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VARIETIES OF REPRESENTATIONS AND THEIR COHOMOLOGY-JUMP SUBVARIETIES FOR KNOT GROUPS

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ABSTRACT. This paper studies the spaces of representations of knot groups into a linear group $\mathrm{GL}_n(\mathbb{C})$, their categorical factor-spaces (i.e., the spaces of all characters of the representations), and their cohomology-jump subspaces. Connections are established between the latter and the spaces of representations of dimension one greater. A complete description is given of these spaces for 2-bridge knots.

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0. INTRODUCTION

In recent years the spaces of all representations of a finitely generated group π into a linear group $\mathrm{GL}_n(\mathbb{C})$ have been studied intensively. These spaces are algebraic subsets in $\mathbb{C}^{n^2p}$ for some $p \in \mathbb{N}$, and with their help many invariants of the given group can be constructed.

Each representation of the given group determines cohomology groups of the latter with coefficients in the representation. The rank of the ith such cohomology group is constant throughout the space of representations except for a certain closed subset, which we call the ith cohomology-jump subvariety.

The study of these cohomology-jump subvarieties was begun by S. P. Novikov (see [1]). Already in the case of 1-dimensional representations there are nontrivial results. If π is a knot group, then the cohomology-jump subvariety in the space of representations of dimension 1 consists of the roots of the Alexander polynomial of the knot. This is a reformulation of classical results (see §2.3 below). A natural question to pose is that of the role of the cohomology-jump subvarieties for a knot group with coefficients in multidimensional representations. The object of this paper is to study these cohomology-jump subvarieties.

In §1 we lay out the theory of the spaces of representations of finitely generated groups and their categorical factor-spaces. In §2 we study the cohomology-jump subvariety, proving in particular that it is Zariski-closed, and we indicate the connection of this subvariety with the space of representations of the same group of dimension one greater. In §3 we describe the spaces of representations for 2-bridge knots; another description of these spaces is given in [8], but we use a different approach to the computation that in our view is more natural and more suitable for our purposes. In §4 we describe in explicit form the cohomology-jump subvarieties for 2-bridge knots and present some examples.

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1. The space of representations of a group, and the corresponding space of characters

1.1. An algebraic variety will be understood in the most naive meaning of the term—simply the set of zeros of a system of polynomial equations in $\mathbb{C}^n$ or an open subset thereof. The topology will always be understood in the sense of Zariski, unless otherwise stated.

Let π be a group, with presentation

$$\pi = \langle a_1, a_2, \dots, a_p \mid r_1, \dots, r_q \rangle;$$

if π is a knot group, we assume that $q = p-1$ and all the generators $a_1, a_2, \dots, a_p$ are conjugate to each other. We denote by $R_n(\pi)$ (resp. $\mathrm{SR}_n(\pi)$) the set of all homomorphisms of π into $\mathrm{GL}_n(\mathbb{C})$ (resp. $\mathrm{SL}_n(\mathbb{C})$). An action of the group $\mathrm{GL}_n(\mathbb{C})$ is defined on these spaces: if $g \in \mathrm{GL}_n$ and $\rho \in R_n(\pi)$, then $g(\rho) = g\rho g^{-1}$.

It is known (see [6]) that if two semisimple representations of the group π into a linear group have the same characters, they are conjugate to each other. The character of any representation coincides with that of its semisimple part. The set of all characters of representations of π into GL_n is somewhat smaller than the set of equivalence classes of representations, but is more convenient to use, because it can be parametrized as an algebraic variety (see [6]). We denote the set of all characters of representations of π into GL_n (resp. SL_n) by $\mathrm{X}_n(\pi)$ (resp. $\mathrm{SX}_n(\pi)$). There is a natural mapping pr: $R_n(\pi) \to \mathrm{X}_n(\pi)$, taking each representation into its character. The mapping pr (and its restriction to SR_n) is regular and also submersive, i.e., a set $Y \subset \mathrm{X}_n(\pi)$ is open if and only if $\mathrm{pr}^{-1}(Y)$ is open. $\mathrm{SR}_n(\pi)$ and $\mathrm{SX}_n(\pi)$ are closed algebraic varieties.

Let $\mathbb{Z}_n = \{z \mid z^n = 1\}$. If ρ is a representation of π into SL_n, then obviously $z\rho$ is likewise such a representation, where $(z\rho)(c) = z\rho(c)$; i.e., the group $\mathbb{Z}_n$ acts on $\mathrm{SR}_n(\pi)$. It also acts on $\mathrm{SX}_n(\pi)$. If π is a knot group, then $H^1(\pi, \mathbb{Z}) = \mathbb{Z}$ and $\mathrm{Hom}(\pi, \mathbb{C}^*) = \mathbb{C}^*$; and if $\rho_0 \in \mathrm{Hom}(\pi, \mathbb{C}^*)$, then $z\rho_0 \in \mathrm{Hom}(\pi, \mathbb{C}^*)$ where $(z\rho_0)(c) = z^{-1}(\rho_0(c))$; i.e., the group $\mathbb{Z}_n$ acts on $\mathbb{C}^*$, and therefore on $\mathrm{SX}_n \times \mathrm{Hom}(\pi, \mathbb{C}^*) = \mathrm{SX}_n \times \mathbb{C}^*$.

1.1.1. **Proposition.** *Let π be a knot group. Then the variety $\mathrm{X}_n(\pi)$ is the quotient of the variety $\mathrm{SX}_n \times \mathbb{C}^*$ by the action of the group $\mathbb{Z}_n$.*

Proof. Consider the mapping h: $\mathrm{SX}_n \times \mathbb{C}^* \to X_n$, $h(x, \rho_0) = \rho_0 x$, where $\rho_0 x = \mathrm{pr}(\rho_0\rho)$, $\rho \in \mathrm{pr}^{-1}(x)$. Note that $\mathrm{tr}[(\rho_0\rho)(c)] = \rho_0(c)\,\mathrm{tr}(\rho(c))$.

Obviously h is surjective, and if $z(x, \rho_0) = (x', \rho_0')$, then $h(x, \rho_0) = h(x', \rho_0')$.

Conversely, suppose $h(x, \rho_0) = h(x', \rho_0')$. Let ρ, ρ' be two representations such that $\mathrm{pr}(\rho) = x$, $\mathrm{pr}(\rho') = x'$. Then $\rho\rho_0 = \rho'\rho_0'$, $\det(\rho\rho_0) = \det(\rho'\rho_0')$, $\det(\rho_0\rho) = \rho_0^n \det\rho$, and therefore $\rho_0^n = (\rho_0')^n$. This means that there exists a $z \in \mathbb{Z}_n$ such that $\rho_0 = z^{-1}\rho_0'$; then $\rho = z\rho'$, i.e., $z(x', \rho_0') = (x, \rho_0)$.

1.1.2. **Proposition.** *Suppose $V \subset R_n$ is closed and invariant under the action of the group GL_n. Then $\mathrm{pr}(V)$ is a closed subset of X_n.*

Proof. Consider an $x \in \mathrm{X}_n$ that does not belong to $\mathrm{pr}(V)$. Then both sets $\mathrm{pr}^{-1}(x)$ and V are closed and invariant. Hence by Lemma 1 in Chapter 4, §1, of [10], there exists an invariant regular function $\tilde{f}$: $R_n \to \mathbb{C}$ such that $\tilde{f}|_V = 0$ and $\tilde{f}|_{\mathrm{pr}^{-1}(x)} = 1$. Dropping to X_n, we obtain a function f: $\mathrm{X}_n \to \mathbb{C}$ such that $f(x) = 1$, $f(\mathrm{pr}(V)) = 0$, and f is regular on X_n (since the ring of regular functions on X_n coincides with the ring of invariant functions on R_n; see [6]). This means that there exists a Zariski-open neighborhood containing the point x and not intersecting the set $\mathrm{pr}(V)$. Thus, $\mathrm{pr}(V)$ is Zariski-closed.

1.2. **The subspaces of reducible representations.** Let τ be a partition of the number n, $\tau = (n_1, n_2, \dots, n_l)$, $n_1 + \dots + n_l = n$, and let μ be a set $(m_1, \dots, m_k)$ such that $0 < m_1 < \dots < m_k = n$. Every such set determines a partition of n, by putting $n_1 = m_1$, $n_2 = m_2 - m_1, \dots, n_k = m_k - m_{k-1}$; we call this the partition induced by the set μ. In general, many different sets of increasing numbers determine the same partition; for example, the sets $(1, 3)$ and $(2, 3)$ determine the same partition of the number 3: $(1, 2) = (2, 1)$.

We say that a representation ρ is subordinate to the set $\mu = (m_1, \dots, m_k)$ if there exists a basis in which ρ has the block form

$$\rho(a_i) = \begin{pmatrix} \boxed{} & * & & \\ & \boxed{} & & \\ & & \ddots & \\ 0 & & & \boxed{*} \end{pmatrix}$$

where the elements below the diagonal are zero and the dimensions of the blocks are, respectively, $m_1 \times m_1$, $(m_2 - m_1) \times (m_2 - m_1), \dots, (m_k - m_{k-1}) \times (m_k - m_{k-1})$. We shall write $\rho \le \mu$.

Let $R_\mu(\pi) = \{\rho \in R_n(\pi), \rho \le \mu\}$.

1.2.1. **Proposition.** *The set $R_\mu(\pi)$ is closed in the Zariski topology of $R_n(\pi)$.*

Proof. Let E be the set of flags of dimensions $(m_1, m_2, \dots, m_k)$ in $\mathbb{C}^n$. Then E is a (closed) projective variety. Consider the mapping

$$f\colon E \times R_n(\pi) \to E^{p+1},$$
$$f(\alpha, \rho) = (\alpha, \rho(a_1)\alpha, \rho(a_2)\alpha, \dots, \rho(a_p)\alpha).$$

The condition $\rho \le \mu$ is equivalent to the requirement that all the mappings $\rho(a_i)$, $i = 1, \dots, p$, preserve some point of E. Let D be the diagonal in E^{p+1}: $D = \{(\alpha, \alpha, \dots, \alpha) \in E^{p+1}\}$; D is closed in E^{p+1}. Let $p_2\colon E \times R_n(\pi) \to R_n(\pi)$ be a projection onto the second factor. Then $R_\mu = p_2(f^{-1}(D))$, since E is a projective variety, p_2 is a closed mapping, and so R_μ is a closed subset of $R_n(\pi)$.

Let τ be a partition of the number n. We say that a representation ρ is subordinate to τ if ρ is subordinate to some set μ that induces τ. Let X_τ be the set of characters of all the representations subordinate to τ. Then $X_\tau = \mathrm{pr}(\bigcup R_\mu)$, where μ runs over the collection of sets that induce the partition τ.

1.2.2. **Corollary.** X_τ *is closed in the space* $X_n(\pi)$.

This follows from the fact that pr is a submersion.

1.2.3. **Corollary.** *The set R_n^s of irreducible representations is open in R_n. Similarly, the set X_n^s is open in X_n.*

2. The cohomology-jump subvariety

2.1. Every representation $\rho\colon \pi \to GL_n$ turns the vector space $\mathbb{C}^n$ into a left $\mathbb{Z}[\pi]$-module. We can therefore define the cohomology groups $H^i(\pi, \rho)$ with coefficients in this module. They are the cohomology groups $H^i(K(\pi, 1), \rho)$ of the space $K(\pi, 1)$ with coefficients in ρ. The groups $H^i(\pi, \rho)$ do not change if ρ is replaced by a conjugate representation. Hence the function $\operatorname{rk}_i(x) = \operatorname{rk} H^i(\pi, \rho)$, where $\operatorname{pr}(\rho) = x$, is well defined on the set $\mathrm{X}_n^s(\pi)$ of characters of all simple (i.e., irreducible) representations. On the rest of the variety $\mathrm{X}_n(\pi)$ this function is no longer well defined, since every point of this part of $\mathrm{X}_n(\pi)$ corresponds, in general, to many nonequivalent representations.

We define $\operatorname{rk}_i(x)$ (for any $x \in \mathrm{X}_n(\pi)$) as the greatest of the ranks of the groups $H^i(\pi, \rho)$, where $\operatorname{pr}(\rho) = x$.

2.1.1. **Proposition.** *Let* $\Pi_n^{i,k}$ *be the subset (in* $\mathrm{X}_n(\pi)$*) of all points* x *such that* $\operatorname{rk}_i(x) \geq k$, $i, k \in \mathbb{Z}$, $i, k \geq 0$. *Then* $\Pi_n^{i,k}$ *is a closed algebraic subset. In other words, the function* rk_i *is upper semicontinuous.*

Proof. Let $\widetilde{\Pi}_n^{i,k} = \{\rho \in R_n(x), \operatorname{rk} H^i(\pi, \rho) \geq k\}$. Then $\widetilde{\Pi}_n^{i,k}$ is a closed subset. This fact appears (in a different form) in [1]. The subset $\widetilde{\Pi}_n^{i,k}$ is obviously invariant under the action of GL_n; therefore, $\operatorname{pr}(\widetilde{\Pi}_n^{i,k})$ is closed in X_n, by Proposition 1.1.3. By definition, $\operatorname{pr}(\widetilde{\Pi}_n^{i,k})$ is precisely $\Pi_n^{i,k}$. Thus, $\Pi_n^{i,k}$ is closed in X_n.

Let k be the smallest value of the function rk_i on the variety X_n. The subset $\Pi_n^{i,k+1}$ will be called the jump subvariety of the ith cohomology group.

2.1.2. **Proposition.** *Suppose* $x \in \mathrm{X}_n(\pi)$, *and let* $\rho \in R_n(\pi)$ *be a representative of the unique class of semisimple representations with character* x. *Then* $\operatorname{rk}_i(x) = \operatorname{rk}_i(\rho)$.

Proof. We must show that if $\tau \in \operatorname{pr}^{-1}(x)$, then $\operatorname{rk}_i(\tau) \leq \operatorname{rk}_i(\rho)$. Consider the orbits $O(\rho)$ and $O(\tau)$ in $R_n(\pi)$ (remember that the group GL_n acts on $R_n(\pi)$). By Lemma 1.2.6 of [6], $O(\rho)$ is closed; $O(\tau)$ is not closed if τ is nonsemisimple, but $\overline{O(\tau)} \supset O(\rho)$, so that $\rho \in \overline{O(\tau)}$. Since the function rk_i is upper semicontinuous and its restriction to $O(\tau)$ is constant, we have $\operatorname{rk}_i(\tau) \leq \operatorname{rk}_i(\rho)$.

2.2. **Cohomology in small dimensions; zero-dimensional cohomology.** We recall the construction for computing the cohomology of our group in small dimensions. Consider a 2-dimensional cell complex X consisting of a single 0-cell O, p 1-cells $a_1, a_2, \dots, a_p$, and q 2-cells $c_1, c_2, \dots, c_q$ such that $\partial c_i = r_i$. Let $\widetilde{X}$ be the universal covering of X, $\widetilde{O}$ a fixed point over O, and $\tilde{a}_i$, $\tilde{c}_j$ liftings of the cells a_i, c_j from $\widetilde{O}$. Then $\widetilde{X}$ is a free complex over the ring $\mathbb{Z}[\pi]$, $C_0(\widetilde{X}) = \mathbb{Z}[\pi]$, $C_1(\widetilde{X}) = (\mathbb{Z}[\pi])^p$ with generators $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_p$, and $C_2(\widetilde{X}) = (\mathbb{Z}[\pi])^q$ with generators $\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_q$. We have the sequence

$$0 \to C_2(\widetilde{X}) \overset{\partial_1}{\to} C_1(\widetilde{X}) \overset{\partial_0}{\to} C_0(\widetilde{X}) \to 0.$$

The boundary operator is given by the formulas

$$\partial_0(\tilde{a}_i) = (a_i - 1)\widetilde{O}, \qquad \partial_1(\tilde{c}_j) = \sum_{i=1}^{p} \left(\frac{\partial r_j}{\partial a_i}\right) \tilde{a}_i$$

($\partial r_j / \partial a_i$ is an element of $\mathbb{Z}[\pi]$), where $\partial / \partial a_i$ is Fox differentiation (see [4]).

Now, if $\rho\colon \pi \to \mathrm{GL}_n$ is a representation of the group, $\mathbb{C}^n$ becomes a left $\mathbb{Z}[\pi]$-module. If $\alpha \in \mathbb{Z}[\pi]$, then $\rho(\alpha)$ is an endomorphism of the module $\mathbb{C}^n$. Consider the complex

$$0 \leftarrow (\mathbb{C}^n)^q \overset{\rho(\partial_1)}{\longleftarrow} (\mathbb{C}^n)^p \overset{\rho(\partial_0)}{\longleftarrow} \mathbb{C}^n \leftarrow 0,$$

where $\rho(\partial_1)$ is an $nq \times np$ matrix, and $\rho(\partial_0)$ an $np \times n$ matrix. The first two cohomology groups of this complex are the groups $H^0(\pi, \rho)$ and $H^1(\pi, \rho)$.

2.2.1. **Proposition.** *For $n \geq 2$, the dimension of the zeroth cohomology group makes no jumps on the set $X_n^s(\pi)$ of characters of simple representations, and* $\operatorname{rk} H^0(\pi, \rho) = 0$ *for all $\rho \in R_n^s(\pi)$.*

Proof. We have

$$\rho(\partial_0) = \begin{pmatrix} \rho(a_1) - 1 \\ \vdots \\ \rho(a_p) - 1 \end{pmatrix},$$

and therefore

$$H^0(\pi, \rho) = \bigcap_{i=1}^{p} \ker(\rho(a_i) - 1).$$

If $v \in H^0(\pi, \rho)$, then v is a common eigenvector of all the operators $\rho(a_i)$. Since ρ is simple, we conclude that $v = 0$. Thus, $H^0(\pi, \rho) = 0$.

2.3. **The case of 1-dimensional representations.** The following results are already known; we give them here for completeness and as constituting an important example.

We have $X_1(\pi) = (\mathbb{C}^*)^m$, where $m = \operatorname{rk} H^1(\pi, \mathbb{Z})$. Let $e_1, \ldots, e_m$ be generators for the group $H^1(\pi, \mathbb{Z})$. Any 1-dimensional representation of π is determined by a set of numbers $(z_1, \ldots, z_m) \in (\mathbb{C}^*)^m$, with $(z_1, \ldots, z_m)(e_i) = z_i$.

In contrast to Proposition 2.2.1, we have:

2.3.1. **Proposition.** *The subvariety of jumps in dimension for zero-dimensional cohomology consists of the point $\{z_1 = z_2 = \cdots = z_m = 1\}$: the trivial representation is the only one that has nonzero $H^0(\pi, \rho)$.*

Proof. If ρ is the trivial representation, then obviously $\rho(\partial_0) = 0$; if ρ is nontrivial, then one of the $\rho(a_i)$ is equal to 1, and then $\ker \rho(\partial_0) = 0$, so that $H^0(\pi, \rho) = 0$.

Let M be a matrix with elements in the ring $\mathbb{Z}[z_1^{\pm 1}, \ldots, z_m^{\pm 1}]$. Denote by $d_k(M)$ the ideal generated by all the minors of order k. Then $d_k(M) \subset d_l(M)$ for $k > l$, and $d_k(M) = 0$ for large k. Let $\Delta(M)$ be the last nonzero ideal in the sequence $d_1(M), d_2(M), \ldots$.

2.3.2. **Proposition.** *The subvariety of jumps in dimension for 1-dimensional cohomology (on $X_1(\pi) = (\mathbb{C}^*)^m$) is the set of zeros of the equations $\Delta(\rho(\partial_1)) = 0$ plus, possibly, the point $\{z_1 = z_2 = \cdots = z_m = 1\}$.*

Proof. We have the complex

$$(\mathbb{C}^q) \overset{\rho(\partial_1)}{\longleftarrow} (\mathbb{C}^p) \overset{\rho(\partial_0)}{\longleftarrow} \mathbb{C} \leftarrow 0;$$

if $(z_1, \ldots, z_m) \neq (1, \ldots, 1)$, then $\dim \rho(\partial_0) = 1$, which means that $\dim H^1(\pi, \rho) = p_1 - \operatorname{rk} \rho(\partial_1)$. If now $\Delta(\rho(\partial_1) = d_k(\rho(\partial_1))$, then $\operatorname{rk}(\rho(\partial_1))$ becomes less than k only when $(z_1, \ldots, z_m)$ is a root of the ideal $\Delta(\rho(\partial_1))$.

When π is the fundamental group of a knot, Δ is the principal ideal generated by the corresponding Alexander polynomial (see [4]). For this group, $H^3(\pi, \rho) = 0$ $\forall \rho \in R_n$, whereas the dimension of the second cohomology group jumps only if that of the first or the zeroth does. Thus, the jump variety consists of the roots of the Alexander polynomial plus the point 1 (which, incidentally, is never a root of the Alexander polynomial).

2.4. **Connection with representations in SL_2.** Let π be a knot group. Consider the subset SX_2^1 (in $SX_2(\pi)$) of all characters of reducible representations. If $x \in SX_2^1 = \mathbb{C}$ and ρ is a semisimple representation with character x, then $\rho = \tau \oplus \tau^{-1}$, where $\tau \in X_1(\pi) = \mathbb{C}^*$, $\tau(a_i) = t \in \mathbb{C}^*$. Let SX_2^* be the subset (in $\mathrm{SX}_2(\pi)$) of all characters of nonabelian representations.

With this notation, we have:

2.4.1. **Proposition.** *Suppose $\tau \neq \pm 1$. Then x belongs to the intersection of $\mathrm{SX}_2^1(\pi)$ and $\mathrm{SX}_2^*(\pi)$ if and only if the representations τ^2 and τ^{-2} lie on the jump subvariety of the first cohomology group.*

Proof. Suppose $x \in \mathrm{SX}_2^1 \cap \mathrm{SX}_2^*$. Then there exists a nonabelian representation ρ with character x. Since $x \in \mathrm{SX}_2^1$, ρ must be reducible. We have then, in some basis:

$$\rho(a_i) = \begin{pmatrix} t & \alpha_i \\ 0 & t^{-1} \end{pmatrix}, \qquad t \in \mathbb{C}^*, \quad \alpha_i \in \mathbb{C}, \quad i = 1, \dots, p.$$

Consider the 1-dimensional representation $\tau\colon \pi \to \mathrm{GL}_1$, $\tau(a_i) = t$. Then

$$\tau\rho(a_i) = \begin{pmatrix} t^2 & t\alpha_i \\ 0 & 1 \end{pmatrix}$$

is a 2-dimensional representation, not into SL_2 but into GL_2. The matrices $\tau\rho(a_i)$ and $\tau\rho(a_j)$ commute if and only if $t\alpha_i(t^2-1) = t\alpha_j(t^2-1)$, i.e., if and only if $(t\alpha_i : t\alpha_j) = (t^2-1 : t^2-1)$. Since the representations are nonabelian, it follows that

$$(\alpha_1 : \alpha_2 : \cdots : \alpha_p) \neq (t^2-1 : t^2-1 : \cdots : t^2-1),$$

which means that the vector $(\alpha_1, \dots, \alpha_p)$ lies outside the domain of values of the differential $\tau^2(\partial_0)$, i.e., is not a coboundary.

2.4.2. **Lemma.** *The condition $r_1 = \cdots = r_q = 1$ is equivalent to the system of equations*

$$\tau_0^2\left(\frac{\partial r_i}{\partial a_1}\right)\alpha_1 + \cdots + \tau^2\left(\frac{\partial r_i}{\partial a_p}\right)\alpha_p = 0, \qquad i = 1, \dots, q.$$

The lemma will be proved in a general form in §2.5. As stated here, it amounts to the assertion that the vector $(\alpha_1, \dots, \alpha_p)$ is a cocycle. This means that the group $H^1(\pi, \tau^2)$ is not 0, i.e., that $\mathrm{rk}_1(\tau^2) \geq 1$. On the other hand, for a knot group the function rk_1 on $\mathrm{X}_1(\pi)$ is zero almost everywhere. Hence if $x \in \mathrm{SX}_2^1(\pi) \cap \mathrm{SX}_2^*(\pi)$, then $\tau^2 \in \Pi_1^1(\pi)$; similarly, $\tau^{-2} \in \Pi_1^1(\pi)$. Conversely, suppose $\tau^2 \in \Pi_1^1(\pi)$, $\tau^2 \neq \pm 1$. Then there exist numbers $\alpha_1, \dots, \alpha_p$ such that

$$\sum_{j=1}^{p} \tau^2\left(\frac{\partial r_i}{\partial a_j}\right)\alpha_j = 0, \qquad i = 1, \dots, q,$$

and by Lemma 2.4.2 the correspondence

$$a_j \to \begin{pmatrix} t & t^{-1}\alpha_j \\ 0 & t^{-1} \end{pmatrix}$$

is a representation of the group. Since $(\alpha_1, \dots, \alpha_p)$ is not a coboundary, we have

$$(\alpha_1 : \alpha_2 : \cdots : \alpha_p) \neq (t^2-1 : t^2-1 : \cdots : t^2-1).$$

If $\tau \neq \pm 1$, then $t \neq \pm 1$ and therefore $t^2 - 1 \neq 0$, which means that the matrices $\rho(a_j)$ do not commute.

Proposition 2.4.1 can be reformulated:

2.4.3. **Corollary.** *Let π be a knot group. Then x belongs to the intersection $\mathrm{SX}_2^1 \cap \mathrm{SX}_2^*$ if and only if there exists a reducible nonabelian representation ρ with character x such that the ratio of the two eigenvalues of the matrix $\rho(m)$ is a root of the Alexander polynomial of the knot, where $m \in \pi$ is an element representing some meridian of the knot.*

2.4.4. **Proposition.** *The set $\mathrm{SX}_2^*(\pi)$ is a closed algebraic subvariety of $\mathrm{SX}_2(\pi)$.*

Proof. We prove that

$$\mathrm{SX}_2^* = \{x \in \mathrm{SX}_2(\pi)\,,\ \dim \mathrm{pr}^{-1}(x) \geq 3\}.$$

Indeed, if $x \in \mathrm{SX}_2^*(\pi)$, then there exists a nonabelian representation with character x. We claim that $\dim O(\rho) \geq 3$, where $O(\rho)$ is the orbit of the element ρ under the action of GL_2. If ρ is irreducible, this is obvious, since $\mathrm{St}(\rho) = \mathbb{C}^*$, where $\mathrm{St}(\rho)$ is the stationary subgroup of the element ρ, and $O(\rho) = \mathrm{GL}_2 / \mathrm{St}(\rho) = \mathrm{PSL}_2$, so that $\dim O(\rho) = 3$. If ρ is reducible, there exists a basis in which

$$\rho(a_i) = \begin{pmatrix} t & \alpha_i \\ 0 & t^{-1} \end{pmatrix},$$

and since ρ is nonabelian, we have $(\alpha_1 : \alpha_2 : \cdots : \alpha_p) \neq (1 : 1 : \cdots : 1)$. It is easily seen that then a matrix that commutes with all the $\rho(a_i)$ must be of the form $c \cdot 1$. Hence $\dim O(\rho) = 3$; and since $O(\rho) \subset \mathrm{pr}^{-1}(x)$, we have $\dim \mathrm{pr}^{-1}(x) \geq 3$.

Conversely, suppose $\dim \mathrm{pr}^{-1}(x) \geq 3$. We must show that there exists a nonabelian representation ρ with character x. Suppose on the contrary that $\mathrm{pr}^{-1}(x)$ contains only abelian representations. It is easily seen that the set SR_2^a of all abelian representations of a knot group is a closed irreducible algebraic variety in $\mathrm{SR}_2(\pi)$, being isomorphic to SL_2; hence if $\dim \mathrm{pr}^{-1}(x) \geq 3$ and $\mathrm{pr}^{-1}(x)$ is entirely contained in this set, then $\mathrm{pr}^{-1}(x) = \mathrm{SR}_2^a$. But $\mathrm{pr}(\mathrm{SR}_2^a) = \mathrm{SX}_2^1 = \mathbb{C}$—a contradiction. Thus,

$$\mathrm{SX}_2^*(\pi) = \{x \in \mathrm{SX}_2(\pi)\,,\ \dim \mathrm{pr}^{-1}(x) \geq 3\},$$

and since the function $\dim \mathrm{pr}^{-1}$ is upper semicontinuous, $\mathrm{SX}_2^*(\pi)$ is a closed algebraic subvariety.

We have the obvious inclusion $\overline{\mathrm{SX}_2^s} \subset \mathrm{SX}_2^*$. In §3 it will be shown that for the groups of 2-bridge knots, $\overline{\mathrm{SX}_2^s} = \mathrm{SX}_2^*$; but in general $\overline{\mathrm{SX}_2^s} \neq \mathrm{SX}_2^*$ (if π is not a knot group).

2.5. **Representations of knot groups of dimension greater than one.** For a knot group π, there exists a presentation $\pi = \langle a_1, a_2, \ldots, a_p \mid r_1, \ldots, r_{p-1} \rangle$ such that the 2-dimensional cell complex X constructed as prescribed in §2.2 is aspherical, i.e., such that $X = K(\pi, 1)$. For example, it suffices to take the standard Wirtinger presentation (see [5]). This means that we always have $H^i(\pi, \rho) = 0$ for $i \geq 3$ (the complex X has no cells of dimension greater than 2). On X_n^s the dimension of the zeroth cohomology group has no jumps, and since $H^0(\pi, \rho) = 0$ and the Euler characteristic is always zero, we have $\mathrm{rk}\, H^2(\pi, \rho) = \mathrm{rk}\, H^1(\pi, \rho)$. It follows that $\Pi_n^1(\pi) = \Pi_n^2(\pi)$; we shall also denote these both simply by $\Pi_n(\pi)$ and cell this the cohomology-jump subvariety on $\mathrm{X}_n^s(\pi)$.

It is easily seen that $\min \mathrm{rk}_1(x) = 0$ for $x \in X_n$, since if ρ is a representation of the form $\rho = \tau_1 \oplus \cdots \oplus \tau_n$, where the $\tau_i \colon \pi \to \mathbb{C}^*$ are such that the ratio τ_i/τ_j is not a root of the Alexander polynomial for any i, j, then $H^1(\pi, \rho) = 0$. It follows that $\Pi_n^1(\pi) = \{x\,,\ \mathrm{rk}_1(x) \geq 1\}$.

We denote by SR_n^1 the set of all representations in $\mathrm{SR}_n(\pi)$ that are the sum of an $(n-1)$-dimensional irreducible representation and a 1-dimensional representation;

let $\mathrm{SX}_n^1 = \mathrm{pr}(\mathrm{SR}_n^1(\pi))$. In contrast to the case of 1-dimensional representations, SX_{n+1}^1 is isomorphic to $X_n^s(\pi)$ for $n \geq 2$.

This isomorphism can be described as follows. Suppose $x \in \mathrm{SX}_{n+1}^1$ and ρ is a semisimple representation with character x. Then $\rho = \tau \oplus \rho_0$, where ρ_0 is 1-dimensional and $\tau \in R_n^s(\pi)$, with $\rho_0 = (\det \tau)^{-1}$. Put $h_0(x) = \mathrm{pr}(\tau)$. Then $h_0 \colon \mathrm{SX}_{n+1}^1(\pi) \to \mathrm{X}_n^s(\pi)$ is the isomorphism.

We consider also an interesting mapping $h_1 \colon \mathrm{X}_n^s \to \mathrm{X}_n^s$, defined as follows: for $x \in \mathrm{X}_n^s$ and $\rho \in \mathrm{pr}^{-1} x$, put $h_1(x) = \mathrm{pr}[(\det \rho)^{-1}\rho]$. It is easily seen that h_1 is an epimorphism, and that $h_1^{-1}(y)$ contains exactly $n+1$ points. Indeed, if $y \in \mathrm{X}_n^s$ and $\rho \in \mathrm{pr}^{-1}(y)$, then $h_1^{-1}(y) = \mathrm{pr}((\det \rho)^{1/n+1} \cdot \rho)$, and $h_1 \colon \mathrm{X}_n^s \to \mathrm{X}_n^s$ is a covering of X_n^s by a second copy of X_n^s.

Let $h \colon \mathrm{SX}_{n+1}^1 \to \mathrm{X}_n^s$ be the composite mapping, $h = h_1 \circ h_0$. For $n \geq 3$ this is a covering with fiber $\mathbb{Z}_{n+1}$. We denote by SX_n^* the set of all $x \in \mathrm{SX}_n$ such that $\dim \mathrm{pr}^{-1}(x) \geq n^2 - 1$. Then SX_n^* is a closed subset of SX_n, and $\overline{\mathrm{SX}_n^s} \subset \mathrm{SX}_n^*$.

2.5.1. **Theorem.** *The image of the intersection* $\mathrm{SX}_{n+1}^1 \cap \mathrm{SX}_{n+1}^*$ *under the mapping* h *is the cohomology-jump subvariety* $\Pi_n(\pi)$.

Proof. We preface the proof with the following important lemma.

2.5.2. **Lemma.** *Let* $r(a_1, \dots, a_p)$ *be a word in the letters* $a_1^{\pm 1}, \dots, a_p^{\pm 1}$, *and* $R(A_1, \dots, A_p)$ *the same word in the letters* $A_1^{\pm 1}, \dots, A_p^{\pm 1}$ *(i.e., we replace* a_i *by* A_i*);* $F(a_1, \dots, a_p)$ *the free group with generators* $a_1, \dots, a_p$*;* $F(A_1, \dots, A_p)$ *the free group with generators* $A_1, \dots, A_p$*; and*

$$\rho \colon F(a_1, \dots, a_p) \to \mathrm{GL}_{n+1} \quad \textit{and} \quad \rho' \colon F(A_1, \dots, A_p) \to \mathrm{GL}_n$$

representations such that in some basis

$$\rho(a_i) = \left(\begin{array}{c|c} \rho'(A_i) & v_i \\ \hline 0 \cdots 0 & 1 \end{array}\right),$$

where v_i *is a vector in* $\mathbb{C}^n$. *Then*

$$\rho(r) = \left(\begin{array}{c|c} \rho'(R) & v \\ \hline 0 & 1 \end{array}\right), \qquad \textit{where} \quad v = \sum_{i=1}^{p} \rho'\left(\frac{\partial R}{\partial A_i}\right) v_i.$$

Proof of the lemma. We use induction on the length of the word r, with $r = a_1 u$ or $r = a_1^{-1} u$. We verify only the case $r = a_1^{-1} u$, the remaining cases being similar. Thus, $r = a_1^{-1} u$ and

$$\rho(a_1^{-1}) = \left(\begin{array}{c|c} \rho'(A_1^{-1}) & -\rho'(A_1^{-1}) v_1 \\ \hline 0 & 1 \end{array}\right).$$

Then

$$\rho(r) = \rho(a_1^{-1})\rho(u) = \left(\begin{array}{c|c} \rho'(R) & v \\ \hline 0 & 1 \end{array}\right),$$

where

$$v = \sum_{i=1}^{p} \rho'\left(A_1^{-1} \frac{\partial U}{\partial A_i}\right) v_i - \rho'(A_1^{-1}) v_1 = \sum_{i=1}^{p} \rho'\left(\frac{\partial R}{\partial A_i}\right) v_i,$$

which was to be proved.

Suppose $x \in \mathrm{SX}_{n+1}^1 \cap \mathrm{SX}_{n+1}^*$ and $\rho_0 \in \mathrm{pr}^{-1}(x)$, ρ_0 semisimple. Then $\rho_0 = \tau_0 \oplus (\det \tau_0)^{-1}$, $\tau_0 \in R_n^s$. Consider the orbit $O(\rho_0)$ and the stationary group $\mathrm{St}(\rho_0)$.

By the results of [6], $O(\rho_0)$ is closed. Regard the group $\mathrm{St}(\rho_0)$ as a subgroup of GL_n. It is then easily seen to be of the form

$$\left(\begin{array}{c|c} d_1 \cdot 1 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & d_2 \end{array}\right), \qquad \text{where} \quad d_1, d_2 \in \mathbb{C}^*,$$

so that $\dim \mathrm{St}(\rho_0) = 2$. Hence $\dim O(\rho_0) = \dim(\mathrm{GL}_n / \mathrm{St}(\rho_0)) = n^2 - 2n - 1$. On the other hand, $\dim \mathrm{pr}^{-1}(x) \geq n^2 + 2n$. This means that there exists a representation $\rho \in \mathrm{pr}^{-1}(x)$ that lies outside the orbit $O(\rho_0)$. But the semisimple part of this representation ρ must coincide with ρ_0, and therefore in some basis we have

$$\rho(a_i) = \left(\begin{array}{c|c} \tau(a_i) & v_i \\ \hline 0 & 1 \end{array}\right) \cdot (\det(\tau_0))^{-1},$$

where $v_i \in \mathbb{C}^n$ and $\tau = (\det \tau_0)^{-1}\tau_0$, $\mathrm{pr}(\tau) = h(x)$. Fix this basis. By Lemma 2.5.2,

$$\sum_{i=1}^{p} \tau\left(\frac{\partial r_j}{\partial a_i}\right) v_i = 0 \quad \text{for} \quad j = 1, \ldots, p-1,$$

i.e., $\tau(\partial_2)(v_1, \ldots, v_p) = 0$. This means that the set $(v_1, \ldots, v_p)$ is a cocycle. We prove that it is not a coboundary. Indeed, if $(v_1, \ldots, v_p)$ is a coboundary, there exists a vector $v \in \mathbb{C}^n$ such that $v_i = (\tau(a_i) - 1)v$. But then ρ is conjugate to ρ_0, since $\rho = P\rho_0 P^{-1}$. where

$$P = \left(\begin{array}{c|c} 1 & v \\ \hline 0 \cdots 0 & 1 \end{array}\right),$$

i.e., $\rho \in O(\rho_0)$, and we have obtained a contradiction. It follows that $H^1(\pi, r) \neq 0$, so that $h(x) \in \Pi_n(\pi)$.

Conversely, if $y \in \Pi_n(\pi)$, then since h is onto, there exists a point $x \in \mathrm{SX}^1_{n+1}$ such that $h(x) = y$. We claim that $x \in \mathrm{SX}^*_{n+1}(\pi)$. Let ρ_0 be a semisimple representation with character x, $\rho_0 = \tau_0 \oplus (\det \tau_0)^{-1}$, $\mathrm{pr}[(\det \tau_0)^{-1}\tau_0] = y$, $(\det \tau_0)^{-1}\tau_0 = \tau$. From Lemma 2.5.2 and the fact that $H^1(\pi, \tau) \neq 0$, it follows that there exists a representation $\rho \in \mathrm{pr}^{-1}(x)$ that lies outside the orbit $O(\rho_0)$. Clearly $O(\rho_0) \subset \mathrm{pr}^{-1}(x)$, $O(\rho) \subset \mathrm{pr}^{-1}(x)$. We must show that $\dim \mathrm{pr}^{-1}(x) \geq n^2 + 2n$. Assume the contrary; $\dim \mathrm{pr}^{-1}(x) < n^2 + 2n$. Then $\dim \mathrm{pr}^{-1}(x) = \dim \overline{O(\rho_0)} = n^2 + 2n - 1$. The sets $\mathrm{pr}^{-1}(x)$ and $\overline{O(\rho_0)}$ are closed, while $O(\rho)$ is not, and $\overline{O(\rho)} \supset O(\rho_0)$ (see [6], Chapter 1). Hence $\dim \overline{O(\rho)} = \dim O(\rho)$. Let V be an irreducible component of maximal dimension of the variety $O(\rho_0)$. Then V is not contained in the closure $(\overline{O(\rho)\backslash O(\rho_0)})$, whereas this closure does contain $\overline{O(\rho)}$ since $O(\rho) \subset \overline{O(\rho)}\backslash O(\rho_0)$, and $\overline{O(\rho)} \supset O(\rho_0) \supset V$—a contradiction. This means that $\dim \mathrm{pr}^{-1}(x) \geq n^2 + 2n$, i.e., $x \in \mathrm{SX}^*_{n+1}$. The theorem is proved.

Let us consider the group $\mathrm{St}(\rho)$, where ρ is a representation.

2.5.3. **Lemma.** *If* $\dim \mathrm{St}(\rho) \leq 1$, *then* $\mathrm{St}(\rho) = \mathbb{C}^* = \{d \cdot 1, d \in \mathbb{C}^*\}$.

Proof. We have $g \in \mathrm{St}(\rho) \Leftrightarrow g\rho(a_i)g^{-1} = \rho(a_i)$, $i = 1, \ldots, p$; i.e., $\mathrm{St}(\rho) = \mathrm{GL}_n \cap \{g \in M_n, \rho(a_i)g = g\rho(a_i)\}$, where M_n is the set of all matrices of order $n \times n$. The second set is a linear space, and it contains the set $\{d \cdot 1, d \in \mathbb{C}\}$. Clearly, $\dim \mathrm{St}(\rho)$ is equal to the dimension of this linear space; therefore, if $\dim \mathrm{St}(\rho) \leq 1$, then $\mathrm{St}(\rho) = \mathbb{C}^*$.

Lemma 2.5.3 indicates that if the orbit $O(\rho)$ has the maximal dimension $n^2 - 1$, then the stationary group $\mathrm{St}(\rho)$ is $\mathbb{C}^*$. Let $\mathrm{SR}^{\max}_n$ be the set of all representations

$\rho \in \mathrm{SR}_n$ whose stationary group is $\mathbb{C}^*$, and $\mathrm{SX}_n^{\max}$ its image under the projection pr. Obviously, $\mathrm{SX}_n^* \supset \mathrm{SX}_n^{\max}$.

2.5.4. **Proposition.** *The intersection* $\mathrm{SX}_{n+1}^1 \cap \mathrm{SX}_{n+1}^{\max}$ *coincides with the intersection* $\mathrm{SX}_{n+1}^1 \cap \mathrm{SX}_{n+1}^*$; *therefore, its image under the mapping* h *of Theorem* 2.5.1 *coincides with the cohomology-jump subvariety* $\Pi_n(\pi)$.

Proof. Clearly,

$$(\mathrm{SX}_{n+1}^{\max} \cap \mathrm{SX}_{n+1}^1) \subset (\mathrm{SX}_{n+1}^* \cap \mathrm{SX}_{n+1}^1),$$

since $\mathrm{SX}_{n+1}^{\max} \subset \mathrm{SX}_{n+1}^*$.

Suppose $x \in \mathrm{SX}_{n+1}^* \cap \mathrm{SX}_{n+1}^1$ and $\tau \in \mathrm{pr}^{-1}(h(x))$. As in the proof of Theorem 2.5.1, take a semisimple representation ρ_0 with character x, and let $\rho \in \mathrm{pr}^{-1}(x)$ be nonsemisimple. From the proof of the theorem we see that $\dim O(\rho_0) = n^2+2n-1$, $O(\rho_0)$ is closed, $\overline{O(\rho)} \supset O(\rho_0)$, and $O(\rho) \cap O(\rho_0) = \varnothing$; it follows that $\dim O(\rho) > \dim O(\rho_0)$. This means that $\dim O(\rho) \geq n^2 + 2n$, i.e., $\dim \mathrm{St}(\rho) \leq 1$; by Lemma 2.5.3 this means that $\mathrm{St}(\rho) = \mathbb{C}^*$, i.e., $x \in \mathrm{SX}_{n+1}^{\max}$. Q.E.D.

It was shown in §2.4 that $\mathrm{SX}_2^* = \mathrm{SX}_2^{\max}$ ($\mathrm{SX}_2^{\max}$ is simply the set of characters of all nonabelian representations). Whether this equality holds for all dimensions, we do not know.

3. The groups of 2-bridge knots

In this section we compute the space of characters of representations into SL_2. The space of representations has been described in [8], but the different approach to the computation we use here is, in our view, more natural and more suitable for our purposes. The proof of Theorem 3.3.1 below is a generalization of the proof in [9] for the case of the figure-eight knot.

3.1. **2-bridge knots (see [4]).** A knot is said to be *2-bridge* if it can be obtained by closing up a braid of four strands with four semicircles (two upper ones and two lower). (See Figure 1.)

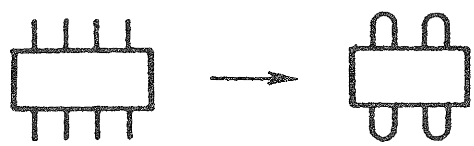

FIGURE 1

The fundamental group of the knot has a simple form:

$$\pi = \langle a, b \mid wa = bw \rangle,$$

where

$$w = a^{\varepsilon_1} b^{\varepsilon_n} a^{\varepsilon_2} b^{\varepsilon_{n-1}} \cdots a^{\varepsilon_n} b^{\varepsilon_1}, \qquad \varepsilon_i = \pm 1.$$

3.2. **Representations into** SL_2. Let F be the free group generated by the two elements a and b. Then $\pi = F/G$, where G is the normal subgroup generated by the element $r = w^{-1}b^{-1}wa$. It is easily seen that $\mathrm{SX}_2(F)$ is isomorphic to $\mathbb{C}^3$; specifically, the numbers $\operatorname{tr}\rho(a)$, $\operatorname{tr}\rho(b)$, $\operatorname{tr}\rho(ab)$ uniquely determine a representation ρ in $\mathrm{SX}_2(F)$ (see [7]). Let T be the subset of $\mathrm{SX}_2(F)$ consisting of the characters of all representations of F for which $\operatorname{tr}\rho(a) = \operatorname{tr}\rho(b)$. Then T is the 2-dimensional complex space $\mathbb{C}^2$. For any element c of F (i.e., any word in the letters $a^{\pm 1}$ and $b^{\pm 1}$) there exists a polynomial P_c in two variables such that

$\operatorname{tr}\rho(c) = P_c(\operatorname{tr}\rho(a), \operatorname{tr}\rho(ab))$ for every representation whose character belongs to the set T (see [7]). Abusing the language, we shall say that a representation ρ belongs to T if its character belongs to T.

We recall the following formulas, to be used below (see [7], [9]). Let C_1 and C_2 be elements of GL_2. Then

$$\operatorname{tr}(C_1C_2) = \operatorname{tr}(C_1)\operatorname{tr}(C_2) - \det C_1 \operatorname{tr}(C_1^{-1}C_2), \tag{1}$$

$$\operatorname{tr}(C_1^{-1}) = (\det C_1)^{-1}\operatorname{tr}(C_1). \tag{2}$$

3.2.1. **Lemma.** *Suppose* $u \in F$, *i.e.,* u *is a word in the letters* $a^{\pm 1}$ *and* $b^{\pm 1}$. *Let* $\mu_1(u)$ *be the word obtained from* u *by interchanging the letters* a *and* b. *Then* $\operatorname{tr}\rho(u) = \operatorname{tr}\rho(\mu_1(u))$ *for any representation in* T.

Proof. Consider the automorphism $\mu_1\colon F \to F$, $\mu_1(a) = b$, $\mu_1(b) = a$. Then $\rho \circ \mu_1$ is likewise a representation of F, and obviously $\operatorname{pr}(\rho \circ \mu_1) = \operatorname{pr}(\rho)$; so $\operatorname{tr}\rho(u) = \operatorname{tr}\rho(\mu_1(u))$ for any $u \in F$.

3.2.2. **Lemma.** *Suppose the word* $\overleftarrow{u}$ *is obtained from* u *by reversing the order of the letters. Then* $\operatorname{tr}\rho(u) = \operatorname{tr}\rho(\overleftarrow{u})$ *for any* ρ *in* T.

Proof. Let $\mu_2\colon F \to F$ be the automorphism

$$\mu_2(a) = a^{-1}, \qquad \mu_2(b) = b^{-1}.$$

Then $\rho \circ \mu_2\colon F \to \mathrm{SL}_2$ is a representation, and $\operatorname{pr}(\rho \circ \mu_2) = \operatorname{pr}(\rho)$. Therefore, $\operatorname{tr}\rho(\mu_2(u)) = \operatorname{tr}\rho(u)$. By formula (2), $\operatorname{tr}\rho(\mu_2(u)) = \operatorname{tr}\rho[(\mu_2(u))^{-1}]$; and since $\mu_2(u)^{-1} = \overleftarrow{u}$ we have $\operatorname{tr}\rho(\overleftarrow{u}) = \operatorname{tr}\rho(u)$.

3.2.3. **Corollary.** *Let* $w = a^{\varepsilon_1}b^{\varepsilon_n}a^{\varepsilon_2}b^{\varepsilon_{n-1}}\cdots a^{\varepsilon_n}b^{\varepsilon_1}$ $(w \in F)$. *Then for any representation* ρ *in* T *we have*

$$\operatorname{tr}\rho(wa) = \operatorname{tr}\rho(bw), \qquad \operatorname{tr}\rho(b^{-1}wa) = \operatorname{tr}\rho(bwa^{-1}).$$

Proof. Observe that $\mu_1(\overleftarrow{w}) = w$. Therefore $\mu_1(\overleftarrow{wa}) = \mu_1(a\overleftarrow{w}) = \mu_1(a)\mu_1(\overleftarrow{w}) = bw$. Lemmas 1 and 2 allow us to conclude that $\operatorname{tr}\rho(wa) = \operatorname{tr}\rho(bw)$. Similarly, $\mu_1(\overleftarrow{bwa^{-1}}) = \mu_1(a^{-1}\overleftarrow{w}b) = b^{-1}wa$, and therefore $\operatorname{tr}\rho(b^{-1}wa) = \operatorname{tr}\rho(bwa^{-1})$.

We define a grading in $\mathbb{Z}[t_1, t_2]$ by putting $\deg t_1 = 1$, $\deg t_2 = 2$, where $t_1 = \operatorname{tr}\rho(a)$, $t_2 = \operatorname{tr}\rho(ab)$.

3.2.4. **Lemma.** $\deg P_u$ *does not exceed the length of the word* u.

Proof. We use induction on the length of u. If the lemma holds for the words x and ax, then it holds also for the word $a^{-1}x$; indeed, $\operatorname{tr}\rho(a^{-1}x) + \operatorname{tr}\rho(ax) = t_1\operatorname{tr}\rho(x)$. Therefore, we can assume that in the word u all the letters a and b have positive powers. If $u = xa^2y$, then $\operatorname{tr}\rho(u) = t_1\operatorname{tr}\rho(xay) - \operatorname{tr}\rho(xy)$, and the lemma holds also for u. It suffices then to consider the remaining case that u contains no expression of the form a^2 or b^2. This means that if the length of u is greater than 3, then u has the form

$$u = (ab)^2x,$$

so that $\operatorname{tr}\rho(u) = t_2\operatorname{tr}\rho(abx) - \operatorname{tr}\rho(x)$, and $\deg P_u \le \deg P_x + 4$. The case that the length of u is less than 4 can be verified directly.

3.3. **The character varieties for nonabelian representations.** We have $wa = bw$, which means that a and b are conjugate. So for all representations ρ of the group π, $\operatorname{tr}\rho(a) = \operatorname{tr}\rho(b)$. This means that the set $\mathrm{SX}_2(\pi)$ of all characters can be imbedded into $T = \mathbb{C}^2$. Put $t_1 = \operatorname{tr}\rho(a)$, $t_2 = \operatorname{tr}\rho(ab)$. These are two independent coordinates for the variety T, and the trace of any element of π is a polynomial in t_1 and t_2: it suffices to regard this element as an element of the group F, and the representation ρ as a representation of F with character in T.

Rewriting the equality $wa = bw$ as $w = bw_a^{-1}$, we obtain

$$P_w(t_1, t_2) = P_{bwa^{-1}}(t_1, t_2).$$

It turns out that this equality determines the character variety $SX_2(\pi)$.

3.3.1. **Theorem.** *The character variety* $SX_2(\pi)$ *is an algebraic subvariety of the variety* $T = \mathbb{C}^2$, *and is determined by the equality* $P_{bwa^{-1}} - P_w = 0$. *Furthermore, we have the factorization* $P_{bwa^{-1}} - P_w = (2 + t_2 - t_1^2)\Phi_w(t_1, t_2)$. *Here, if* x' *is the word obtained from* x *by deleting the two end letters, then*

$$\Phi_w = P_w - P_{w'} + \cdots + (-1)^{n-1}P_{w^{n-1}} + (-1)^n.$$

The first factor $t_1^2 - t_2 - 2$ *determines the character variety for abelian representations, i.e.,* $SX_2^1(\pi)$; *the second factor* Φ_w *determines the character variety for nonabelian representations, i.e.,* $SX_2^*(\pi)$.

Proof. We prove first the equality

$$P_{bwa^{-1}} - P_w = (t_1^2 - t_2 - 2)\Phi_w(t_1, t_2)$$

by induction on the length of the word w. Suppose this equality holds for the word $u = w'$, i.e.,

$$P_{aub^{-1}} - P_u = (t_1^2 - t_2 - 2)\Phi_u$$

(for u the letters a and b interchange). We consider separately the two cases $w = aub$ and $w = a^{-1}ub^{-1}$.

a) $w = aub$. Everywhere below we write $\operatorname{tr} x$ for $\operatorname{tr}\rho(x)$, which should not cause any confusion. We have $\operatorname{tr}(bwa^{-1}) = \operatorname{tr}(bauba^{-1})$. Applying formula (1) with $C_1 = a^{-1}$, $C_2 = baub$, we obtain

$$\operatorname{tr}(bwa^{-1}) = t_1 \operatorname{tr}(baub) - \operatorname{tr}(bauba) = t_1 \operatorname{tr}(b^2au) - \operatorname{tr}[(ba)^2u].$$

Again applying (1) to the factorizations $\operatorname{tr}(b \cdot bau)$ and $\operatorname{tr}(ba)(bau)$, we get

$$\begin{aligned}\operatorname{tr}(bwa^{-1}) &= t_1[t_1 \operatorname{tr}(bau) - \operatorname{tr}(au)] - t_2 \operatorname{tr}(bau) + \operatorname{tr} u\\ &= t_1^2 \operatorname{tr} w - t_1 \operatorname{tr}(au) - t_2 \operatorname{tr} w + \operatorname{tr} u\\ &= (t_1^2 - t_2 - 2)\operatorname{tr} w + 2\operatorname{tr} w - t_1 \operatorname{tr}(au) + \operatorname{tr} u\\ &= (t_1^2 - t_2 - 2)\operatorname{tr} w + \operatorname{tr} w + [\operatorname{tr}(aub) - t_1 \operatorname{tr}(au)] + \operatorname{tr} u.\end{aligned}$$

By formula (1),

$$\operatorname{tr}(aub) - t_1 \operatorname{tr}(au) = -\operatorname{tr}[(au)b] + \operatorname{tr}(au)\operatorname{tr}(b) = -\operatorname{tr}(aub^{-1}).$$

Therefore,

$$\operatorname{tr}(bwa^{-1}) = \operatorname{tr} w + (t_1^2 - t_2 - 2)\operatorname{tr} w + \operatorname{tr} u - \operatorname{tr}(aub^{-1}).$$

By the induction assumption,

$$\operatorname{tr} u - \operatorname{tr}(aub^{-1}) = (t_1^2 - t_2 - u)\Phi_u,$$
$$\operatorname{tr}(bwa^{-1}) - \operatorname{tr} w = (t_1^2 - t_2 - 2)(\operatorname{tr} w - \Phi_u) = (t_1^2 - t_2 - 2)\Phi_w,$$

which was to be proved.

b) $w = a^{-1}ub^{-1}$. Then

$$\begin{aligned}
\operatorname{tr}(bwa^{-1}) - \operatorname{tr} w &= \operatorname{tr}(ba^{-1}ub^{-1}a^{-1}) - \operatorname{tr} w \\
&= t_1 \operatorname{tr}(a^{-1}ub^{-1}a^{-1}) - \operatorname{tr} w - \operatorname{tr}(b^{-1}a^{-1}ub^{-1}a^{-1}) \\
&= t_1[t_1 \operatorname{tr}(ub^{-1}a^{-1}) - \operatorname{tr}(ub^{-1})] - \operatorname{tr} w - t_2 \operatorname{tr}(ub^{-1}a^{-1}) + \operatorname{tr} u \\
&= (t_1^2 - t_2 - 2)\operatorname{tr} w + \operatorname{tr} w - t_1 \operatorname{tr}(ub^{-1}) + \operatorname{tr} u \\
&= (t_1^2 - t_2 - 2)\operatorname{tr} w + \operatorname{tr} u - \operatorname{tr} aub^{-1} \\
&= (t_1^2 - t_2 - 2)(\operatorname{tr} w - \Phi_u) \\
&= (t_1^2 - t_2 - 2)\Phi_w,
\end{aligned}$$

and the second case is complete.

To continue with the proof of Theorem 3.3.1, we need a lemma from [9]:

3.3.2. **Lemma.** *Let R, A, B be three matrices in SL_2 such that $\operatorname{tr} A = \operatorname{tr} B = t_1$, $\operatorname{tr}(AB) \neq 2$, $\operatorname{tr}(AB) \neq t_1^2 - 2$, $\operatorname{tr} R = 2$, $\operatorname{tr}(RA) = \operatorname{tr}(RB) = t_1$. Then $R = 1$ (the identity matrix).*

We prove several further lemmas, generalizing Lemma 2 of [9].

3.3.3. **Lemma.** *For any representation ρ of F with character in T,*

$$\operatorname{tr}(w^{-1}b^{-1}wa) = 2 + (t_1^2 - t_2 - 2)\Phi_w^2.$$

Proof. We have

$$\operatorname{tr}(w^{-1}b^{-1}wa) - 2 = \operatorname{tr}(w^{-1}b^{-1}wa - 1) = \operatorname{tr}[w^{-1}b^{-1}(wa - bw)].$$

Applying formula (1) with $C_1 = w^{-1}b^{-1}$, $C_2 = wa - bw$, we obtain

$$\operatorname{tr}(w^{-1}b^{-1}wa) - 2 = \operatorname{tr}(w^{-1}b^{-1})\operatorname{tr}(wa - bw) - \operatorname{tr}[bw(wa - bw)]).$$

By Corollary 3.2.3, $\operatorname{tr}(wa - bw) = 0$. Hence

$$\operatorname{tr}(w^{-1}b^{-1}wa) - 2 = \operatorname{tr} bwbw - \operatorname{tr} bw^2a.$$

We prove the lemma by induction.

a) Suppose $w = aub$. Then

$$\begin{aligned}
\operatorname{tr}(bwbw) - \operatorname{tr}(bw^2a) &= \operatorname{tr}[a(ubaauba)] - \operatorname{tr}[(ba)(ubauba)] \\
&= t_1 \operatorname{tr}(ubaauba) - \operatorname{tr}(ubaaub) - t_2 \operatorname{tr}(ubauba) + \operatorname{tr}(ubau) \\
&= t_1 \operatorname{tr}(a^2ubauba) - \operatorname{tr}(a^2ubub) - t_2 \operatorname{tr}(w^2) + \operatorname{tr}(u^2ba) \\
&= t_1[t_1 \operatorname{tr}(aubaub) - \operatorname{tr}(ubaub)] - t_1 \operatorname{tr}(aubub) \\
&\quad + \operatorname{tr}(ubub) - t_2 \operatorname{tr}(w^2) + \operatorname{tr}(u^2ba) \\
&= (t_1^2 - t_2 - 2)\operatorname{tr}(w^2) + \operatorname{tr}(ubub) - \operatorname{tr}(u^2ba) \\
&\quad + 2[\operatorname{tr}(w^2) - t_1 \operatorname{tr}(aubub) + \operatorname{tr}(u^2ba)].
\end{aligned}$$

By formula (1), $\operatorname{tr}(w^2) = (\operatorname{tr} w)^2 - 2$. Using the induction assumption, we obtain:

$$\begin{aligned}
\operatorname{tr}(bwbw) - \operatorname{tr}(bw^2a) &= (t_1^2 - t_2 - 2)[(\operatorname{tr} w)^2 + \Phi_u^2] \\
&\quad + 2[\operatorname{tr}(w^2) - t_1 \operatorname{tr}(aubub) + \operatorname{tr} u^2ba - t_1^2 + t_2 + 2].
\end{aligned}$$

Consider the last term:

$$\begin{aligned}
t_1 \operatorname{tr}(aubub) + t_1^2 &= t_1[\operatorname{tr}(aub \cdot ub) + \operatorname{tr}(ub)^{-1}(uba)] \\
&= t_1 \operatorname{tr}(ub)\operatorname{tr}(aub) = t_1 \operatorname{tr}(w)\operatorname{tr}(ub),
\end{aligned}$$

$$\operatorname{tr}(w^2) + 2 = (\operatorname{tr} w)^2 = \operatorname{tr} w \operatorname{tr}(aub),$$
$$\operatorname{tr}(u^2ba) + t_2 = \operatorname{tr}(uuba) + \operatorname{tr}(u^{-1}uba) = \operatorname{tr}(w)\operatorname{tr}(u).$$

Therefore,

$$\begin{aligned}\operatorname{tr}(bwbw) - \operatorname{tr}(bw^2a) &= (t_1^2 - t_2 - 2)[(\operatorname{tr} w)^2 + \Phi_u^2] \\ &\quad + 2\operatorname{tr} w[\operatorname{tr} aub - t_1 \operatorname{tr} ub + \operatorname{tr} u],\end{aligned}$$

$$\begin{aligned}&(t_1^2 - t_2 - 2)[(\operatorname{tr} w)^2 + \Phi_u^2] + 2\operatorname{tr} w[\operatorname{tr} u - \operatorname{tr} a^{-1}ub] \\ &\quad = (t_1^2 - t_2 - 2)[(\operatorname{tr} w)^2 + \Phi_u^2] + 2\operatorname{tr} w(t_1^2 - t_2 - 2)\Phi_u,\end{aligned}$$

$$(t_1^2 - t_2 - 2)[(\operatorname{tr} w)^2 + \Phi_u^2 - 2\operatorname{tr} w\Phi_u] = (t_1^2 - t_2 - 2)\Phi_w^2.$$

b) Suppose $w = a^{-1}ub^{-1}$. Then

$$\begin{aligned}\operatorname{tr}(bwbw) - \operatorname{tr}(wabw) &= \operatorname{tr}(ba^{-1}ua^{-1}ub^{-1}) - \operatorname{tr}(a^{-1}ub^{-1}aba^{-1}ub^{-1}) \\
&= \operatorname{tr}(a^{-1}ua^{-1}u) - t_2\operatorname{tr}(w^2) + \operatorname{tr}(a^{-2}ub^{-1}a^{-1}ub^{-2}) \\
&= \operatorname{tr}(a^{-1}ua^{-1}u) - t_2(\operatorname{tr}(w^2) + t_1\operatorname{tr}(a^{-1}ub^{-1}a^{-1}ub^{-2}) - \operatorname{tr}(ub^{-1}a^{-1}ub^{-2}) \\
&= \operatorname{tr}(a^{-1}ua^{-1}u) - t_2\operatorname{tr}(w^2) + t_1[t_1\operatorname{tr}(w^2) - \operatorname{tr}(a^{-1}ub^{-1}a^{-1}u)] \\
&\quad - t_1\operatorname{tr}(ub^{-1}a^{-1}ub^{-1}) + \operatorname{tr}(u^2b^{-1}a^{-1}) \\
&= (t_1^2 - t_2 - 2)\operatorname{tr}(w^2) + 2\operatorname{tr}(w^2) - 2t_1[\operatorname{tr}(a^{-1}u)\operatorname{tr}(b^{-1}a^{-1}u) - t_1] \\
&\quad + \operatorname{tr}(u^2b^{-1}a^{-1}) + \operatorname{tr}(a^{-1}ua^{-1}u) \\
&= (t_1^2 - t_2 - 2)[(\operatorname{tr} w)^2 + \Phi_u^2] \\
&\quad + 2[(\operatorname{tr} w)^2 - t_1\operatorname{tr}(a^{-1}u)\operatorname{tr} w + t_2 + \operatorname{tr}(u^2b^{-1}a^{-1})] \\
&= (t_1^2 - t_2 - 2)[(\operatorname{tr} w)^2 + \Phi_u^2] + 2\operatorname{tr} w[\operatorname{tr}(w) + \operatorname{tr}(u) - t_1\operatorname{tr}(a^{-1}u)] \\
&= (t_1^2 - t_2 - 2)[(\operatorname{tr} w)^2 + \Phi_u^2] + 2\operatorname{tr}(w)[\operatorname{tr}(u) - \operatorname{tr}(a^{-1}ub)] \\
&= (t_1^2 - t_2 - 2)[(\operatorname{tr} w)^2 + \Phi_u^2 - 2\operatorname{tr}(w)\Phi_u] \\
&= (t_1^2 - t_2 - 2)[\operatorname{tr} w - \Phi_u]^2 \\
&= (t_1^2 - t_2 - 2)\Phi_w^2.\end{aligned}$$

This completes the proof of Lemma 3.3.3.

3.3.4. **Lemma.** *For all representations in* T,

$$\operatorname{tr}(w^{-1}b^{-1}wab) - t_1 = (t_1^2 - t_2 - 2)\Phi_w[t_1\Phi_w - \operatorname{tr}(aw)].$$

Proof. We have

$$\begin{aligned}\operatorname{tr}(w^{-1}b^{-1}wab) - t_1 &= \operatorname{tr}[w^{-1}(b^{-1}wab - wa)] \\
&= \operatorname{tr} w[\operatorname{tr}(b^{-1}wab) - \operatorname{tr}(wa)] - \operatorname{tr}[w(b^{-1}wab - wa)] \\
&= \operatorname{tr} w^2a - \operatorname{tr} wb^{-1}wab = \operatorname{tr} w^2a - \operatorname{tr}[(bw)(b^{-1}wa)] \\
&= \operatorname{tr} w\operatorname{tr}(wa) - t_1 - \operatorname{tr}(bw)\operatorname{tr}(b^{-1}wa) + \operatorname{tr}(w^{-1}b^{-2}wa) \\
&= \operatorname{tr}(wa)[\operatorname{tr} w - \operatorname{tr}(b^{-1}wa)] - t_1 + \operatorname{tr}(b^{-2}waw^{-1}) \\
&= \operatorname{tr}(wa)\Phi_w(t_1^2 - t_2 - 2) + t_1\operatorname{tr}(b^{-1}waw^{-1}) - 2t_1 \\
&= \operatorname{tr}(wa)\Phi_w(t_1^2 - t_2 - 2) + t_1(\operatorname{tr}(w^{-1}b^{-1}wa) - 2) \\
&= \text{(by Lemma 3.3.3) } (t_1^2 - t_2 - 2)\Phi_w[t_1\Phi_w - \operatorname{tr}(wa)].\end{aligned}$$

3.3.5. **Lemma.** *For an*
such that $\operatorname{tr} A = \operatorname{tr} B =$
chosen so as not to con
will not commute.

Proof. We choose A a

Then $t_1 = \operatorname{tr} A = \operatorname{tr} B =$

$AB = ($

and

Clearly, $\forall t_1, t_2 \in \mathbb{C}$
$\operatorname{tr} AB = t_2$. If $t_2 \neq t_1^2$
and B commute, then

where either $\lambda = 0$ or
means that if $t_2 \neq 2$ a

3.3.6. **Lemma.** $\Phi_w(2$

Proof. Consider the tr
Hence for any represen
that

$\Phi_w($

We proceed now wi
tation $\rho(a) = A$, $\rho(b)$
if t_1, t_2 satisfy the eq
such that $W^{-1}B^{-1}WA$
or $\Phi_w(t_1, t_2) = 0$. If

where $t + t^{-1} = t_1$. Th
$\rho(b) = B$ is as require
Suppose $t_1^2 - t_2 - 2 \neq$
if two noncommuting
then $W^{-1}B^{-1}WA = 1$
A and B with this pr
By Lemma 3.3.3, $\operatorname{tr} R$
Let us compute $\operatorname{tr}(RA$
$\operatorname{tr}(A^{-1}R) + \operatorname{tr}(AR) = t_1$
of Lemma 3.3.2 are sati

3.3.5. **Lemma.** *For any pair of numbers* $t_1, t_2 \in \mathbb{C}$, *there exist matrices* $A, B \in \mathrm{SL}_2$ *such that* $\operatorname{tr} A = \operatorname{tr} B = t_1$ *and* $\operatorname{tr} AB = t_2$. *If* $t_2 \neq t_1^2 - 2$, *then* A *and* B *can be chosen so as not to commute; and if in addition* $t_2 \neq 2$, *then any two such matrices will not commute.*

Proof. We choose A and B in the form

$$A = \begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix}, \qquad B = \begin{pmatrix} t & 0 \\ \beta & t^{-1} \end{pmatrix}.$$

Then $t_1 = \operatorname{tr} A = \operatorname{tr} B = t + t^{-1}$,

$$AB = \begin{pmatrix} t^2+\beta & t^{-1} \\ t^{-1}\beta & t^{-2} \end{pmatrix}, \qquad BA = \begin{pmatrix} t^2 & t \\ t\beta & t^{-2}+\beta \end{pmatrix},$$

and

$$t_2 = \operatorname{tr} AB = t^2 + t^{-2} + \beta = t_1^2 - 2 + \beta.$$

Clearly, $\forall t_1, t_2 \in \mathbb{C}$ there exist matrices A, B such that $\operatorname{tr} A = \operatorname{tr} B = t_1$ and $\operatorname{tr} AB = t_2$. If $t_2 \neq t_1^2 - 2$, then $\beta \neq 0$, and obviously $AB \neq BA$. Moreover, if A and B commute, then there exists a basis in which

$$A = \begin{pmatrix} t & \lambda \\ 0 & t^{-1} \end{pmatrix}, \qquad B = \begin{pmatrix} t^{\pm 1} & * \\ 0 & t^{\mp 1} \end{pmatrix},$$

where either $\lambda = 0$ or $\lambda = 1$, and then either $t_2 = \operatorname{tr} AB = 2$ or $t_2 = t_1^2 - 2$. This means that if $t_2 \neq 2$ and $t_1^2 - 2 \neq t_2$, then A and B do not commute.

3.3.6. **Lemma.** $\Phi_w(2, 2) = (-1)^{n-1}$.

Proof. Consider the trivial representation $\rho(a) = \rho(b) = 1$. Here $t_1 = t_2 = 2$. Hence for any representation with $t_1 = t_2 = 2$ we have $\operatorname{tr} w = 2$ $\forall w$. This means that

$$\Phi_w(2, 2) = 2 - 2 + 2 - \cdots + (-1)^n = (-1)^{n-1}.$$

We proceed now with the proof of Theorem 3.3.1. Clearly, if ρ is the representation $\rho(a) = A$, $\rho(b) = B$, then $P_{bwa^{-1}} - P_w = 0$. We must prove the converse: if t_1, t_2 satisfy the equation $P_{bwa^{-1}} - P_w = 0$, then there exist matrices A and B such that $W^{-1}B^{-1}WA = 1$, where $W = w(A, B)$. We have either $t_1^2 - t_2 - 2 = 0$ or $\Phi_w(t_1, t_2) = 0$. If $t_1^2 - t_2 - 2 = 0$, consider the matrices

$$A = B = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},$$

where $t + t^{-1} = t_1$. Then obviously the representationn ρ such that $\rho(a) = A$ and $\rho(b) = B$ is as required.

Suppose $t_1^2 - t_2 - 2 \neq 0$ and $\Phi_w = 0$. We prove a stronger assertion, namely, that if two noncommuting matrices satisfy the conditions $\operatorname{tr} A = \operatorname{tr} B = t_1$, $\operatorname{tr} AB = t_2$, then $W^{-1}B^{-1}WA = 1$ (by Lemma 3.3.5 there always exist noncommuting matrices A and B with this property). Let $R = W^{-1}B^{-1}WA$. Suppose first that $t_2 \neq 2$. By Lemma 3.3.3, $\operatorname{tr} R = 2$. By Lemma 3.3.4, $\operatorname{tr} RB = \operatorname{tr}(W^{-1}B^{-1}WAB) = t_1$. Let us compute $\operatorname{tr}(RA)$. We have $\operatorname{tr}(RA^{-1}) = \operatorname{tr}(W^{-1}BW) = t_1$. By formula (1), $\operatorname{tr}(A^{-1}R) + \operatorname{tr}(AR) = t_1 \operatorname{tr} R = 2t_1$. It follows that $\operatorname{tr} RA = t_1$. Since all the conditions of Lemma 3.3.2 are satisfied, we have $R = 1$. Now consider the case $t_2 = 2$. Observe that in view of Lemma 3.3.6 the polynomial Φ_w is not divisible by t_2-2, so that the intersection of the varieties $\{\Phi_w = 0\}$ and $\{t_2 - 1 = 0\}$ consists of a finite number of points; by continuity (or the fact that the set $SX_2(\pi)$ is closed) we conclude that $R = 1$. This completes the proof of the theorem.

3.4. Some properties of the variety of representations.

3.4.1. Proposition. *The polynomial $\Phi_w \in \mathbb{Z}[t_1, t_2]$ has no quadratic divisor; in other words, the ideal (Φ_w) in $\mathbb{Z}[t_1, t_2]$ is radical.*

Proof. For any element $u \in F$ and any representation $\rho \in T$ we have:

$$\begin{aligned} &\operatorname{tr}(aub) + \operatorname{tr}(a^{-1}ub) = t_1 \operatorname{tr}(ub), \operatorname{tr}(a^{-1}ub) + \operatorname{tr}(a^{-1}ub^{-1}) = t_1 \operatorname{tr}(a^{-1}u) \\ &\quad\Rightarrow \operatorname{tr}(aub) \equiv \operatorname{tr}(a^{-1}ub^{-1})(\operatorname{mod} t_1). \end{aligned}$$

It follows that $\Phi_w = \Phi + t_1 H$, where $\Phi = \Phi_{w_0}$, $w_0 = (ab)^n$, $H \in \mathbb{Z}[t_1, t_2]$.

Below in §4.3 (Proposition 4.3.2) we show that Φ is a polynomial in t_2 (independent of t_1) and has no multiple divisor other than unity. Let $\Phi_w = \Phi_1^{k_1} \cdots \Phi_m^{k_m}$, where the $\Phi_i \in \mathbb{Z}[t_1, t_2]$ are not constants. Write Φ_i in the form

$$\Phi_i = G_i(t_2) + t_1(H_i(t_1, t_2)).$$

It is easily seen that $\Phi = G_1^{k_1} \cdots G_m^{k_m}$. Observe that $\deg \Phi_w = \deg \Phi = 2n$, and therefore $\deg G_i = \deg \Phi_i$. This means that the G_i are likewise not constants, and therefore $k_1 = k_2 = \cdots = k_m = 1$; q.e.d.

The following corollary will be useful. We call two elements u and v in $\mathbb{Z}[F]$ congruent ($u \equiv v$) if they determine the same element of the factor-ring $\mathbb{Z}[F]/(wa = bw)$.

3.4.2. Corollary. *If $u \equiv v$, then $P_u - P_v$ is divisible by Φ_w.*

Proof. Consider a pair of numbers t_1, t_2 such that $\Phi_w(t_1, t_2) = 0$, $t_2 \neq 2$, $t_2 \neq t_1^2 - 2$. The set of such pairs is everywhere dense on the curve $\{\Phi_w = 0\}$. For any such pair there exist matrices A and B such that putting $\rho(a) = A$, $\rho(b) = B$ gives a representation of our group π. Then $\operatorname{tr} \rho(u) = \operatorname{tr} \rho(v)$. This means that $P_u(t_1, t_2) = P_v(t_1, t_2)$ if $\Phi_w = 0$ and $t_2 = 2$, $t_2 \neq t_1^2 - 2$. Proposition 3.4.1 allows us to conclude that $P_u - P_v$ is divisible by Φ_w.

4. The cohomology-jump subvariety of a representation into GL_2

4.1. We have $\pi = \langle a, b \mid wa = bw \rangle$. It is easily seen that the word $r = w^{-1}b^{-1}wa$ is not a power; therefore, by Lyndon's theorem (see [11], Chapter 3) the complex X constructed in §2.2 is aspherical, i.e., X is homotopic to the complement of our knot in S^3. Consider the complex of modules

$$0 \leftarrow \mathbb{C}^2 \overset{\rho(\partial_1)}{\longleftarrow} \mathbb{C}^4 \overset{\rho(\partial_0)}{\longleftarrow} \mathbb{C}^2 \leftarrow 0,$$

where $\partial_1 = (\partial r/\partial a, \partial r/\partial b)$. Clearly,

$$H^2(\pi, \rho) = 2 - \dim(\rho(\partial r/\partial a), \rho(\partial r/\partial b)),$$

and therefore

$$\operatorname{rk}_2(\rho) \geq 1 \Leftrightarrow \dim(\rho(\partial r/\partial a), \rho(\partial r/\partial b)) \leq 1.$$

REPRESEN

Let δ = (det $\rho(a))^1$
SL_2. Let $\widetilde{T}$ be the s
that tr $\rho(a)$ = tr $\rho(b)$.
polynomials Q_u, Q'_u s

tr $\rho(u)$ = (

where t_1 = tr A, t_2 = t
Similarly to Corolla

4.1.1. **Corollary.** *If u*

$Q_u - Q$

4.1.2. **Corollary.** det(
$X_2^s(\pi)$.

Proof. We have the fu

$\frac{\partial r}{\partial a}(a-1)$ -

⇒ (1 -

where $f \in \mathbb{Z}[\delta^{\pm 1}, t_1,$
is not divisible by (1 -

q.e.d.
We recall that w =
Define

Put

4.2. We have

$\frac{\partial r}{\partial a} \equiv$

Clearly, a necessary

It turns out that this is

Let $\delta = (\det\rho(a))^{1/2}$, $\delta A = \rho(a)$, $\delta B = \rho(b)$. Then A and B belong to SL_2. Let $\widetilde{T}$ be the set of all characters of representations of π into GL_2 such that $\operatorname{tr}\rho(a) = \operatorname{tr}\rho(b)$. Then for any element u of the group ring $\mathbb{Z}[F]$ there exist polynomials Q_u, Q'_u such that

$$\operatorname{tr}\rho(u) = Q_u(\delta^{\pm 1}, t_1, t_2), \qquad \det\rho(u) = Q'_u(\delta^{\pm 1}, t_1, t_2),$$

where $t_1 = \operatorname{tr} A$, $t_2 = \operatorname{tr}(AB)$, $\rho \in \mathrm{pr}^{-1}(\widetilde{T})$.

Similarly to Corollary 3.4.2, we have:

4.1.1. **Corollary.** *If* $u, v \in \mathbb{Z}[F]$, $u \equiv v$, *then*

$$Q_u - Q_v \equiv 0 \ (\mathrm{mod}\,\Phi_w), \qquad Q'_u - Q'_v \equiv 0 \ (\mathrm{mod}\,\Phi_w).$$

4.1.2. **Corollary.** $\det(\partial r/\partial a) \equiv \det(\partial r/\partial b)$ $(\mathrm{mod}\,\Phi_w)$ *for any representation* $\rho \in X_2^s(\pi)$.

Proof. We have the fundamental identity (see [5])

$$\frac{\partial r}{\partial a}(a-1) + \frac{\partial r}{\partial b}(b-1) = r - 1$$
$$\Rightarrow (1 - \delta t_1 + \delta^2)\det\frac{\partial r}{\partial a} = (1 - \delta t_1 + \delta^2)\det\frac{\partial r}{\partial b} + f\Phi_w,$$

where $f \in \mathbb{Z}[\delta^{\pm 1}, t_1, t_2]$. Since $(1 - \delta t_1 + \delta^2)$ is irreducible in $\mathbb{Z}[\delta^{\pm 1}, t_1, t_2]$, Φ_w is not divisible by $(1 - \delta t_1 + \delta^2)$. It follows that

$$\det\frac{\partial r}{\partial a} \equiv \det\frac{\partial r}{\partial b} (\mathrm{mod}\,\Phi_w),$$

q.e.d.

We recall that $w = a^{\varepsilon_1} b^{\varepsilon_n} \cdots a^{\varepsilon_n} b^{\varepsilon_1}$.

Define

$$e(w) = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n,$$
$$e(w') = \varepsilon_2 + \varepsilon_3 + \cdots + \varepsilon_n,$$
$$e(w^{(k)}) = \varepsilon_{k+1} + \cdots + \varepsilon_n.$$

Put

$$|w| \equiv \delta^{-2e(w)}\left(\det\frac{\partial w}{\partial a} + \det\frac{\partial w}{\partial b}\right).$$

4.2. We have

$$\frac{\partial r}{\partial a} \equiv w - (a-1)\frac{\partial w}{\partial a}, \qquad \frac{\partial r}{\partial b} \equiv 1 + (b-1)\frac{\partial w}{\partial b}.$$

Clearly, a necessary condition for a cohomology jump is

$$\det\frac{\partial r}{\partial a} = \det\frac{\partial r}{\partial a} = 0.$$

It turns out that this is also sufficient.

4.2.1. **Theorem.** *The polynomial* $\delta^{-2e(w)}\det[1+(b-1)\partial w/\partial b]-\Phi_w$ *is the product of two factors,* $(1-\delta t_1+\delta^2)$ *and* $S_w(\delta^{\pm 1}, t_1, t_2)$. *The second factor determines the cohomology-jump subvariety on the character variety* $X_2^s(\pi)$ *of the irreducible representations; the polynomial* S_w *has the form*

$$(*) \qquad S_w = |w| - |w'| + \cdots + (-1)^{n-1}|w^{(n-1)}|.$$

Proof. We first prove the formula

$$\delta^{-2e(w)}\det\left[1+(b-1)\frac{\partial w}{\partial b}\right]-\Phi_w=(1-\delta t_1+\delta^2)S_w,$$

where S_w is defined by $(*)$, using induction on the length of the word w. Suppose the formula holds for $u=w'$, $w=aub$ or $w=a^{-1}ub^{-1}$.

a) For $w=aub$, $\partial w/\partial b=a\partial u/\partial b+au$,

$$\begin{aligned}
&\det\left(1+(b-1)\frac{\partial w}{\partial b}\right)-\delta^{2e(w)}\Phi_w\\
&=1+\operatorname{tr}\left[(b-1)\frac{\partial w}{\partial b}\right]+\det(b-1)\det\frac{\partial w}{\partial b}-\delta^{2e(w)}\Phi_w\\
&=1+\operatorname{tr}\left[a\frac{\partial u}{\partial b}(b-1)+aub-au\right]+\det(b-1)\det\frac{\partial w}{\partial b}\\
&\quad-\operatorname{tr}w+\delta^{2e(w)}\Phi_u\\
&=1+\operatorname{tr}\left[a\left(u-1-\frac{\partial u}{\partial a}(a-1)\right)-au\right]\\
&\quad+\det(b-1)\det\frac{\partial w}{\partial b}+\delta^{2e(w)}\Phi_u\\
&=1+\delta^{2e(w)}\Phi_u-\delta t_1-\operatorname{tr}\left[a\frac{\partial u}{\partial a}(a-1)\right]+\det(1-b)\det\frac{\partial w}{\partial b}\\
&=(1-\delta t_1+\delta^2)-\delta^2\left(1+\operatorname{tr}\frac{\partial u}{\partial a}(a-1)-\delta^{2e(u)}\Phi_u\right)\\
&\quad+\delta^2\operatorname{tr}\frac{\partial u}{\partial a}(a-1)-\operatorname{tr}a\frac{\partial u}{\partial a}(a-1)+(1-\delta t_1+\delta^2)\det\frac{\partial w}{\partial b}\\
&=(1-\delta t_1+\delta^2)\left[1+\det\frac{\partial w}{\partial b}-S_u\delta^{2e(w)}+\delta^2\det\frac{\partial u}{\partial a}\right]\\
&\quad+\delta^2\operatorname{tr}\frac{\partial u}{\partial a}a-\delta^2\operatorname{tr}\frac{\partial u}{\partial a}-\operatorname{tr}\left(a^2\frac{\partial u}{\partial a}\right)+\operatorname{tr}a\frac{\partial u}{\partial a}\\
&=(1-\delta t_1+\delta^2)\left[\det\frac{\partial w}{\partial a}+\det\frac{\partial w}{\partial b}-S_u\delta^{2e(w)}\right]\\
&=(1-\delta t_1+\delta^2)S_w\delta^{2e(w)}.
\end{aligned}$$

b) For $w=a^{-1}ub^{-1}$, $\partial w/\partial b=a^{-1}\partial u/\partial b-a^{-1}ub^{-1}$, $\partial w/\partial a=-a^{-1}+a^{-1}\partial u/\partial a$,

4.2.2. **Lemma.** *Let*

Proof. Denote the left

and by Corollary 4.1.2

by a multiple of Φ_w:

$$
\begin{aligned}
\det &\left[1 + (b-1)\frac{\partial w}{\partial b}\right] - \delta^{2e(w)}\Phi_w \\
&= 1 + \operatorname{tr}\left[(b-1)a^{-1}\frac{\partial u}{\partial b} - a^{-1}u + a^{-1}ub^{-1}\right] \\
&\quad + \det(b-1)\det\frac{\partial w}{\partial b} - \operatorname{tr} w + \delta^{2e(w)}\Phi_u \\
&= 1 + \operatorname{tr}\left[a^{-1}\frac{\partial u}{\partial b}(b-1) - a^{-1}u\right] + \det(b-1)\det\frac{\partial w}{\partial b} + \delta^{2e(w)}\Phi_u \\
&= 1 - \delta t_1 + \delta^{2e(w)}\Phi_u - \operatorname{tr}\left[a^{-1}\frac{\partial u}{\partial a}(a-1)\right] + \det(b-1)\det\frac{\partial w}{\partial b} \\
&= (1 - \delta^{-1}t_1 + \delta^{-2}) - \delta^{-2}\left[1 + \operatorname{tr}\frac{\partial u}{\partial a}(a-1) - \delta^{2e(u)}\Phi_u\right] \\
&\quad + \delta^{-2}\operatorname{tr}\frac{\partial u}{\partial a}(a-1) - \operatorname{tr}\frac{\partial u}{\partial a} + \operatorname{tr} a^{-1}\frac{\partial u}{\partial a} \\
&\quad + (1 - \delta t_1 + \delta^2)\det\frac{\partial w}{\partial b} \\
&= (1 - \delta t_1 + \delta^2)\left(\delta^{-2} + \det\frac{\partial w}{\partial b} - \delta^{2e(w)}S_u + \delta^{-2}\det\frac{\partial u}{\partial a}\right) \\
&\quad - \delta^{-2}\operatorname{tr}\frac{\partial u}{\partial a} - \operatorname{tr}\frac{\partial u}{\partial a} + t_1\delta^{-1}\operatorname{tr}\frac{\partial u}{\partial a} \\
&= (1 - \delta t_1 + \delta^2)\left(\delta^{-2} + \delta^{-2}\det\frac{\partial u}{\partial a} - \delta^{-2}\operatorname{tr}\frac{\partial u}{\partial a} + \det\frac{\partial w}{\partial b} - \delta^{2e(w)}S_u\right) \\
&= (1 - \delta t_1 + \delta^2)\left(\det\frac{\partial w}{\partial a} + \det\frac{\partial w}{\partial b} - \delta^{2e(w)}S_u\right) \\
&= (1 - \delta t_1 + \delta^2)S_u\delta^{2e(w)}.
\end{aligned}
$$

4.2.2. **Lemma.** *Let ρ be a representation in $\widetilde{T}$. Then*

$$
\det\left[1 + (b-1)\frac{\partial w}{\partial b} + (b-1)\frac{\partial w}{\partial a} - w\right] \equiv (t_1^2 - t_2 - 2)S_w \pmod{\Phi_w}.
$$

Proof. Denote the left-hand side by M. Then

$$
\begin{aligned}
M(1 - \delta t_1 + \delta^2) &= M\det(b-1) \\
&= \det\left[(1-w)(b-1) + (b-1)\left(\frac{\partial w}{\partial b} + \frac{\partial w}{\partial a}\right)(b-1)\right] \\
&= \det\left\{(1-w)(b-1) + (b-1)\left[w - 1 + \frac{\partial w}{\partial a}(b-a)\right]\right\},
\end{aligned}
$$

and by Corollary 4.1.2 this expression differs from

$$
\det\left[wa - wb + (b-1)\frac{\partial w}{\partial a}(b-a)\right]
$$

by a multiple of Φ_w:

$$
\begin{aligned}
M(1 - \delta t_1 + \delta^2) &= \det\left[wa - wb + (b-1)\frac{\partial w}{\partial a}(b-a)\right] + f\cdot\Phi_w \\
&= \det(b-a)\det\left[w - (b-1)\frac{\partial w}{\partial a}\right] + f\cdot\Phi_w.
\end{aligned}
$$

We have

$$\det(b-a)=\delta^2(t_1^2-t_2-2)\,,\ \det\left[w-(b-1)\frac{\partial w}{\partial a}\right]\equiv\det\frac{\partial r}{\partial a}\ (\operatorname{mod}\Phi_w)$$

$$\Rightarrow M(1-\delta t_1+\delta^2)=(1-\delta t_1+\delta^2)(t_1^2-t_2-2)S_w+g\cdot\Phi_w.$$

Since Φ_w depends only on t_1 and t_2, g is divisible by $(1-\delta t_1+\delta^2)$, and $M\equiv(t_1^2-t_2-2)S_w\pmod{\Phi_w}$ as claimed.

We proceed with the proof of the theorem. We have

$$\operatorname{rk}H^2(\pi\,,\rho)=2-\dim\left(\rho\left(\frac{\partial r}{\partial a}\right),\rho\left(\frac{\partial r}{\partial b}\right)\right),$$

and therefore

$$\operatorname{rk}H^2(\pi\,,\rho)\geq 1\Rightarrow\dim\left(\rho\left(\frac{\partial r}{\partial a}\right),\rho\left(\frac{\partial r}{\partial b}\right)\right)\leq 1$$

$$\Rightarrow\det\rho\left(\frac{\partial r}{\partial a}\right)=\det\left(\rho\left(\frac{\partial r}{\partial a}+\frac{\partial r}{\partial b}\right)\right)=0.$$

It follows that in this case $S_w=0$. This means that the cohomology-jump subvariety lies in $\{S_w=0\}$. We prove now the converse: if $\rho\in\{S_w=0\}$, then $\dim H^2(\pi\,,\rho)\geq 1$. Since $\rho\in\{S_w=0\}$, we have $\det(\partial r/\partial a)=\det(\partial r/\partial b)=0$. If $\det\rho(a-1)\neq 0$, then from the identity

$$\frac{\partial r}{\partial a}(a-1)+\frac{\partial r}{\partial b}(b-1)=r-1\equiv 0$$

it follows that $\operatorname{Im}\rho(\partial r/\partial a)=\operatorname{Im}\rho(\partial r/\partial b)$, which means that $\operatorname{Im}\rho(\partial r/\partial a\,,\partial r/\partial b)\leq 1$, i.e., $\operatorname{rk}_2(\rho)\geq 1$.

We must still consider the case $\det\rho(a-1)=0$. Choose a vector $v\in\mathbb{C}^2$ such that $e_1=\rho(1-a)v\neq 0$ and $e_2=\rho(1-b)v\neq 0$. Such a vector exists, since $\rho(a)\neq 1$, $\rho(b)\neq 1$, $\rho\in X_2^s(\pi)$. Then

$$\rho\left(\frac{\partial r}{\partial a}\right)e_1+\rho\left(\frac{\partial r}{\partial b}\right)e_2=0.$$

If $\operatorname{Im}\rho(\partial r/\partial a)\neq\operatorname{Im}\rho(\partial r/\partial b)$, then $\rho(\partial r/\partial a)e_1=0=\rho(\partial r/\partial b)e_2$. There are two possibilities: $e_1=e_2$ and $e_1\neq e_2$.

First case: $e_1\neq e_2$. Then in the basis e_1, e_2 the matrices $\rho(\partial r/\partial a)$ and $\rho(\partial r/\partial b)$ are of the form

$$\rho\left(\frac{\partial r}{\partial a}\right)=\begin{pmatrix}0&\alpha_1\\0&\alpha_2\end{pmatrix},\qquad\rho\left(\frac{\partial r}{\partial b}\right)=\begin{pmatrix}\beta_1&0\\\beta_2&0\end{pmatrix},$$

while $\det[\rho(\partial r/\partial a+\partial r/\partial b)]=0$. This means that $(\alpha_1:\alpha_2)=(\beta_1:\beta_2)$, i.e., $\operatorname{Im}\rho(\partial r/\partial a)=\operatorname{Im}\rho(\partial r/\partial b)$.

Second case: $e_1=e_2$. Then

$$\rho(1-a)v=\rho(1-b)v\,,\qquad\rho(a-b)v=0\,,\qquad\det\rho(a-b)=t_1^2-t_2^2-2=0.$$

We know that the intersection of this variety with the variety $\{\Phi_w=0\}$ of representations of π into SL_2 is discrete, while δ must satisfy the equality $\delta^2-\delta t_1+1=0$. This means that only finitely many points belonging to $\{S_w=0\}\cap X_2^s(\pi)$ fall into this situation, while $\Pi_2(\pi)$ is a closed set in X_2^s. Therefore, again $\dim H^2(\pi\,,\rho)\geq 1$. This completes the proof of the theorem.

4.3. **Examples.** a) Our first example is the torus knot series (type $T(2\,,2n+1)$). These are 2-bridge knots; in the standard notation of [4], they are the knots

$\mathfrak{b}(2n+1, 1)$. The fundamental group is $\pi = \langle a, b \mid wa = bw \rangle$, where $w = (ab)^n$. We denote the corresponding polynomials Φ_w and S_w by Φ_n and S_n.

Let $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}[t_1, t_2]$ be the mapping determined by the formulas

$$f(0, 0) = 2, \qquad f(0, 1) = t_2, \qquad f(n, m) = f(m, n),$$
$$f(1, 1) = t_2^2 - 2 + t_1^2(t_2 - 2), \qquad f(n+1, m) = t_2 f(n, m) - f(n-1, m).$$

4.3.1. **Proposition.** *The polynomials Φ_n and S_n are of the form*

$$\Phi_n = \frac{[f(0, n) + f(0, n+1)]}{t_2 + 2}, \qquad S_n = \frac{\delta^{4n+2} + 1}{\delta^4 - \delta^2 t_2 + 1} \cdot \delta^{-2n},$$

and Φ_n is independent of t_1.

Proof. By Theorem 2.2.1, $\Phi_n = \operatorname{tr}(AB)^n - \operatorname{tr}(AB)^{n-1} + \cdots + (-1)^n$. By formula (1), $\operatorname{tr}(AB)^n = t_2 \operatorname{tr}(AB)^{n-1} - \operatorname{tr}(AB)^{n-2}$; furthermore, $\operatorname{tr} AB = t_2$, $\operatorname{tr}(1) = 2$. Therefore $\operatorname{tr}(AB)^n = f(0, n)$. Let $t_2 = u + u^{-1}$. Then $f(0, n) = u^n + u^{-n}$ and

$$\begin{aligned}
\Phi_n &= f(0, n) - f(0, n-1) + \cdots + (-1)^n \\
&= (u^n - u^{n-1} + \cdots + (-1)^n) + (u^{-n} - u^{-n+1} + \cdots + (-1)^n) - (-1)^n \\
&= \frac{u^{n+1} + (-1)^n}{u+1} + \frac{u^{-n-1} + (-1)^n}{u^{-1}+1} - (-1)^n \\
&= \frac{u^n + u^{-n} + u^{n+1} + u^{-n-1}}{u + u^{-1} + 2} = \frac{f(0, n) + f(0, n+1)}{t_2 + 2},
\end{aligned}$$

which proves the first part.

By Theorem 4.2.1,

$$S_n = |(ab)^n| - |(ab)^{n-1}| + \cdots + (-1)^{n-1}|ab|,$$

$$\begin{aligned}
|(ab)^n| &= \delta^{-2n}\left[\det\rho\left(\frac{\partial(ab)^n}{\partial a}\right) + \det\rho\left(\frac{\partial(ab)^n}{\partial b}\right)\right] \\
&= \delta^{-2n}\left[\det\rho\left(\frac{1-(ab)^n}{1-ab}\right) + \delta^2 \det\rho\left(\frac{1-(ab)^n}{1-ab}\right)\right] \\
&= \frac{\delta^{-2n}(1+\delta^2)}{1 - \delta^2 t_2 + \delta^4}[1 - \delta^{2n} f(0, n) + \delta^{4n}] \\
&= \frac{1+\delta^2}{1 - \delta^2 t_2 + \delta^4}[\delta^{-2n} + \delta^{2n} - f(0, n)].
\end{aligned}$$

Therefore

$$\begin{aligned}
S_n &= \frac{1+\delta^2}{1 - \delta^2 t_2 + \delta^4}[(\delta^{2n} - \delta^{2n-2} + \cdots + (-1)^n) \\
&\qquad + (\delta^{-2n} - \delta^{-2n-2} + \cdots + (-1)^n) - \Phi_n - (-1)^n] \\
&= \frac{1+\delta^2}{1 - \delta^2 t_2 + \delta^4} \cdot \frac{\delta^{2n} + \delta^{-2n} + \delta^{-2n-2} + \delta^{2n+2}}{\delta^2 + \delta^{-2} + 2} \\
&= \delta^{-2n} \cdot \frac{1 + \delta^{4n+2}}{1 - \delta^2 t_2 + \delta^4}.
\end{aligned}$$

The following proposition was used in §3.4.

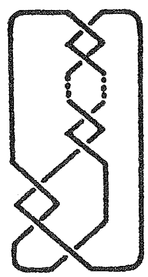

FIGURE 2

4.3.2. **Proposition.** *Φ_n is independent of t_1 and has no multiple divisor (other than unity) in the ring $\mathbb{Z}[t_2]$.*

Proof. The first part is obvious, since $f(0, n) = u^n + u^{-n}$, where $u + u^{-1} = t_2$, and

$$\Phi_n(u) = \frac{(1+u)(u^n + u^{-n-1})}{u + u^{-1} + 2}.$$

Since the polynomial $u^n + u^{-n-1}$ has no multiple roots over $\mathbb{C}$, $\Phi_n(t_2)$ has no multiple divisors, q.e.d.

Observe the identity

$$S_{n-1} + S_{n+1} = (\delta^2 + \delta^{-2})S_n,$$

which is easily verified. There is a similar identity for HOMFLY polynomials. Let $P_n(x, y)$ be the HOMFLY polynomial of the torus link $T(2, 2n+1)$, where n is a half-integer: $n = 1/2, 1, 3/2, \dots$. When n is an integer, $T(2, 2n+1)$ is a link with two components. We have (see [12])

$$x^{-1}P_n + xP_{n+1} = yP_{n+1/2},$$
$$x^{-1}P_{n+1/2} + xP_{n+3/2} = yP_{n+1},$$
$$x^{-1}P_{n+1} + xP_{n+2} = yP_{n+3/2}.$$

Multiplying the first equality by x^{-1}, the second by y, the third by x, and adding, we obtain:

$$x^{-2}P_n + x^2P_{n+2} = (y^2 - 2)P_{n+1}.$$

(See Figure 2.)

b) Our second example is the knot series $\mathfrak{b}(6n+1, 3)$ (see [4]). Here $w = (ab)^n(a^{-1}b^{-1})^n(ab)^n$.

4.3.3. **Proposition.** *The following relations hold:*

$$\Phi_n = \frac{1}{t_2+2}\{f(n, n)[f(0, n) + f(0, n+1)] - f(0, n) - f(0, n-1)\},$$

$$S_n = \frac{1}{\delta^4 - \delta^2 t_2 + 1}\{[2 + f(n, n)](\delta^{2n+1} + \delta^{-2n-1}) - 2t_1(\delta^{2n} + \delta^{-2n}) + (\delta^{2n-1} + \delta^{-2n+1}) - 2f(0, n)(\delta + \delta^{-1}) + t_1 f(0, n)\}.$$

We omit the proof. We indicate certain properties of the cohomology-jump variety. If P_n is the HOMFLY polynomial of the knot, then

$$x^{-2}P_n + x^2P_{n+2} = (y^2 - 2)P_{n+1}.$$

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Let us consider the possible recurrence relations among the S_n. Observe that for $f(n, n)$ we have

$$f(n+3, n+3) - f(n, n) = (t_2^2 - 2)[f(n+2, n+2) - f(n+1, n+1)].$$

Hence there exist constants c_1, c_2, c_3 such that $f(n, n) = c_1 + c_2 u^n + c_3 u^{-n}$, where $u + u^{-1} = t_2$. From this it is easily such that for any nine successive polynomials S_n there is a connection of the form

$$S_{n+8} + A_7 S_{n+7} + \cdots + A_1 S_{n+1} + S_n = 0,$$

where $A_1, \ldots, A_7$ are polynomials in t_2 and $(\delta + \delta^{-1})$ and are independent of n. If we take a more complicated knot series—for example, $w_n = (ab)^n(a^{-1}b^{-1})^n(ab)^n \times (a^{-1}b^{-1})^n(ab)^n$—we find that the recurrence sequence for S_n becomes longer, while the knots are obtained one from the other by sequences of Conway surgeries (see [4], [12]), and the relation between their HOMFLY polynomials looks exactly the same as in the case of torus knots. Thus, a connection between the S_n and the HOMFLY polynomials seems improbable.

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