Estimating maximal perimeters of convex sets with respect to probability measures

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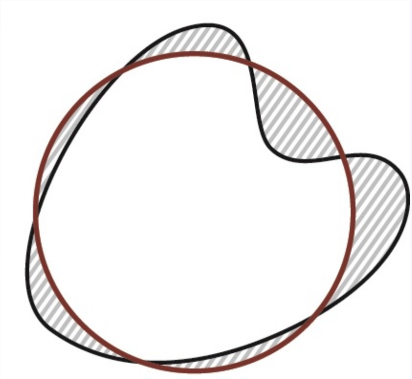
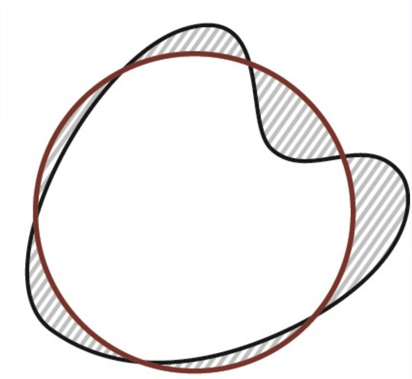
The classical isoperimetric inequality





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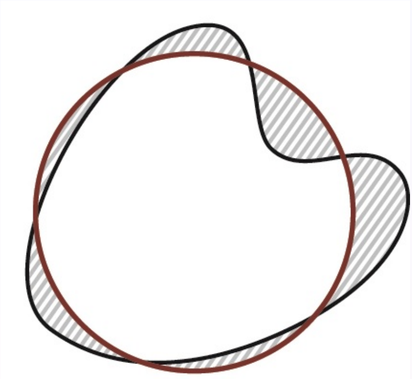

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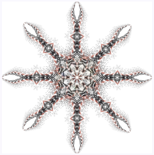
Reverse inequality?

And what about a reverse estimate?

- There exists a set K with volume 1 and arbitrarily large perimeter:



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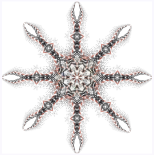

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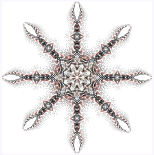

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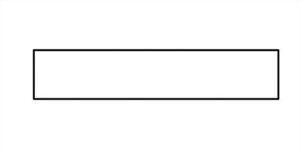

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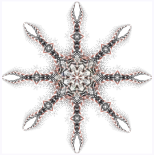

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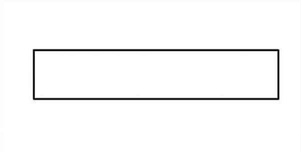

Ball's theorem from 1990s

For any convex set K there exists a linear transformation T such that $|TK| = |S|$ and $|\partial(TK)|_{n-1} \leq |\partial S|_{n-1}$, where S is the regular simplex.

- E. Markessinis, G. Paouris, Ch. Saroglou in 2012 considered $\max_K |TK + B_2^n|^{\frac{1}{n}}$ and showed that it is between $n^{\frac{1}{8}}$ and $n^{\frac{1}{8}} L_K$;

Reverse isoperimetric inequalities: history

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Replacing Lebesgue volume with a measure

Consider a (probability) measure μ on $\mathbb{R}^n$ and define a μ-perimeter of a convex body K as follows:

$$\mu^+(\partial K) := \liminf_{\epsilon \to 0} \frac{\mu(K + \epsilon B_2^n) - \mu(K)}{\epsilon}.$$

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- Example: if μ is uniform on another convex body L with $|L| = 1$, then for any convex K, one has $\mu^+(\partial K) = |\partial K \cap L|_{n-1} \leq |\partial L|_{n-1}$.

The main question of the talk

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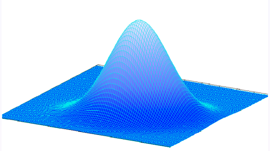
Main question

Depending on μ, what is the value of $\Gamma(\mu)$, in terms of the dimension, when $n \to \infty$?

Consider Gaussian measure γ with density $\frac{1}{\sqrt{2\pi}^n} e^{-\frac{|x|^2}{2}}$.



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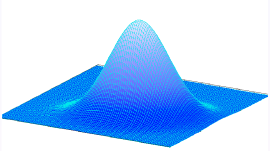

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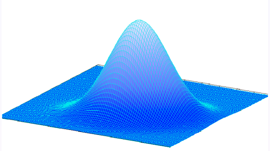

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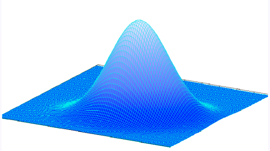

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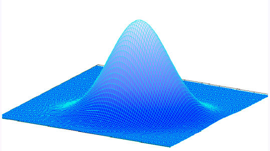

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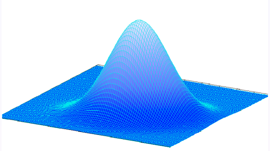

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- Kane in 2010: $\gamma^+(M) \leq \frac{d}{\sqrt{2}}$ where M is a degree d polynomial surface.

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Some results about rotation-invariant log-concave measures

- L. 2014: for any log-concave rotation-invariant random vector X,

$$C_2\frac{\sqrt{n}}{\sqrt[4]{Var|X|}\sqrt{\mathbb{E}|X|}} \leq \Gamma(X) \leq C_1\frac{\sqrt{n}}{\sqrt[4]{Var|X|}\sqrt{\mathbb{E}|X|}}.$$

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- L. 2015: for convex polytopes P with N sides (for appropriate values of N),

$$C_2\frac{\sqrt{n}}{\mathbb{E}|X|}\sqrt{\log N}\leq\max_P\gamma^+(\partial P)\leq C_1\frac{\sqrt{n}}{\mathbb{E}|X|}\sqrt{\log N}\log n.$$

Theorem 1 (L. 2019+)

Let X be a random vector in $\mathbb{R}^n$ with an absolutely continuous distribution. Suppose that $\sqrt{Var(|X|)} \leq \alpha \mathbb{E}|X|$, for $\alpha \in [0,1)$. Then

$$\Gamma(X) \geq C \frac{\sqrt{n}}{\sqrt[4]{Var(|X|)}\sqrt{\mathbb{E}|X|}}, \tag{1}$$

where $C > 0$ depends only on α. Namely, $C = C(\alpha) \to_{\alpha \to 0} 0.06$, and $C(\alpha) \to_{\alpha \to 1} 0$.

Isotropic log-concave vectors: history

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- Klartag 2007: $\sigma < o(1)\sqrt{n}$
- Klartag 2008: $\sigma < n^{-\alpha}\sqrt{n}$
- Fleury 2009: $\sigma < n^{-\beta}\sqrt{n}$ (improved)
- Guedon, Milman 2012: $\sigma < n^{1/3}$
- Lee, Vempala 2017: $\sigma < n^{1/4}$
- Klartag 2010: $\sigma < C$ for unconditional convex bodies
- Radke, Vritsiou 2016: $\sigma < C$ for Schatten classes
- Many more results and connections – Ball, Bobkov, Eldan, Giannopolous, Koldobsky, Paouris, Perissinaki....

General results: lower bound

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Corollary 1 of Theorem 1 (follows from Lee-Vempala's estimate)

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- In other words, the standard Gaussian is among the minimizers of $\Gamma(X)$ (up to a constant multiple), if the thin shell conjecture is true.

Corollary 3

For all isotropic log-concave unconditional vectors X on $\mathbb{R}^n$ with density e^{-V}, such that $0 \leq HessV \leq Id$, one has

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So what about upper bounds?

Theorem 2 (L. 2019+)

Let X be a random vector on $\mathbb{R}^n$ with an absolutely continuous unimodule density f. Then there exists a linear volume preserving transformation T such that

$$\Gamma(TX) \leq Cn||f||_{\infty}^{\frac{1}{n}},$$

where $C > 0$ is an absolute constant.

- The previous theorem includes all log-concave distributions;
- It also generalizes Ball's theorem.

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Conclusion

$$\Gamma(X) \in [n^{1/8}, n^2]$$

for isotropic log-concave vectors (up to constant multiples).

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Conjecture

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- n is attained e.g. for X uniform on B^n_∞.
- $n^{\frac{1}{4}}$ is attained e.g. for X being standard Gaussian.

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One can show that the upper bound of n holds for:

- Uniform distributions on convex sets;
- 1-symmetric log-concave distributions;
- Densities $e^{-||\cdot||}$ and all measures with homothetic level sets.

Classical fact

Suppose $RB_2^n \subset K$. Then $|\partial K|_{n-1} \leq \frac{n|K|}{R}$.

Proof

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- Our assumption implies that $\epsilon B_2^n \subset \frac{\epsilon}{R} K$.

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Conclusion: Theorem 2

There exists a linear volume preserving transformation T such that

$$\Gamma(TX) \leq Cn||f||_\infty^{\frac{1}{n}}.$$

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Conclusion: Theorem 3

For isotropic log-concave X we have $\Gamma(X) \leq Cn^2$.

A strange question

Question

Let μ be an isotropic log-concave measure with density f. Does there exist a level set K_t of μ such that

$$|K_t| \leq \frac{C_1}{||f||_\infty},$$

and $C_2 B_2^n + y \subset K_t$, for some absolute constants C_1 and C_2 and a vector y?

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The answer is affirmative for:

- Uniform distributions on convex sets;
- 1-symmetric log-concave distributions;
- Densities $e^{-||\cdot||}$ and all measures with homothetic level sets.

Thanks for your attention!