regular convex polygons which tile the plane:

In particular regular pentagons cannot tile the plane, although there is a pleasing approximation known as Cairo tilings:

Tomb of I'timād-ud-Daulah
Agra, India.



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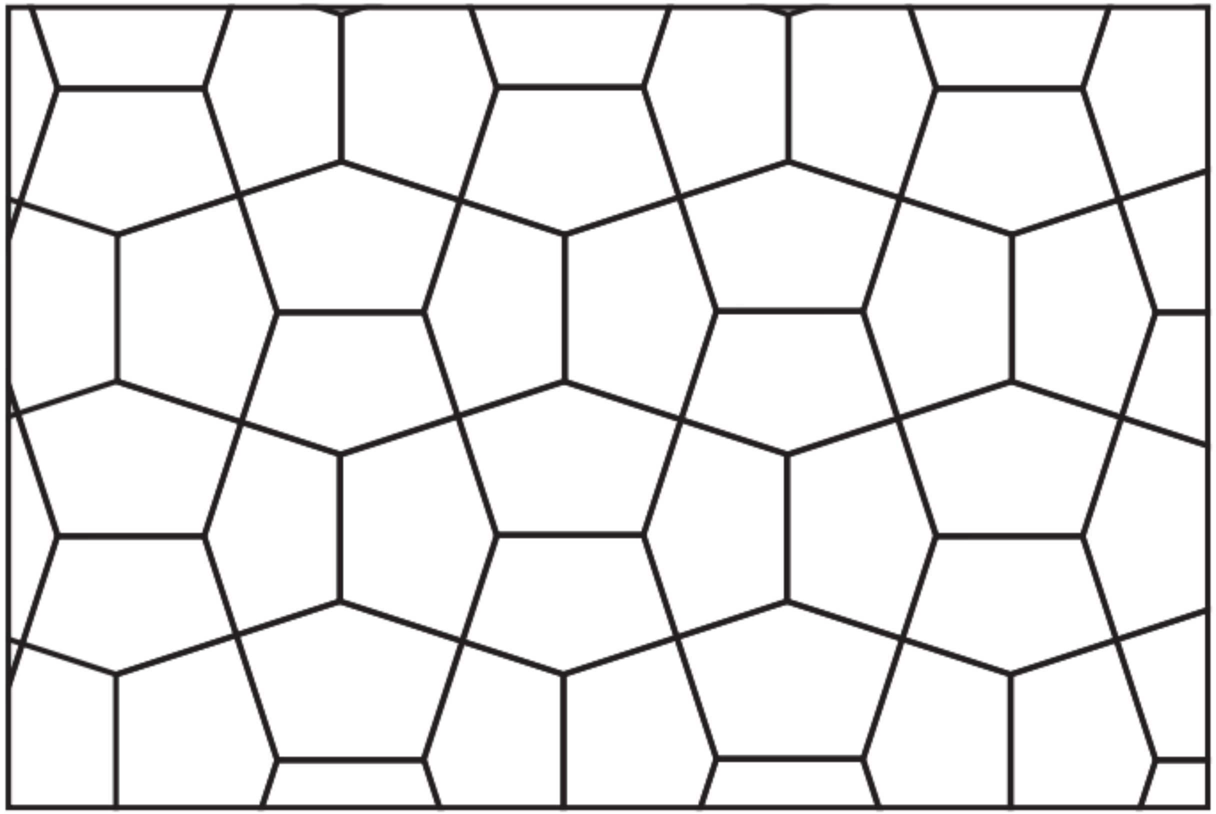
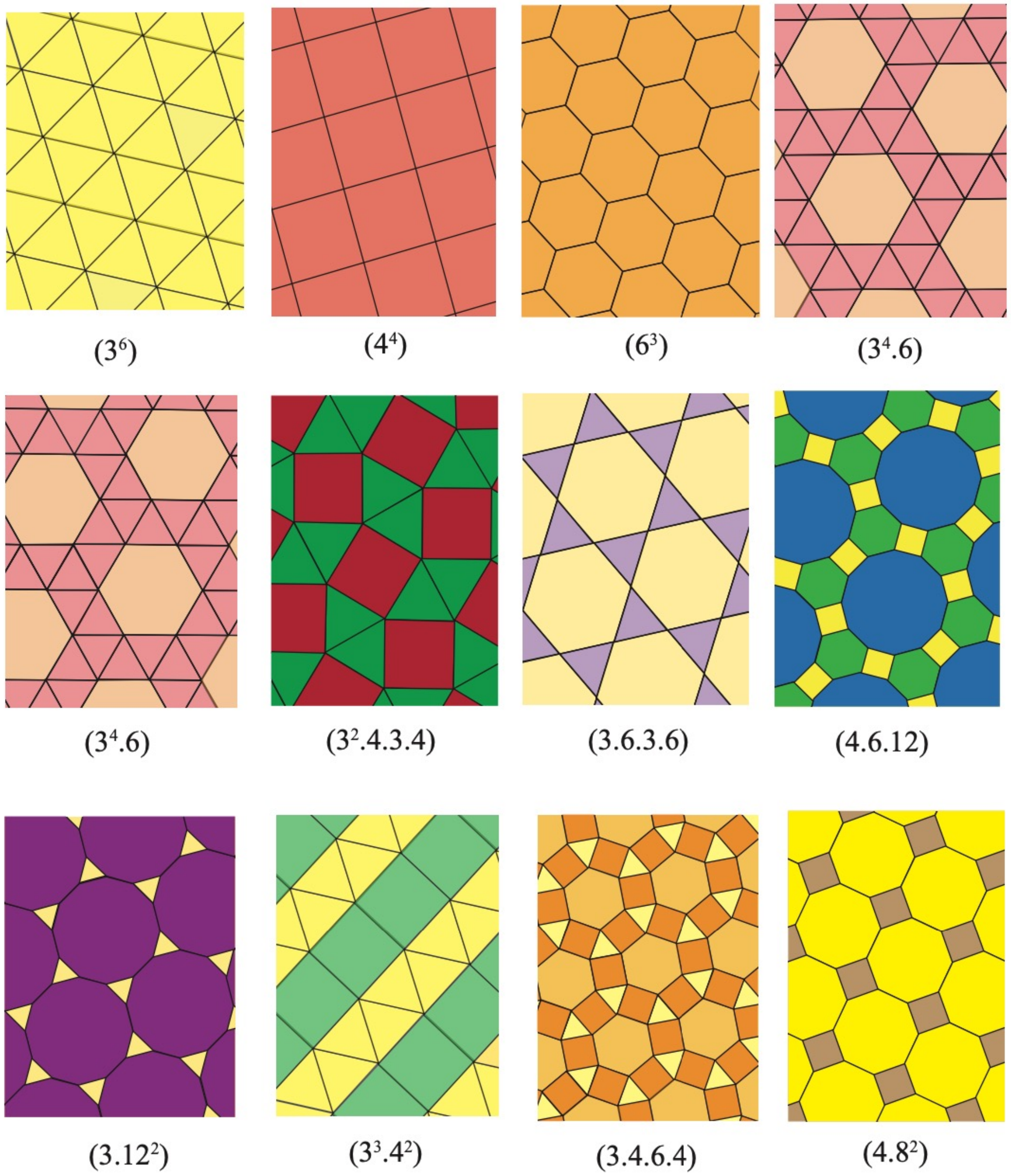
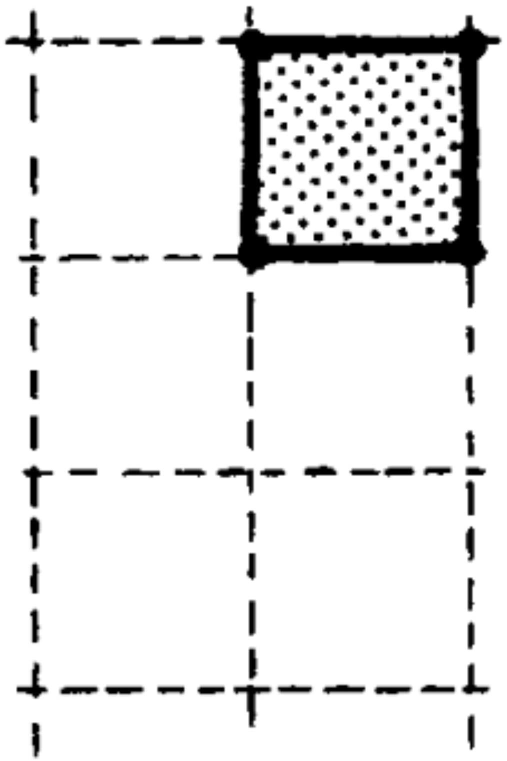


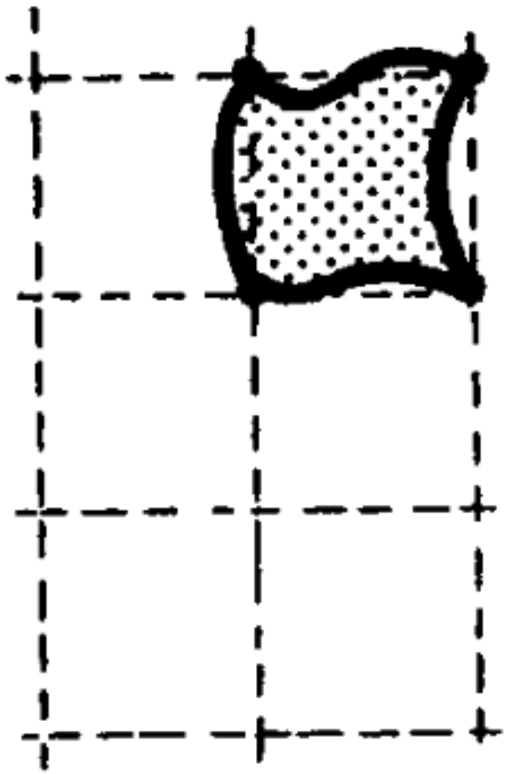
There are exactly 12 ways to tile the plane *uniformly* with multiple regular convex polygons:

(The 3.4.6.4 arrangement appears in the Cosmati pavement in Westminster Abbey.)



But there are an unlimited number of other shapes which tile the plane.

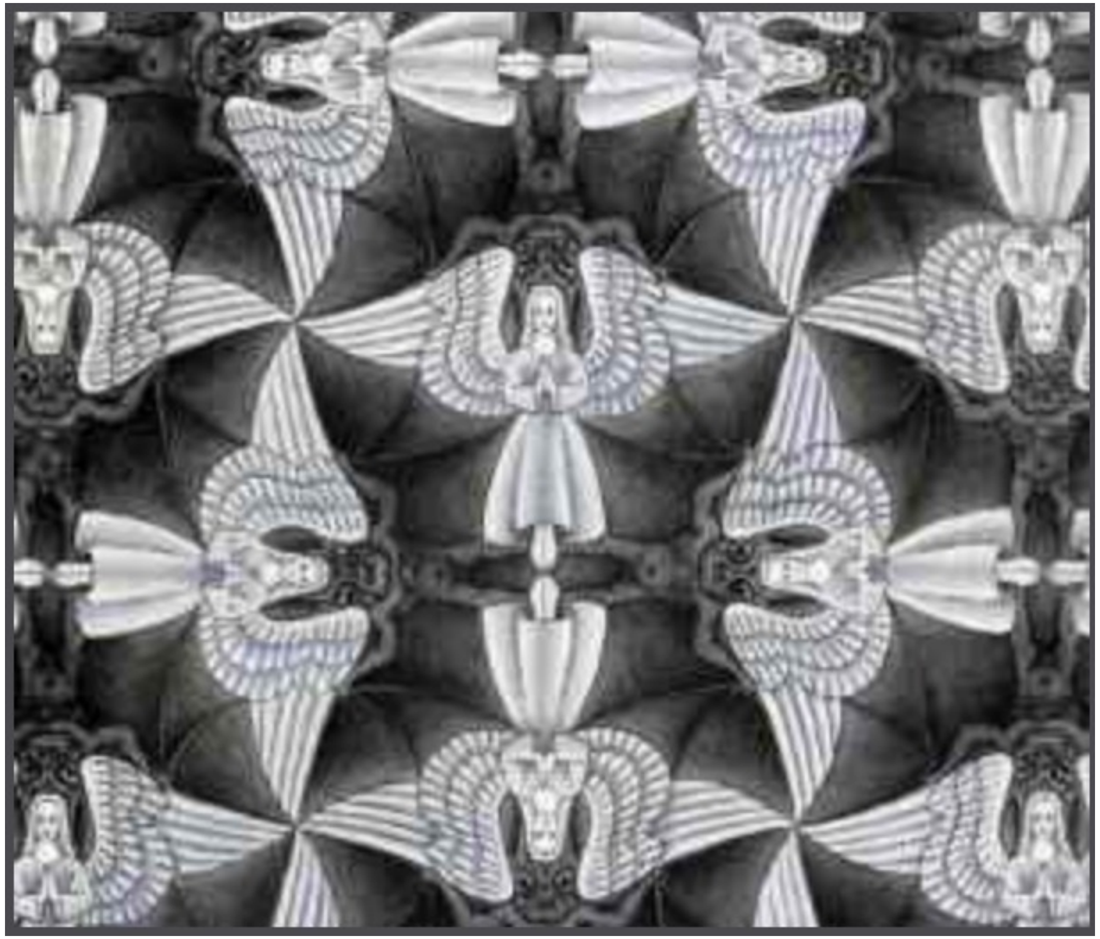




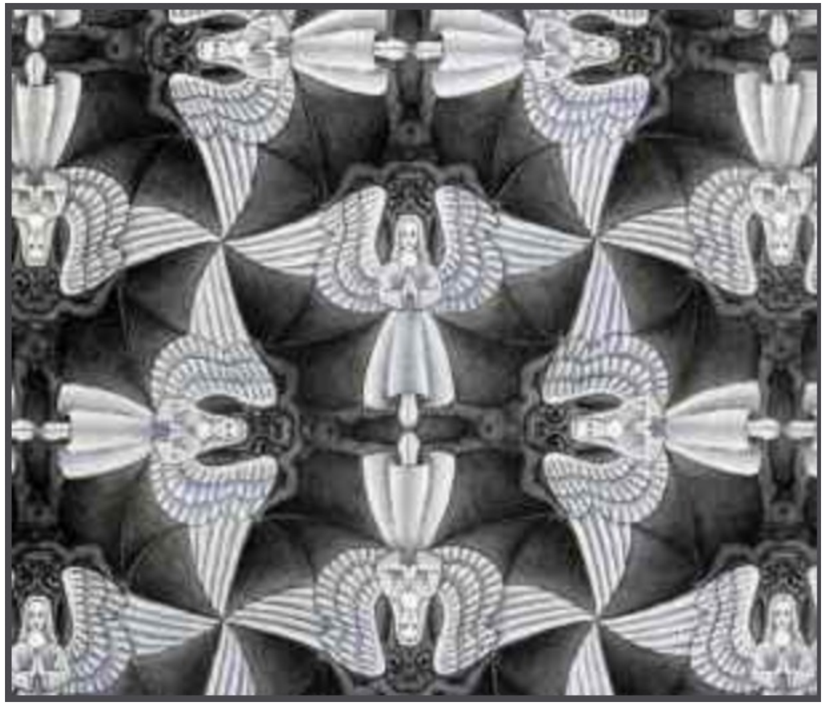
But there are an unlimited number of other shapes which tile the plane.

Famous examples by Escher

Another famous example by Escher:

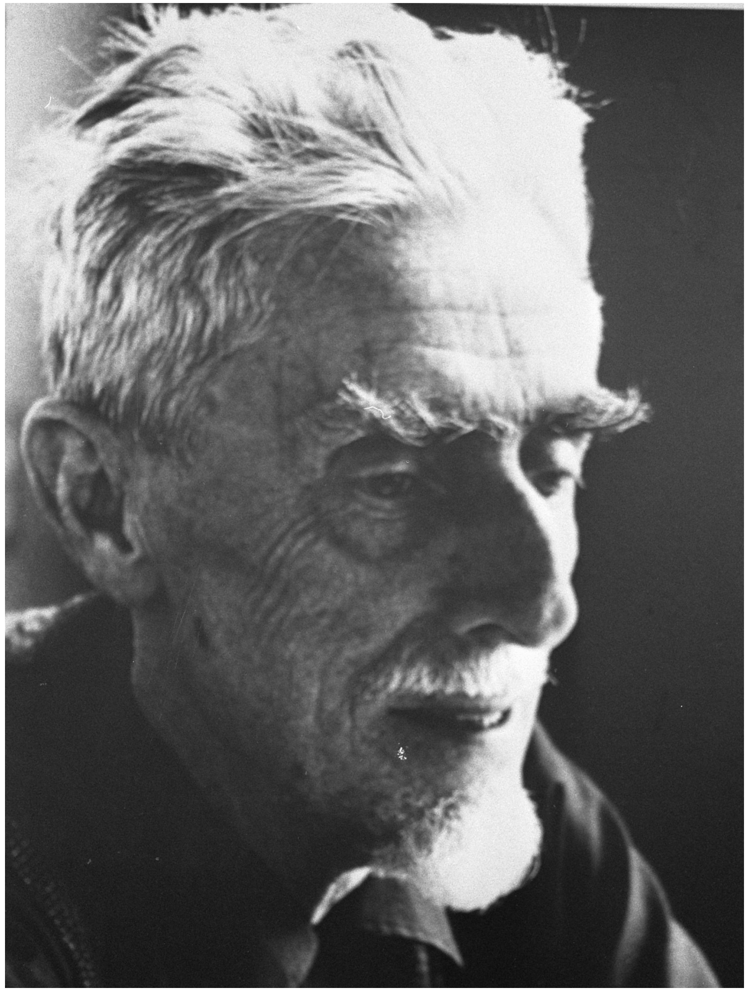


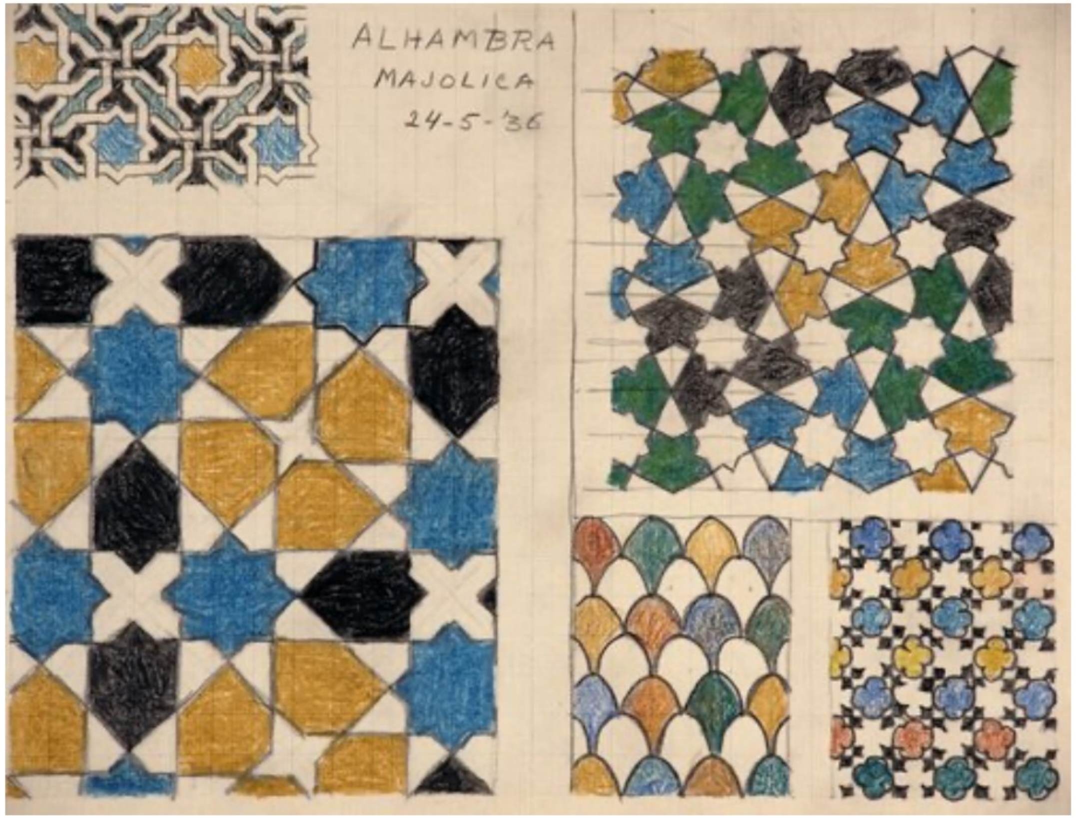
Does it look familiar?



"I can't say how my interest in the regular division of planes originated and whether outside influences had a primary effect on me. My first intuitive step in that direction had already been taken as a student ... before I got to know the Moorish majolica mosaics in the Alhambra, which made a profound impression on me."

—M.C. Escher, 1941



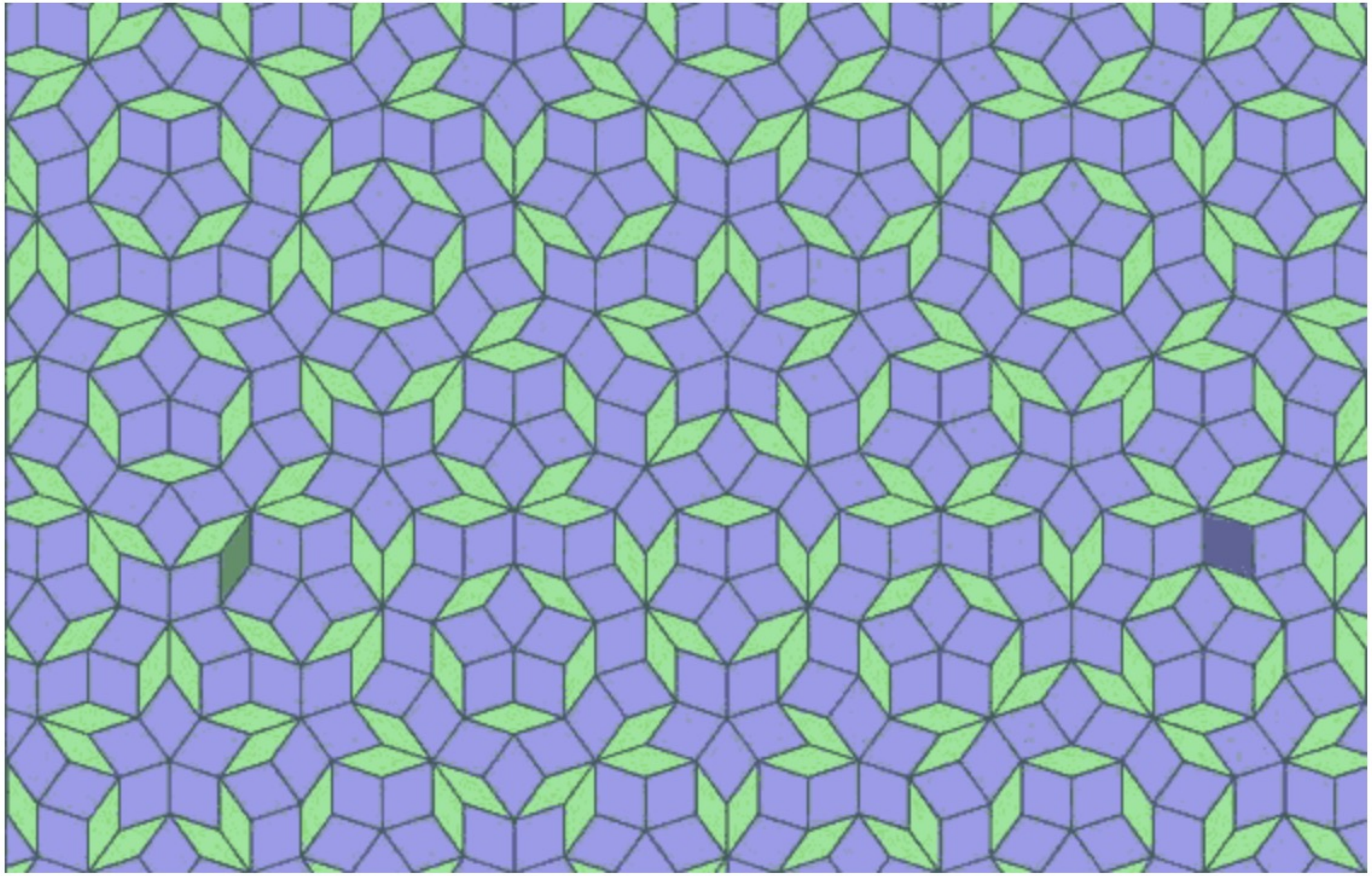


Hand drawings of Alhambra tilings by Echer

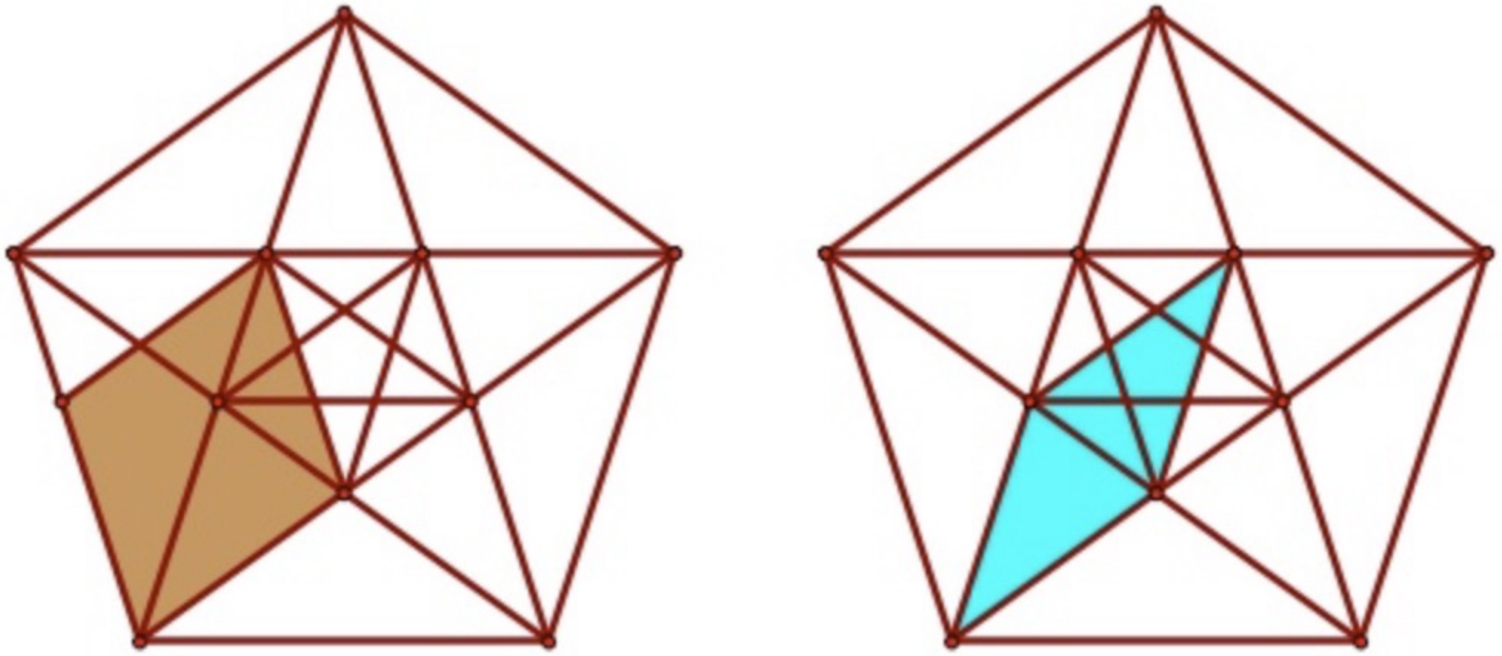
In this talk we are interested in ***periodic*** tilings, i.e., tilings which follow some repetitive pattern.

Not all tilings are periodic:

Penrose tiles,
discovered by
Roger Penrose in 1970s,
composed of only two
diamond Shapes!

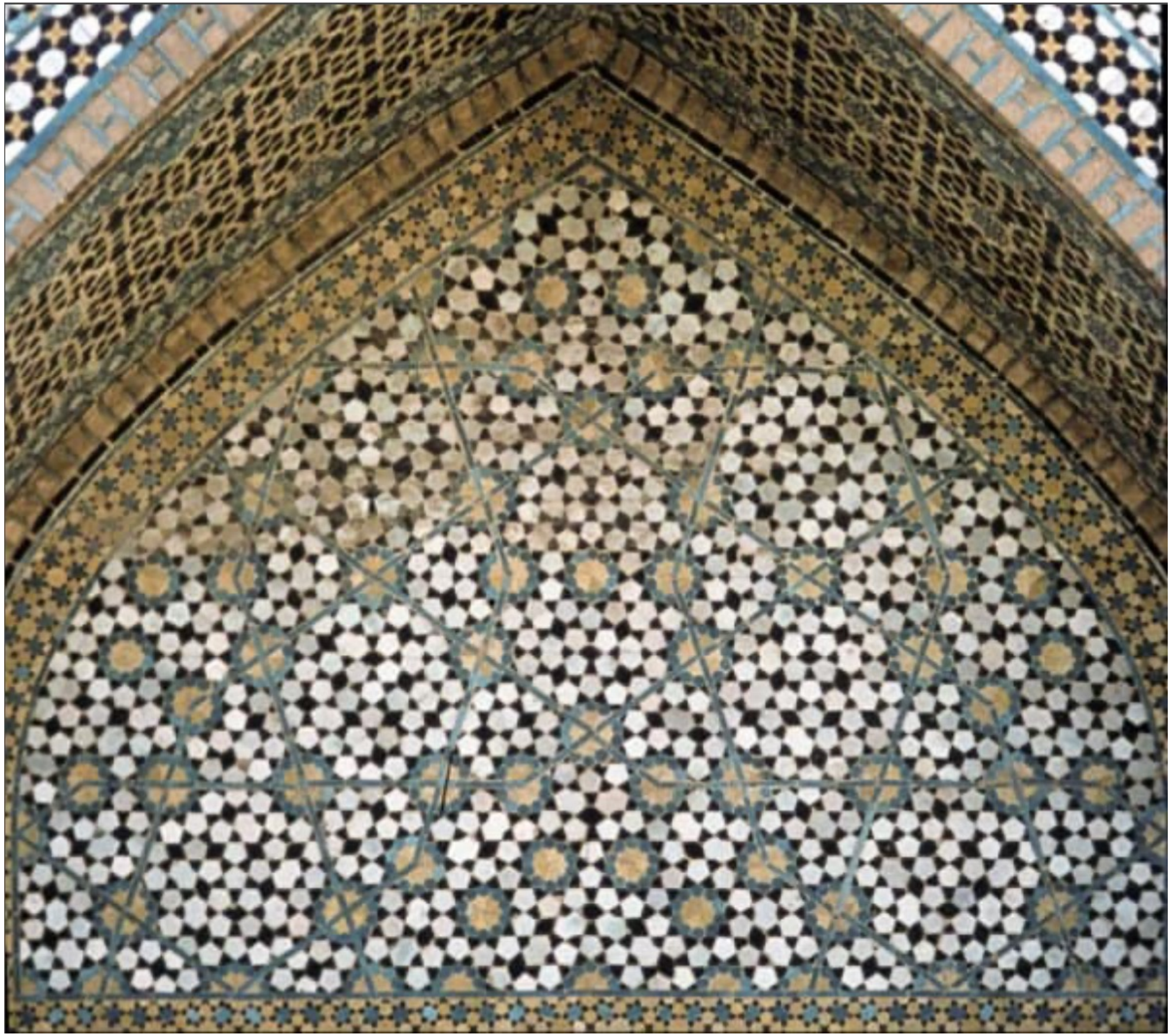


The two tiles must have exact measurements:



Examples of Penrose type non-periodic tilings had already appeared in Darb-i-Iman shrine in Isfahan, Iran

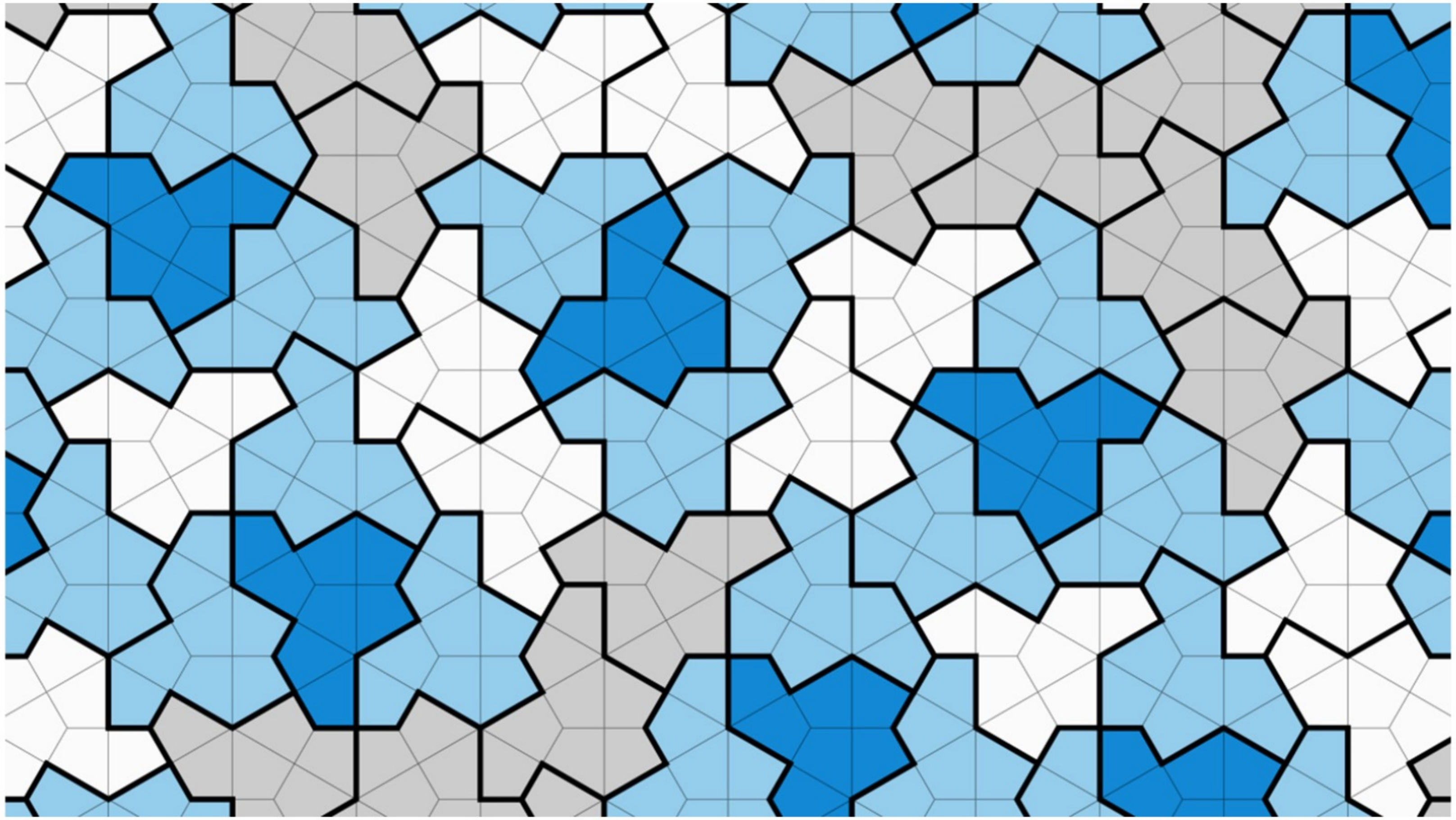
in the 15th century!



Not all tilings are periodic:

There are even examples with a single tile!!

Discovered by David Smith in **2023**.

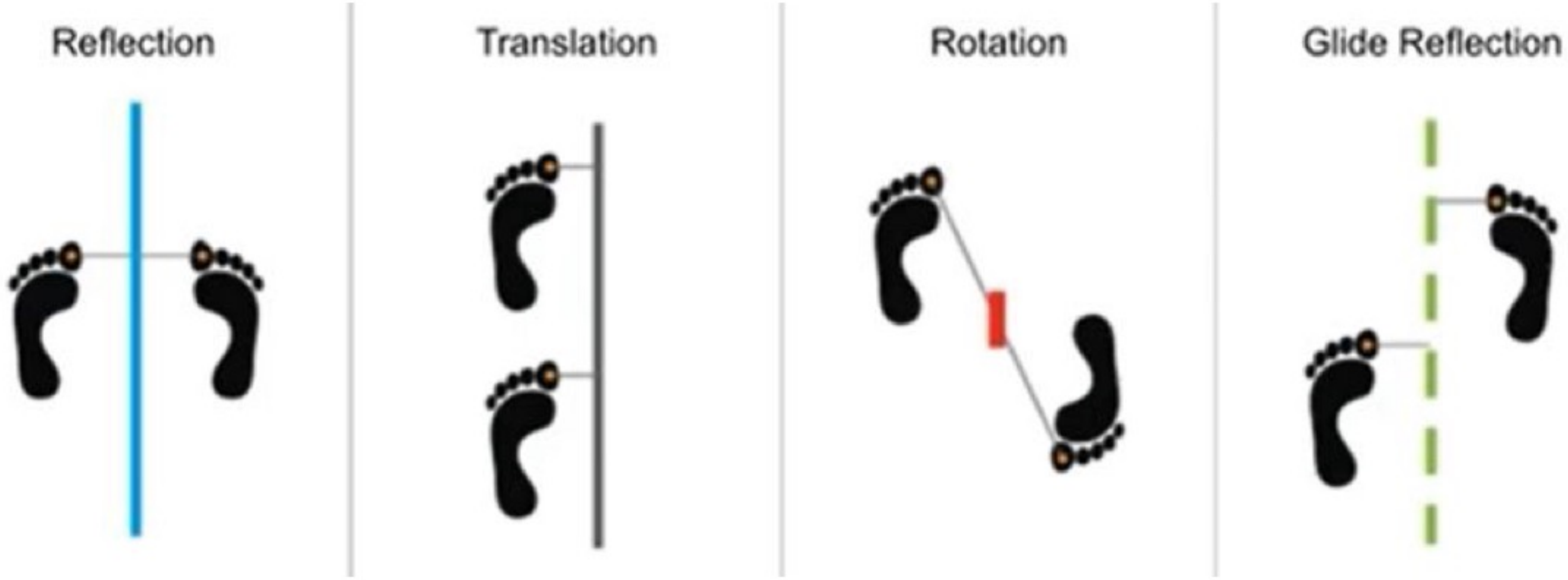


A tiling is ***periodic*** if it is invariant under a group of

isometric transformations of the plane.

An isometric transformation is a mapping of the plane to itself which preserves the distance between all pairs of points.

Isometric transformations consist of ***translations, rotations, reflections,*** and any combination of these operations.

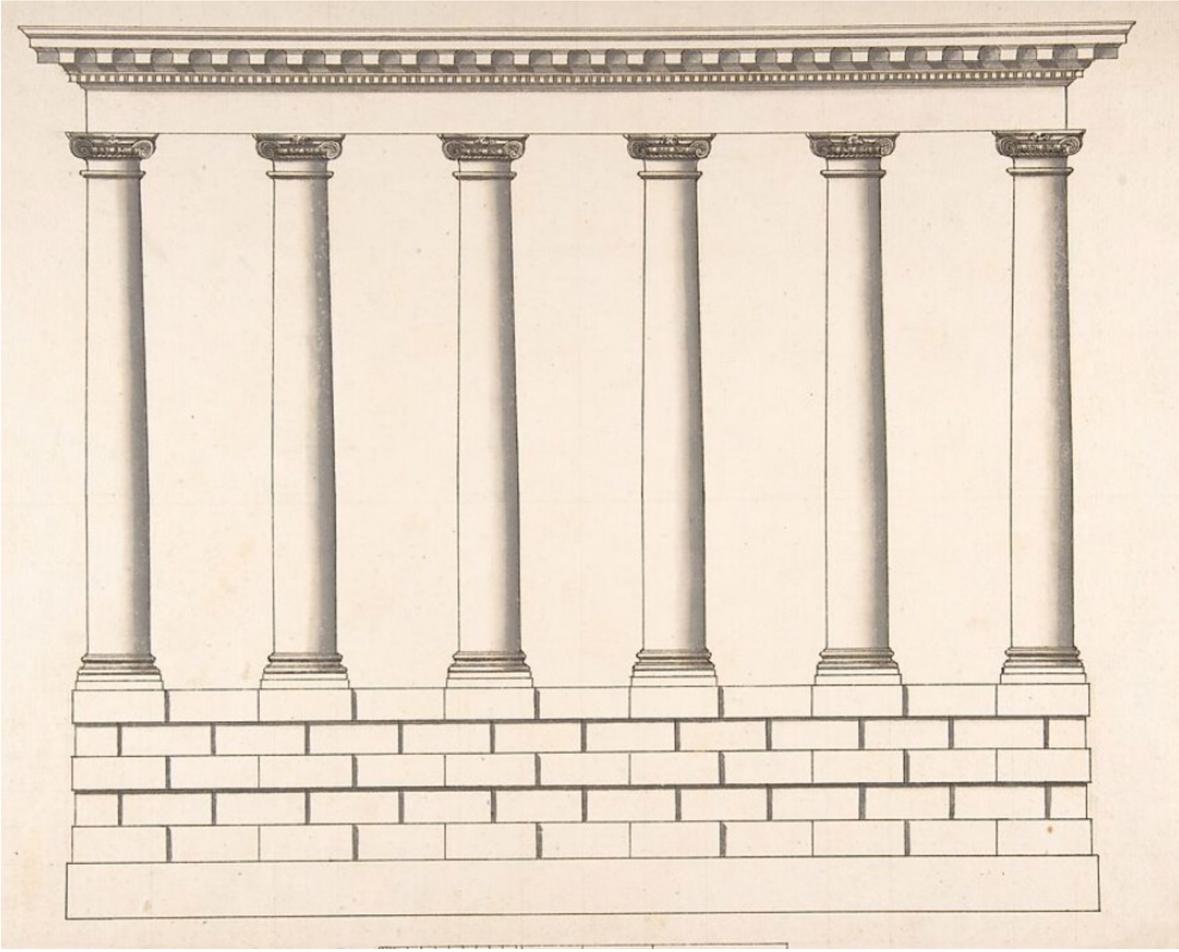


Reflection
Translation
Rotation
Glide Reflection

A collections of transformations form a ***group***, if the composition of any two transformations belongs to that collection.

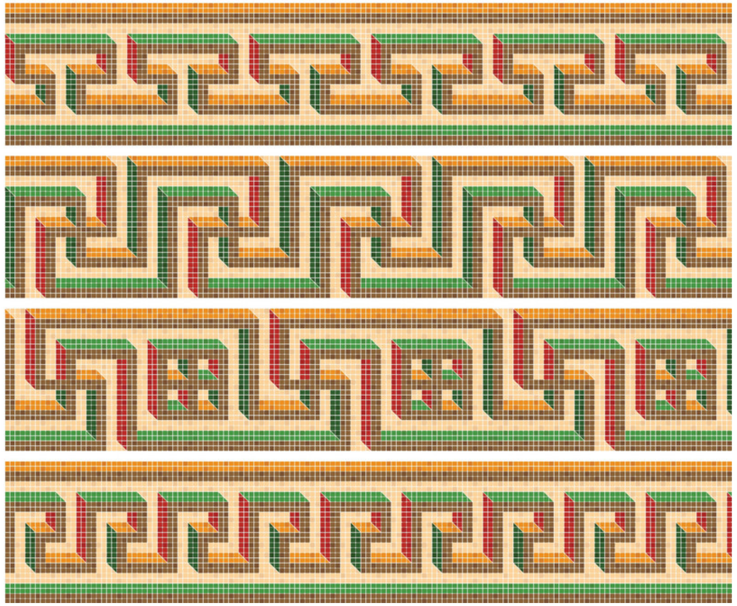
Symmetry is invariance under a group of isometric deformations.

Translational Symmetry



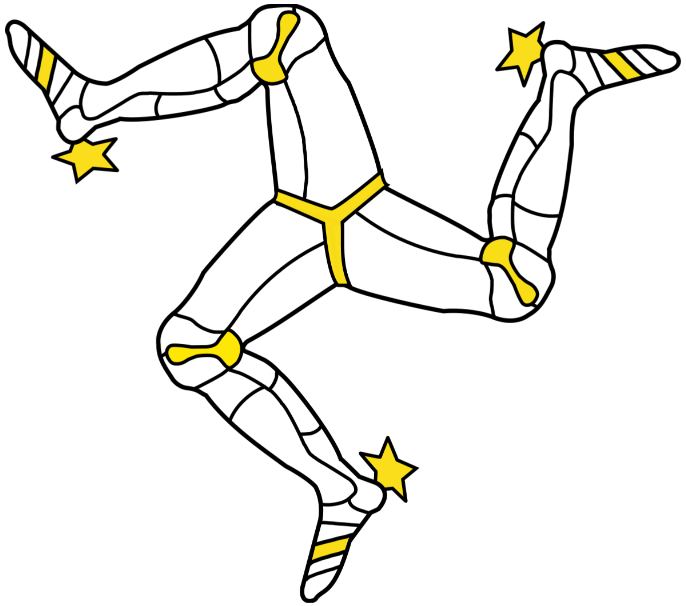
Translational Symmetry

Translational symmetry



Parthenon, Athens

Rotational Symmetry


Reflective (or bilateral) symmetry

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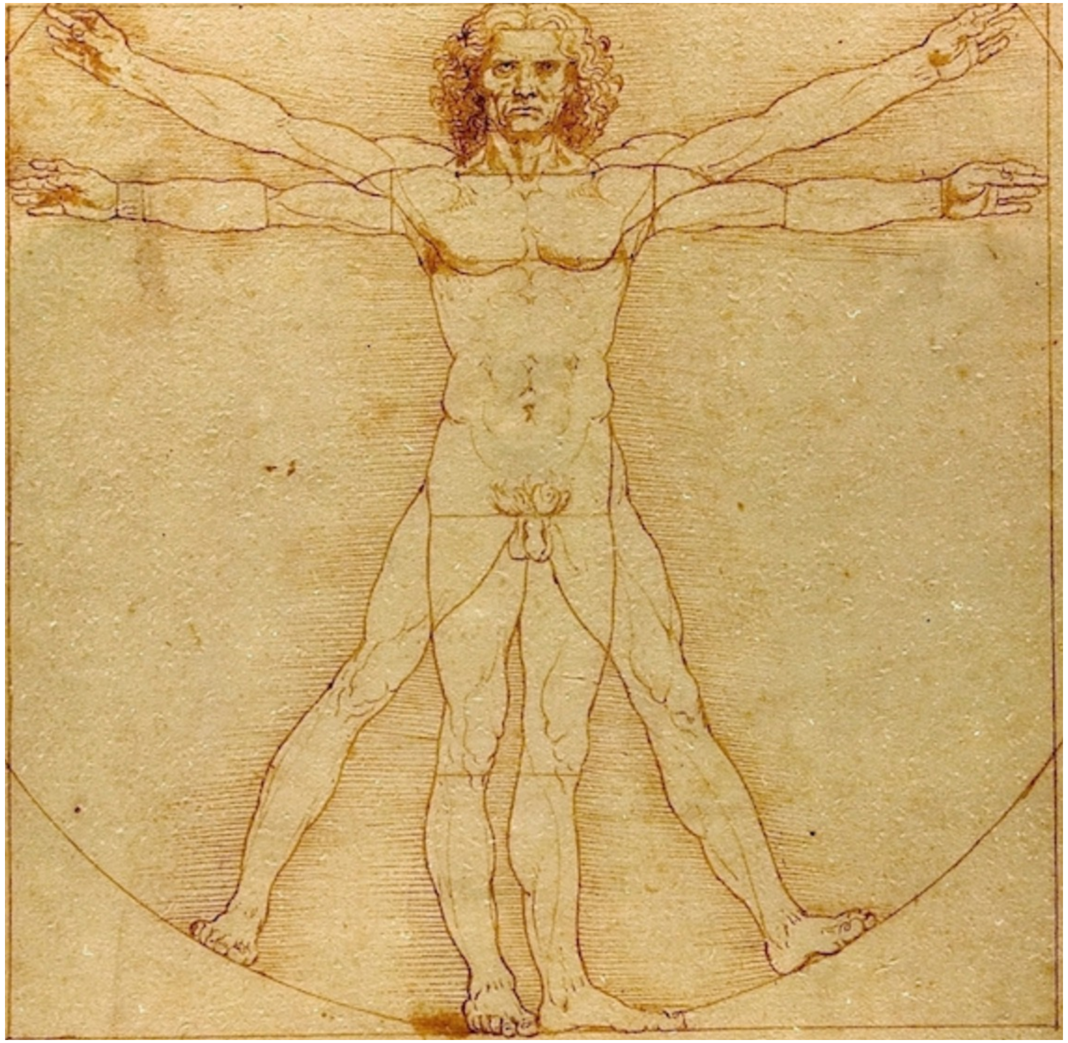
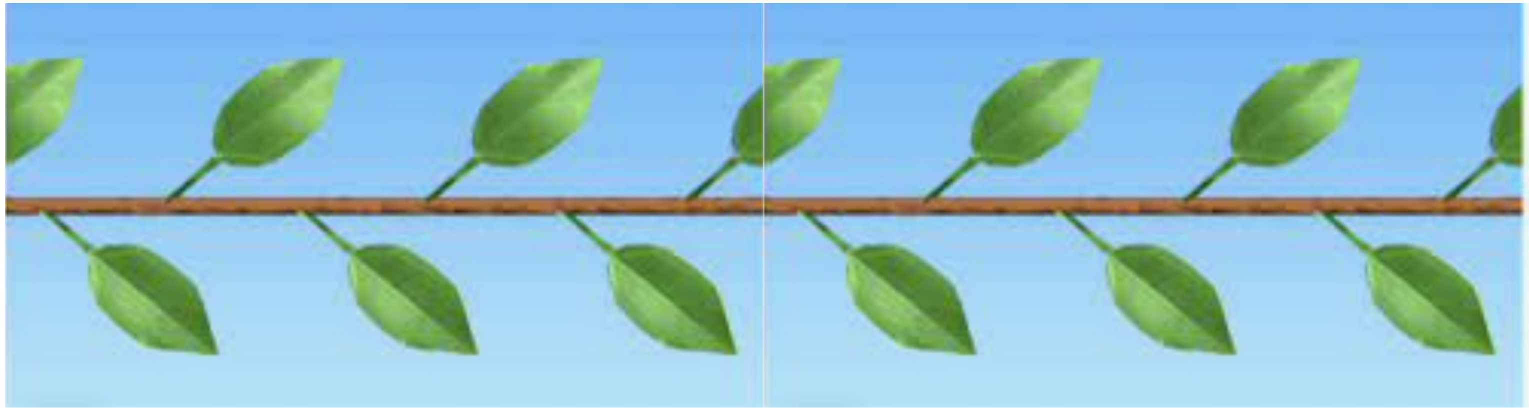
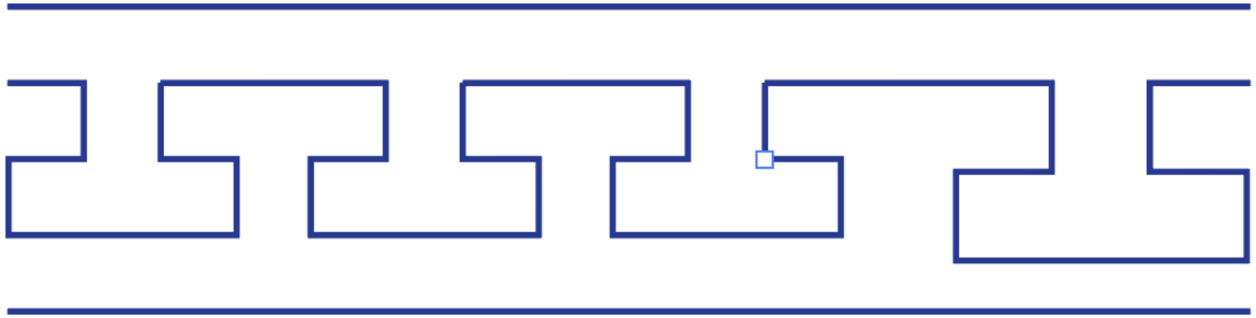
Reflective (or bilateral) symmetry



Reflective (or bilateral) symmetry

Glide reflection symmetry





A group of isometries is called a ***crystallographic*** or ***a wallpaper group*** if when applied to a tile, called the ***fundamental tile***, we obtain a tiling of the whole plane.

The fundamental tile can always be taken to be a triangle or a parallelogram.

So, any periodic tiling, no matter how complicated, can be generated by a triangle, or a parallelogram!

And there are only 17 possible ways of doing that.

There are only 5 wallpaper groups if one disallows reflections (so tiles are painted only one side, or we do not have tiles which are mirror images of each other):

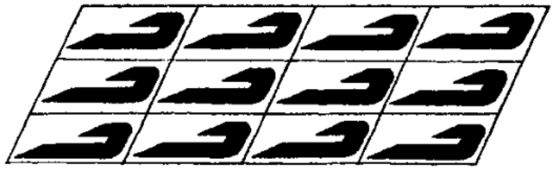


Figure 1.7.4.1

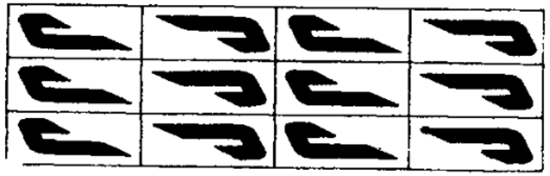


Figure 1.7.4.2

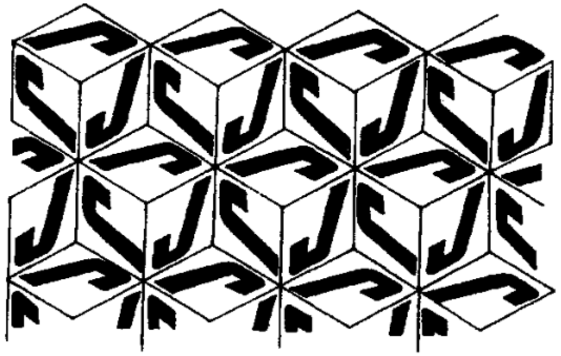


Figure 1.7.4.3

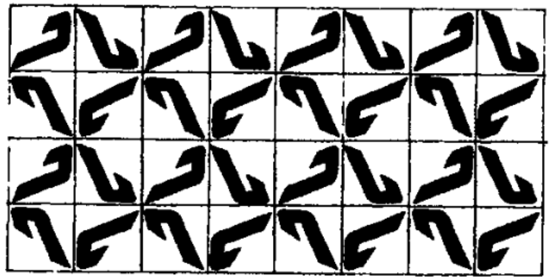


Figure 1.7.4.4

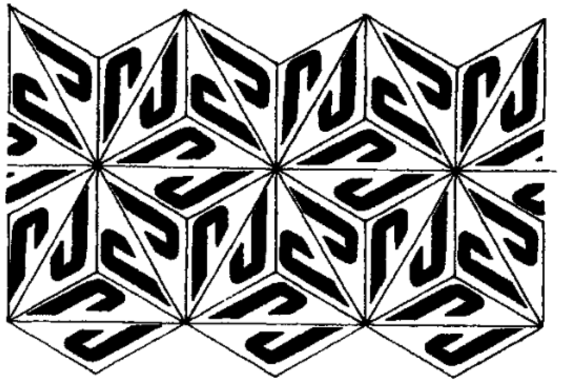


Figure 1.7.4.5

If one allows reflections, then we obtain 12 more symmetry groups:

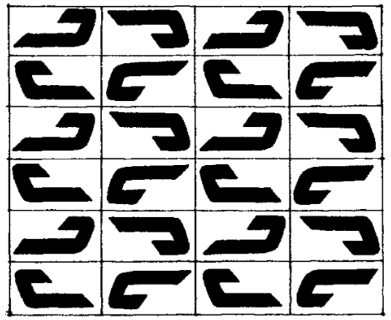


Figure 1.7.6.7

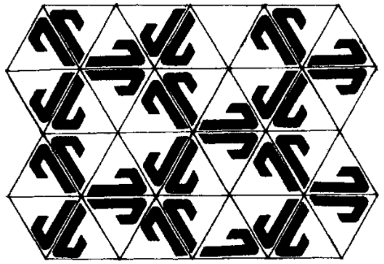


Figure 1.7.6.8

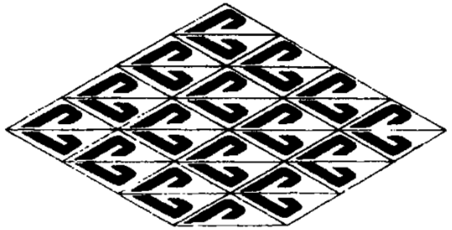


Figure 1.7.6.1

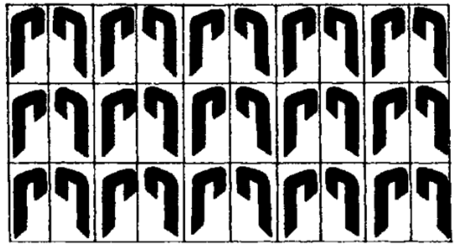


Figure 1.7.6.2

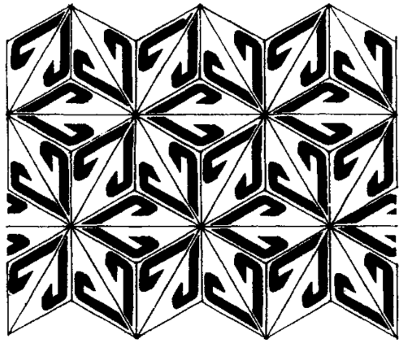


Figure 1.7.6.9

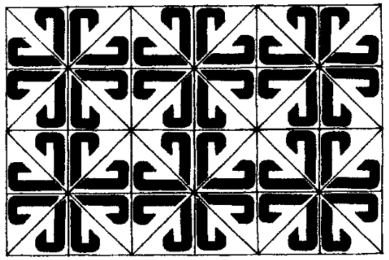


Figure 1.7.6.10

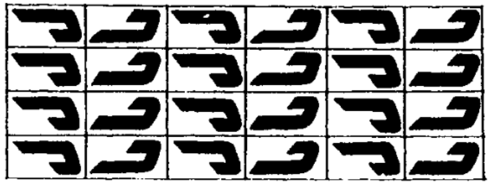


Figure 1.7.6.3

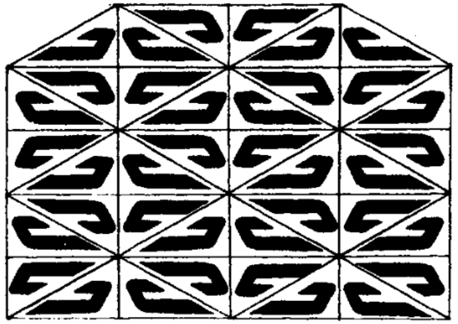


Figure 1.7.6.4

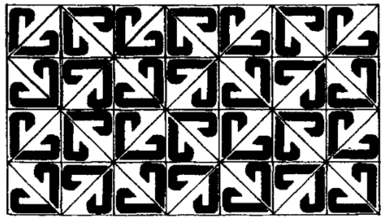


Figure 1.7.6.11

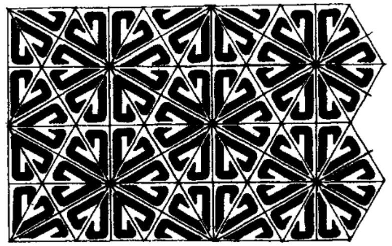


Figure 1.7.6.12

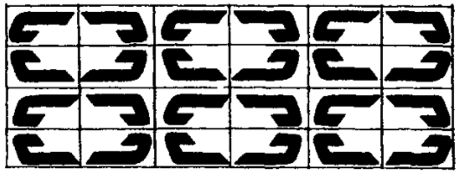


Figure 1.7.6.5

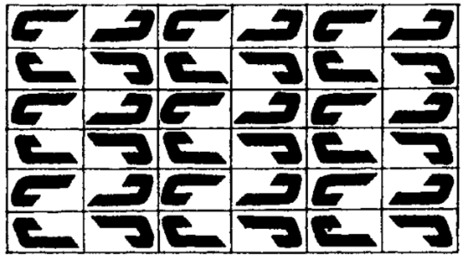
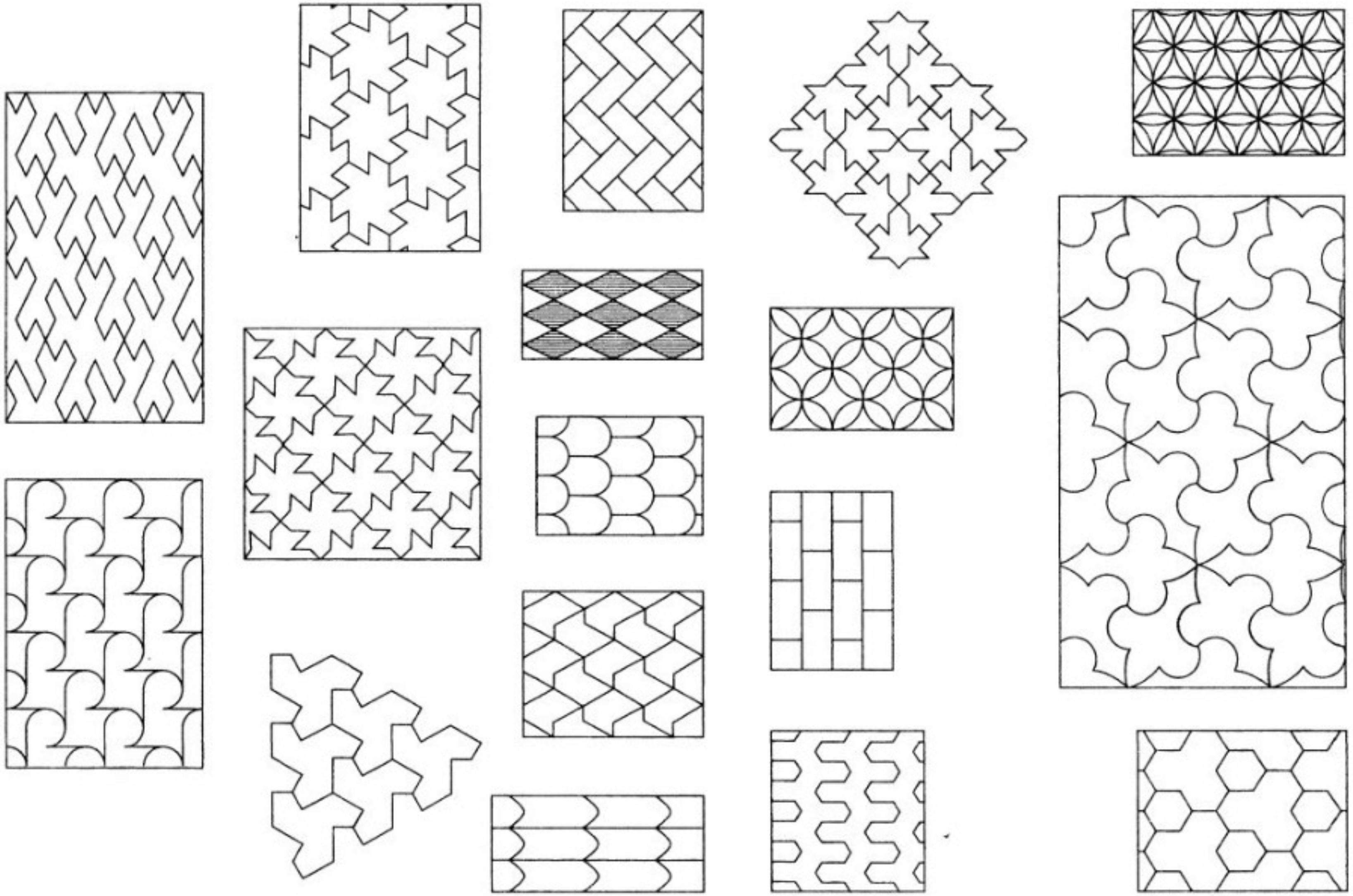


Figure 1.7.6.6

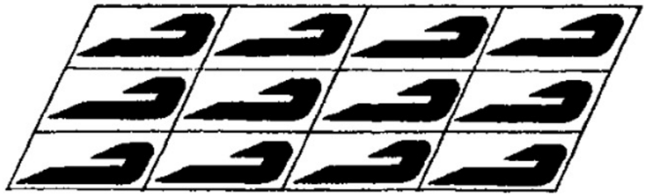
The Alhambra rosette pattern corresponds to Figure 1.7.6.5

Examples from each of the 17 wallpaper groups:

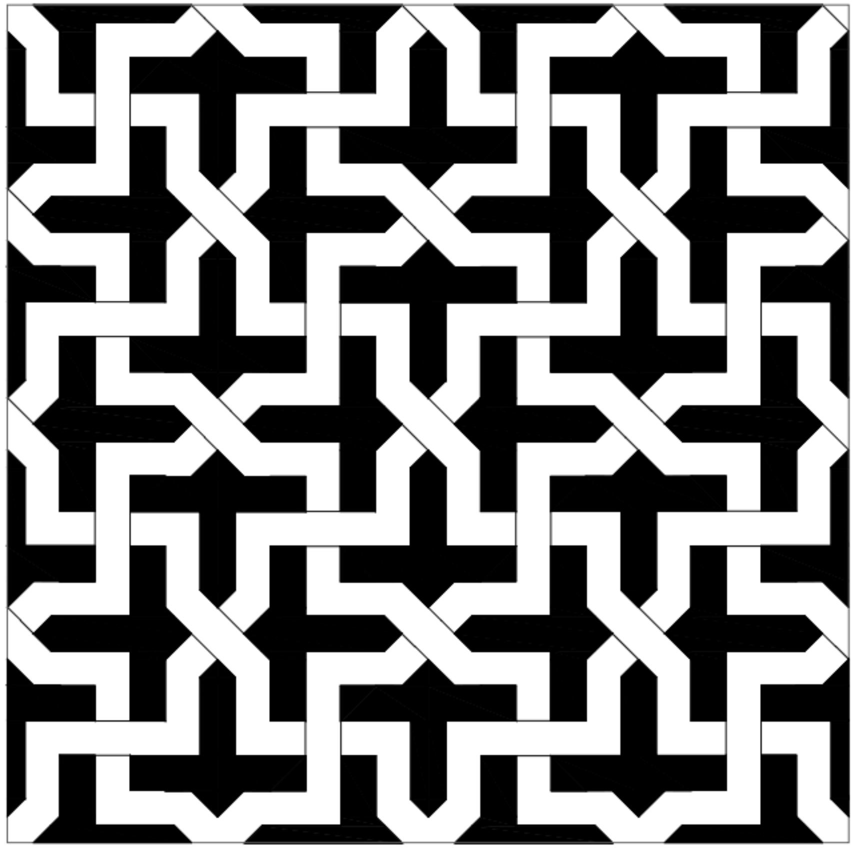


Examples from Alhambra:

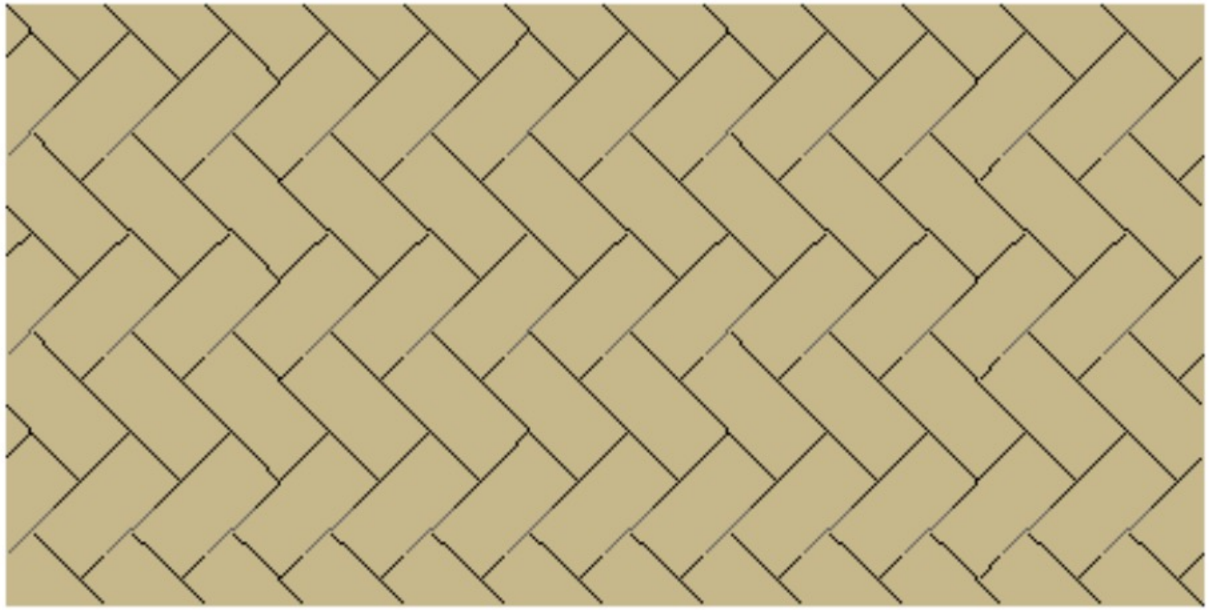
p



Examples from Alhambra:



Examples from Alhambra:



Thank You