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ON A UNIVERSAL PERTURBATIVE INVARIANT OF 3-MANIFOLDS

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0. INTRODUCTION

Using finite type invariants (or Vassiliev invariants) of framed links and the Kirby calculus we construct an invariant of closed oriented three-dimensional manifolds with values in a graded Hopf algebra of certain kinds of 3-valent graphs (of Feynman diagrams). The degree 1 part of the invariant is essentially the Casson–Lescop–Walker invariant of 3-manifolds. A generalization for links in 3-manifolds is also given.

The theory of this invariant can be regarded as a mathematically rigorous realization of part of Witten’s theory of quantum invariants in [33]. For a 3-manifold M , a compact Lie group G , and an integer k , Witten claimed that

$$Z_k(M, G) = \int e^{\sqrt{-1}kCS(A)} \mathcal{D}A$$

is a topological invariant of a 3-manifold M , where the integral, a kind of Feynman path integral, is over all G -connections A , and the Chern–Simons functional $CS(A)$ is given by

$$CS(A) = \frac{1}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3}A^3).$$

The CS functional is well-defined, but so far, there is still no rigorous method to regularize the path integral. We call $Z_k(M, G)$ the quantum G invariant of M .

There are two approaches to Witten’s quantum invariants: perturbative and non-perturbative.

In the perturbative approach, first one uses formal perturbation theory to derive an asymptotic formula of $Z_k(M, G)$ for large k limit, see [3, 4, 14]; then one tries to mathematically define the coefficients of the asymptotic formula. The formal asymptotic formula is given by a sum over flat connections ρ as

$$Z_k(M, G) \sim \sum_{\rho} e^{kCS(\rho)} \frac{k^{d_{\rho}}}{R(\rho)} \exp\left(\sum_{\Gamma} \left(\frac{2\pi\sqrt{-1}}{k}\right)^{d(\Gamma)} I_{\Gamma}(M, \rho)\right).$$

Here d_{ρ} is the dimension of the cohomology group of the adjoint local system given by ρ and $R(\rho)$ is Reidemeister torsion. The second sum is over all trivalent graphs Γ , where $d(\Gamma)$

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is half the number of vertices of Γ , and $I_\Gamma(M, \rho)$ is defined by an integral of a product of certain g -valued 2-forms over a non-compact manifold. The convergence of this integral is a very difficult (analytical) problem (see [4]). The calculation of these invariants may seem even more difficult. If one wants to get rid of Lie algebras, one can use graphical calculus, as suggested by Drinfeld [7]. Then one sees that the perturbative invariants should take values in the vector space of 3-valent graphs described in Section 1.

In the non-perturbative approach, which was initiated by the work of Reshetikhin and Turaev [25] at almost the same time as Witten's work, a precise value of the quantum invariant $Z_k(M, G)$ is defined, but one does not regularize the path integral. Instead of path integral ones uses finite linear combinations of quantum invariants of *links*. The non-perturbative theory involves deep considerations in modular Hopf algebras, quantum groups at roots of unity, etc. (see [25, 30, 29, 12]).

Reshetikhin and Turaev [25] used quantum invariants of links (and the Kirby calculus) to construct quantum invariants of 3-manifolds. Here, instead of quantum invariants of links we use finite type (or Vassiliev) invariants of links. More precisely, we use a modification of the Kontsevich integral [13] which is a universal Vassiliev invariant of framed links. There is no root of unity in the construction. The computation of our invariant is combinatorial and straightforward.

After Khono's pioneering work [11] on representations of braid groups using iterated integrals, Drinfeld [7, 8] developed the theory of quasi-Hopf algebras in which important elements, called associators, were constructed using analytical or perturbative methods. Using Drinfeld's theory of quasi-Hopf algebras Kontsevich defined his famous knot invariant, expressed by iterated integrals. The Kontsevich integral can be regarded as perturbative invariants of links; and our invariant of 3-manifolds can be regarded as the perturbative invariant which corresponds to the trivial connection.

The main idea of the construction of the invariant is as follows. First we have a modification of the Kontsevich integral, which is an invariant \hat{Z} of framed links with values in a graded vector space of chord diagrams. The degree n part of \hat{Z} is a Vassiliev invariant of order n which dominates all other Vassiliev invariants of the same order. We found in [21] that another normalization \check{Z} of \hat{Z} behaves well, in some sense, under the second Kirby move (handle slide move). We would like to impose an equivalence relation on the space of chord diagrams so that the equivalence class of \check{Z} does not change under the second Kirby move. But it seems that there is no such equivalence relation, if we want all the degrees of the invariant to survive. So, for each positive integer n , we define an equivalence relation, modulo which \check{Z} is invariant under the second Kirby move if we ignore terms of degree greater than n . To deal with the first Kirby move, we map the values of \check{Z} (modulo the equivalence relation) to the vector space $\mathcal{A}(\phi)$ of 3-valent graphs, which has the structure of an algebra. A standard normalization will then take care of the first Kirby move, and we get an invariant Ω_n of 3-manifolds with values in $\mathcal{A}(\phi)$, but with only terms of degree less than or equal to n . We then show that one can unify all the Ω_n into a single invariant Ω , without loss of information, with values in the completion $\hat{\mathcal{A}}(\phi)$ (with respect to the grading) of $\mathcal{A}(\phi)$. It turns out that Ω is always a group-like element in the commutative co-commutative Hopf algebra $\hat{\mathcal{A}}(\phi)$. For rational homology 3-spheres, Ω is well-behaved under connected sum and change of orientation. With some modification one can define an invariant $\Omega(M, L)$ for a link L in a closed 3-manifold M , which can be considered as generalizations for both the Kontsevich integral and $\Omega(M)$. One can substitute simple Lie algebras into 3-valent graphs; and from Ω one gets a formal power series in one variable. The relationship between this power series and quantum invariants is conjectured in Section 7.3. If we use Lie algebras sl_N , from

$\Omega(M, L)$ we get a generalization of the Homfly polynomial for links in rational homology 3-spheres.

The paper is organized as follows. In Section 1 we review the definition of the Kontsevich invariant as well as some of its properties. In Section 2 we describe a map from the space of chord diagrams to the algebra $\mathcal{A}(\phi)$ of trivalent graphs. We define the invariant Ω in Sections 3 and 4. We give some properties of the invariant in Section 5, an extension of the invariant to the case of links in 3-manifolds in Section 6, and some calculations of the invariant in Section 7.

In the previous papers [20, 21] only the degree 1 part was obtained from the Kontsevich invariant.

1. THE MODIFIED KONTSEVICH INVARIANT

We will review the theory of the Kontsevich invariant of framed oriented links. Some properties of the invariant, needed later, will be presented.

1.1. Chord diagrams

Note that our definition of a chord diagram is more general than that of [6, 16].

A *uni-trivalent graph* is a graph every vertex of which is either univalent or trivalent. A uni-trivalent graph is *vertex-oriented* if at each trivalent vertex a cyclic order of edges is fixed. Trivalent vertices are also called external, and univalent vertices — internal.

Let X be a compact oriented one-dimensional manifold whose components are ordered. A *chord diagram* with support X is the manifold X together with a vertex-oriented univalent graph whose univalent vertices are on X . The class of chord diagrams in our definition contains all chord diagrams and Chinese character diagrams in [6, 16], and even more. There may be connected components of the dashed graph which do not have univalent vertices, and hence are disjoint from X .

In all figures the components of X will be depicted by solid lines, the graph by dashed lines, and the orientation at every internal vertex is given by the counterclockwise direction.

Each chord diagram has a natural topology. Two chord diagrams D, D' with the same support X are regarded as equal if there is a homeomorphism $f: D \rightarrow D'$ such that $f|_X$ is a homeomorphism of X which preserves components and orientation and the restriction of f to the dashed graph preserves the orientation at each vertex.

Let $\mathcal{A}(X)$ be the vector space over \mathbb{C} spanned by chord diagrams, whose dashed graphs do not contain any loop component, with support X , subject to the AS, IHX and STU relations shown in Fig. 1.

The *degree* of a chord diagram is half the number of vertices of the dashed graph. Since the relations AS, IHX and STU respect the degree, there is a grading on $\mathcal{A}(X)$ induced by this degree. We denote by $\hat{\mathcal{A}}(X)$ the completion of $\mathcal{A}(X)$ with respect to the degree.

Suppose C is a component of X . We will define the following linear maps on $\mathcal{A}(X)$.

Reversing the orientation of C , from X we get X' . Let $S_{(C)}: \mathcal{A}(X) \rightarrow \mathcal{A}(X')$ be the linear map which transfers every chord diagram D in $\mathcal{A}(X)$ to $S_{(C)}(D)$ obtained from D by reversing the orientation of C and multiplying by $(-1)^m$, where m is the number of vertices of the dashed graph on the component C .

Replacing C by 2 copies of C , from X we get $X^{(2,C)}$, with a projection $p: X^{(2,C)} \rightarrow X$. If x is a point on C then $p^{-1}(x)$ consists of 2 points, while if x is a point of other components, then $p^{-1}(x)$ consists of one point. Let D be a chord diagram on X , with the dashed graph G which has m univalent vertices on C . Consider all possible new chord diagrams on $X^{(2,C)}$

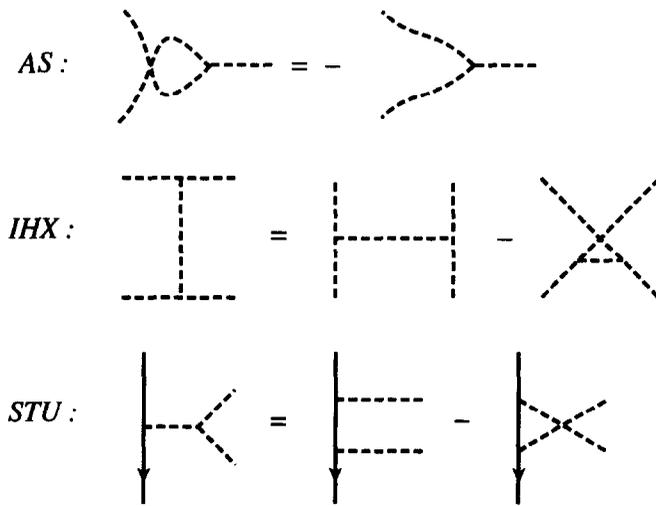


Fig. 1. The AS, IHX and STU relations.

with the same dashed graph G such that if a univalent vertex of G is attached to a point x on X in D , then this vertex is attached to a point in $p^{-1}(x)$ in the new chord diagram. There are 2^m such chord diagrams, and their sum is denoted by $\Delta_{(C)}(D)$, an element of $\mathcal{A}(X^{(2,C)})$.

Removing the component C , from X we get X'' . We define a map $\varepsilon_{(C)}: \mathcal{A}(X) \rightarrow \mathcal{A}(X'')$ as follows. If a chord diagram D has no dashed vertices on C , we put $\varepsilon_{(C)}(D)$ to be the chord diagram obtained by removing the solid component C from D , and put $\varepsilon_{(C)}(D) = 0$ otherwise.

It is easy to check that these linear maps $S_{(C)}$, $\Delta_{(C)}$ and $\varepsilon_{(C)}$ are well-defined on $\mathcal{A}(X)$, i.e. they are compatible with the AS, IHX, and STU relations. These maps are naturally extended to $\hat{\mathcal{A}}(X)$.

Suppose that X and X' have distinguished components C and C' respectively, and that X consists of circle components only. Let $D \in \mathcal{A}(X)$ and $D' \in \mathcal{A}(X')$ be two chord diagrams. From each of C and C' we remove a small arc which does not contain any vertices of the dashed graphs. The remaining part of C is an arc which we glue to C' in the place of the removed arc such that the orientations are compatible. The new chord diagram is called the *connected sum of D and D' along the distinguished components*; it does not depend on the locations of the removed arc, which follows from the STU relation and the fact that all components of X are circles. The proof is the same as in the case $X = X' = S^1$ as in [6]. In particular, in $\mathcal{A}(S^1)$, where the support is simply a circle, the connected sum is a well-defined multiplication which turns $\mathcal{A}(S^1)$ into a commutative algebra (see [6]).

We define a co-multiplication $\hat{\Delta}$ in $\mathcal{A}(X)$ and $\hat{\mathcal{A}}(X)$ as follows. A *chord sub-diagram* of a chord diagram D with dashed graph G is any chord diagram obtained from D by removing some connected components of G . The *complement chord sub-diagram* of a chord sub-diagram D' is the chord sub-diagram obtained by removing components of G which are in D' . We define

$$\hat{\Delta}(D) = \sum D' \otimes D''.$$

Here the sum is over all chord sub-diagrams D' of D , and D'' is the complement of D' . This co-multiplication is co-commutative.

Of special interest is the case $X = \phi$, the empty set. This space $\mathcal{A}(\phi)$ is generated by vertex-oriented 3-valent graphs (no solid lines or circles, no dashed loop components),

subject to the AS and IHX relations. For two chord diagrams in $\mathcal{A}(\phi)$ we define their product to be their disjoint union. Then $\mathcal{A}(\phi)$ is a graded algebra, where the degree 0 part is, by definition, simply the ground field \mathbb{C} . This product is compatible with the co-product introduced above, hence $\mathcal{A}(\phi)$ is a commutative co-commutative Hopf algebra.

Note that all the vector spaces $\mathcal{A}(X)$ are graded $\mathcal{A}(\phi)$ -modules, where the action of $\mathcal{A}(\phi)$ on $\mathcal{A}(X)$ is again the disjoint union.

1.2. Associator

Let $\mathbb{C}\langle\langle A, B \rangle\rangle$ be the algebra over \mathbb{C} of all formal power series in two non-commuting variables A, B . We are going to define an element $\varphi \in \mathbb{C}\langle\langle A, B \rangle\rangle$, first introduced by Drinfeld. From φ one can construct an associator and invariants of framed links (see [18]).

For positive integers i_1, \dots, i_k satisfying $i_k \geq 2$, let

$$\zeta(i_1, \dots, i_k) = \sum_{0 < n_1 < \dots < n_k \in \mathbb{Z}} \frac{1}{n_1^{i_1} \dots n_k^{i_k}}$$

These values, called multiple zeta values, have recently gained much attention among number theorists. In what follows, bold letters, $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ stand for non-negative multi-indices. For a multi-index $\mathbf{p} = (p_1, \dots, p_k)$, we call k the length of \mathbf{p} . Let $\mathbf{1}_k$ be the multi-index consisting of k letters 1. We denote $\sum p_i$ by $|\mathbf{p}|$. For two multi-indices \mathbf{p} and \mathbf{q} of the same length k , we put $\eta(\mathbf{p}; \mathbf{q}) = 0$ if one of p_i, q_i is 0, otherwise

$$\eta(\mathbf{p}; \mathbf{q}) = \zeta(\mathbf{1}_{p_1-1}, q_1 + 1, \mathbf{1}_{p_2-1}, q_2 + 1, \dots, \mathbf{1}_{p_k-1}, q_k + 1).$$

Furthermore, we set two notations by

$$(A, B)^{(\mathbf{p}, \mathbf{q})} = A^{p_1} B^{q_1} A^{p_2} B^{q_2} \dots A^{p_k} B^{q_k},$$

$$\binom{\mathbf{p}}{\mathbf{q}} = \binom{p_1}{q_1} \binom{p_2}{q_2} \dots \binom{p_k}{q_k}.$$

Using the above notations we define $\varphi \in \mathbb{C}\langle\langle A, B \rangle\rangle$ by

$$\varphi(A, B) = 1 + \sum_{\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s} \geq 0} (-1)^{|\mathbf{r}| + |\mathbf{q}|} \eta(\mathbf{p} + \mathbf{r}; \mathbf{q} + \mathbf{s}) \binom{\mathbf{p} + \mathbf{r}}{\mathbf{r}} \binom{\mathbf{q} + \mathbf{s}}{\mathbf{s}} B^{|\mathbf{s}|} (A, B)^{(\mathbf{p}, \mathbf{q})} A^{|\mathbf{r}|}.$$

Here the sum is over all multi-indices $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ of the same length k where $k = 1, 2, 3, \dots$

This is an explicit formula of the power series φ , first introduced by Drinfeld in [7] as the solution of the Knizhnik–Zamolodchikov equation with specific asymptotic behaviors. The formula was obtained in [16]. Note that there exists the inverse of $\varphi(A, B)$; we denote it by $\varphi^{-1}(A, B)$.

1.3. The Kontsevich invariant for framed oriented links

We will define the Kontsevich invariant $\hat{Z}(L)$ of a framed oriented link L . For details, see [6, 16, 18]. The original definition of Kontsevich is for unframed knots only. Note that we will choose the normalization for which the value \hat{Z} of an empty knot is 1, while the value of the unknot with framing 0 is v , a non-trivial element.

We will first define an invariant $Z(\mathcal{D})$ of a link diagram \mathcal{D} .

Suppose \mathcal{D} is an oriented l -component link diagram in a plane with a fixed coordinate system. Using horizontal lines one can decompose \mathcal{D} into elementary tangle diagrams. Here an elementary tangle diagram is one of the tangle diagrams shown in Fig. 2, may be with the reverse orientation on some components. We number the components of the elementary

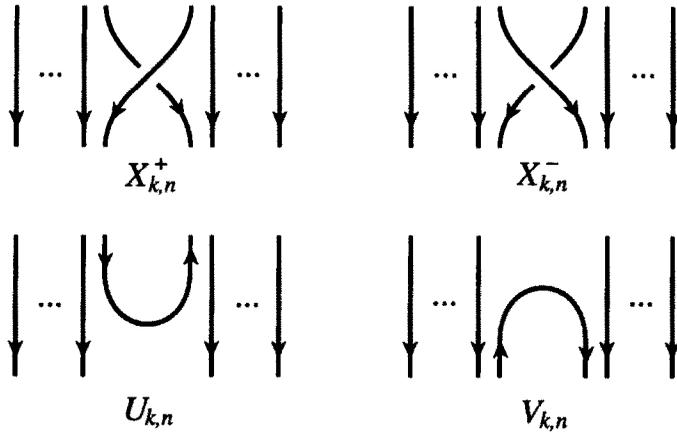


Fig. 2. Elementary tangle diagrams.

tangle diagrams from left to right, using the bottom boundary points. There are n components in $X_{k,n}^\pm$, and the crossing is on the k th and $(k + 1)$ th components. There are $n - 1$ components in $U_{k,n}$ and $V_{k,n}$, and the non-straight component is numbered by k . Before defining $Z(\mathcal{D}) \in \hat{\mathcal{A}}(\coprod^l S^1)$ we will first define $Z(T)$ for each elementary tangle diagram T .

When X is n vertical straight numbered strands with downward orientations, $\hat{\mathcal{A}}(X)$ is denoted by \mathcal{P}_n . All the \mathcal{P}_n are algebras: the product of two chord diagrams D_1 and D_2 is obtained by placing D_1 on top of D_2 . The algebra \mathcal{P}_1 is commutative (see [6, 13]). Consider the following element $x_{k,n}$ in \mathcal{P}_n :

$$x_{k,n}^+ = \varphi^{-1} \left(\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{k-1} \Omega_{i,k}, \frac{\Omega_{k,k+1}}{2\pi\sqrt{-1}} \right) \exp \left(\frac{\Omega_{k,k+1}}{2} \right) \varphi \left(\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{k-1} \Omega_{i,k+1}, \frac{\Omega_{k,k+1}}{2\pi\sqrt{-1}} \right).$$

Here φ is defined in the previous subsection, and $\Omega_{i,j}$ is the chord diagram in \mathcal{P}_n with the dashed graph being a line connecting the i th and the j th strands. Similarly, we put

$$\begin{aligned} x_{k,n}^- &= \varphi^{-1} \left(\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{k-1} \Omega_{i,k}, \frac{\Omega_{k,k+1}}{2\pi\sqrt{-1}} \right) \exp \left(-\frac{\Omega_{k,k+1}}{2} \right) \\ &\quad \times \varphi \left(\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{k-1} \Omega_{i,k+1}, \frac{\Omega_{k,k+1}}{2\pi\sqrt{-1}} \right), \\ u_{k,n} &= \varphi^{-1} \left(\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{k-1} \Omega_{i,k}, \frac{\Omega_{k,k+1}}{2\pi\sqrt{-1}} \right), \\ v_{k,n} &= \varphi^{-1} \left(\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{k-1} \Omega_{i,k}, \frac{\Omega_{k,k+1}}{2\pi\sqrt{-1}} \right). \end{aligned}$$

We define $Z(X_{k,n}^+)$ as the element in $\hat{\mathcal{A}}(X_{k,n}^+)$ obtained by placing $X_{k,n}^+$ (without any dashed graph) on top of $x_{k,n}^+$. Here we also use the notation $X_{k,n}^+$ for the set of solid lines of $X_{k,n}^+$. Similarly, $Z(X_{k,n}^-)$ is the element in $\hat{\mathcal{A}}(X_{k,n}^-)$ obtained by placing $X_{k,n}^-$ on top of $x_{k,n}^-$. Further, $Z(U_{k,n})$ is obtained by placing $U_{k,n}$ at the bottom of $S_{C_{k+1}}(u_{k,n})$, where C_{k+1} is the $(k + 1)$ th strings of the support of chord diagrams in \mathcal{P}_n ; and S_{C_i} is defined in Section 1.1. The element $Z(V_{k,n})$ is obtained by placing $V_{k,n}$ on top of $S_{C_k}(v_{k,n})$. We show pictures of the definition in Fig. 3.

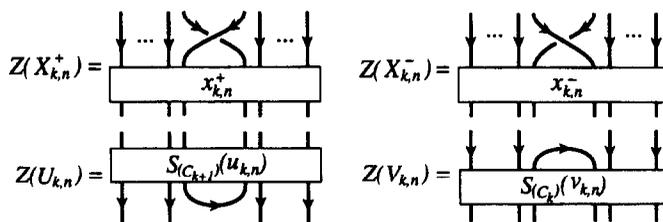


Fig. 3. The definition of $Z(T)$ for each elementary tangle diagram T .

If T is obtained from T' by reversing the orientation of a component C , then we put $Z(T) = S_C(Z(T'))$. Now we can define $Z(T)$ for every elementary tangle diagram.

Then we define $Z(\mathcal{D}) \in \mathcal{A}(\prod^l S^1)$ for an oriented l -component link diagram \mathcal{D} as follows. We can decompose \mathcal{D} into elementary tangle diagrams T_1, T_2, \dots, T_m by horizontal lines, counting from top to bottom. We set $Z(\mathcal{D}) = Z(T_1) \times Z(T_2) \times \dots \times Z(T_m)$. Here, for chord diagrams D_i composing $Z(T_i)$ ($i = 1, 2, \dots, m$), we mean by $D_1 \times \dots \times D_m$ the chord diagram on $\prod^l S^1$ obtained by placing D_1 on top of D_2 , and then placing the union on top of D_3 , and so on. Note that the supports of D_1, D_2, \dots, D_m can be glued together, and the result consists of l solid loops. It is known that $Z(\mathcal{D})$ is well-defined for an oriented link diagram \mathcal{D} and invariant under isotopy of the plane preserving the number of maximal points of each component.

Suppose now L is a framed oriented link, represented by a link diagram \mathcal{D} with black-board framing. Furthermore, suppose that the i th components of \mathcal{D} has s_i maximal points with respect to the height function of the plane. We define an invariant $\hat{Z}(L)$ by

$$\hat{Z}(L) = Z(\mathcal{D}) \# (v^{s_1} \otimes \dots \otimes v^{s_l}) \in \mathcal{A}\left(\prod^l S^1\right),$$

where we put $v = Z(U)^{-1}$ for the link diagram U shown in Fig. 4, and we mean by the above formula that $\hat{Z}(L)$ is obtained from $Z(\mathcal{D})$ by successively taking connected sum with v^{s_i} along the i th component. Note that the notation $\hat{Z}_f(L)$ was used instead of $\hat{Z}(L)$ in [16, 18, 19]. The invariant $\hat{Z}(L)$ of a framed link L is well-defined, not depending on the choice of its link diagram \mathcal{D} , see [16]. For the trivial knot K with framing 0, $\hat{Z}(K)$ is not trivial; in fact, we have $\hat{Z}(K) = v$.

The invariant $\hat{Z}(L)$ is an invariant of framed oriented links which contains in itself all Vassiliev invariants of framed oriented links, and it is a generalization of the Kontsevich integral in [13]. We call this invariant *the Kontsevich invariant* of framed links.

Although the formula of Z of elementary tangles involves complex numbers, it was shown in [18] that $\hat{Z}(L)$, for a framed link L , has rational coefficients. So one can define everything in this paper over the rationals.

1.4. Parallel of framed links

The following was proved in [18, 19].

PROPOSITION 1.1. *Let C be a component of an oriented frame link L .*

1. *Let $L^{(2,C)}$ be the link obtained from L by replacing C by two push-offs of C using the framing. Then*

$$\hat{Z}(L^{(2,C)}) = \Delta_{(C)}(\hat{Z}(L)). \tag{1.1}$$

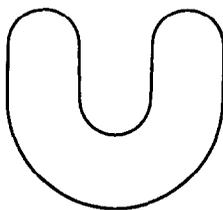


Fig. 4. The link diagram U .

2. Let L' be obtained from L by reversing the orientation of C . Then

$$\hat{Z}(L') = S_{(C)}(\hat{Z}(L)). \tag{1.2}$$

3. Let L'' be obtained from L by removing the component C . Then

$$\hat{Z}(L'') = \varepsilon_{(C)}(\hat{Z}(L)). \tag{1.3}$$

Now, we put

$$\check{Z}(L) = \hat{Z}(L) \# (v \otimes \dots \otimes v) \in \hat{\mathcal{A}}\left(\coprod^l S^1\right).$$

This formula means that $\check{Z}(L)$ is obtained from $\hat{Z}(L)$ by successively taking connected sum with v along every component of L . It is easy to see that Proposition 1.1 is also valid if we replace \hat{Z} by \check{Z} in the statements of the proposition.

Suppose that X consists of n components C_1, \dots, C_n . We denote by $X \sqcup X$ the disjoint union of two copies of X . We define a linear map

$$p: \hat{\mathcal{A}}(X \sqcup X) \rightarrow \hat{\mathcal{A}}(X)^{\otimes 2}$$

as follows. If D is a chord diagram having dashed graph connecting the two copies of X , then we put $p(D) = 0$. Otherwise, D splits into a disjoint union of two chord diagrams D_1 and D_2 on the first and the second copies of X , respectively, and we put $p(D) = D_1 \otimes D_2$.

Then by definition we have

$$p \circ \Delta_{(C_1, \dots, C_n)}(D) = \hat{\Delta}(D), \tag{1.4}$$

where $\Delta_{(C_1, \dots, C_n)}: \hat{\mathcal{A}}(X) \rightarrow \hat{\mathcal{A}}(X \sqcup X)$ is the map obtained by successively applying $\Delta_{(C_i)}$, $i = 1, \dots, n$.

THEOREM 1.2. *For an oriented framed link L , the value $\check{Z}(L)$ is a group-like element in the co-algebra $\hat{\mathcal{A}}(L)$. In other words,*

$$\hat{\Delta}(\check{Z}(L)) = \check{Z}(L) \otimes \check{Z}(L).$$

Proof. By equations (1.1) and (1.4), the left-hand side of the required formula is equal to $p(\check{Z}(L^{(2)}))$. Identifying $\hat{\mathcal{A}}(X) \otimes \hat{\mathcal{A}}(X)$ with a subset of $\hat{\mathcal{A}}(X \sqcup X)$ in the natural way, the right hand side of the required formula becomes equal to $\check{Z}(L \circ L)$, where $L \circ L$ is the split union of two copies of L .

Hence, it is sufficient to show that $p(\check{Z}(L^{(2)}))$ is equal to $\check{Z}(L \circ L)$. Note that we can obtain $L \circ L$ from $L^{(2)}$ by taking crossing changes between the first and second copies of L . It follows that $\check{Z}(L^{(2)}) - \check{Z}(L \circ L)$ consists of only terms which have dashed graphs

connecting the two copies of X , since the difference of a crossing change is locally given by

$$\begin{aligned} x_{k,n}^+ - x_{k,n}^- &= \varphi^{-1}(\dots) \left(\exp\left(\frac{\Omega_{k,k+1}}{2}\right) - \exp\left(-\frac{\Omega_{k,k+1}}{2}\right) \right) \varphi(\dots) \\ &= \varphi^{-1}(\dots) (\Omega_{k,k+1} + \text{higher terms with respect to } \Omega_{k,k+1}) \varphi(\dots). \end{aligned}$$

Any term with dashed graphs connecting the two copies of X is annihilated by p . Hence $p(\check{Z}(L^{(2)})) = \check{Z}(L \circ L)$. \square

1.5. The change under KII move

Throughout the present paper we mean by KII the handle slide move defined in [10]; we show a picture of the move in Fig. 5.

The following fact, proved in [21], helps us to control the change of $\check{Z}(L)$ under KII moves. Roughly speaking, it says that if L' is a framed oriented link obtained from L by a KII move which preserves the orientation, then $\check{Z}(L')$ can be obtained from $\check{Z}(L)$ by replacing the left picture in Fig. 6 with the right picture.

The precise formulation is as follows. Let X be l loops and X' be the union of two strings C_1, C_2 and $(l - 2)$ loops. By gluing the two end points of C_1 , then the two end points of C_2 , we get a linear map $f: \mathcal{A}(X') \rightarrow \mathcal{A}(X)$.

For a chord diagram x in $\mathcal{A}(X')$ let us define $g(x) \in \mathcal{A}(X)$ as follows. Consider $\Delta_{(C_2)}(x)$; the support now is C_1, C_2', C_2'' and $(l - 2)$ loops, where C_2' and C_2'' come from C_2 . Begin with $\Delta_{(C_2)}(x)$, glue the terminal (with respect to the orientation of the string) boundary point of C_1 with the initial boundary point of C_2' , then map the resulting string homeomorphically to C_1 and map C_2'' homeomorphically to C_2 . The result is $g(x)$, an element of $\mathcal{A}(X)$. Extend g linearly on $\hat{\mathcal{A}}(X')$.

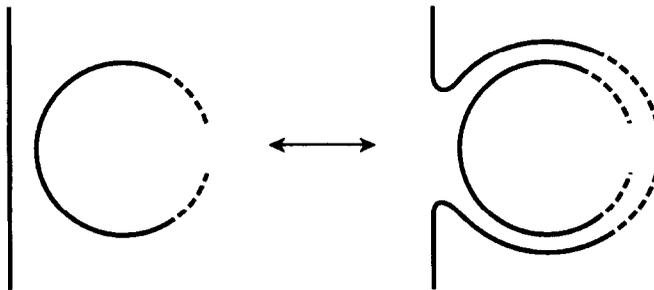


Fig. 5. The KII move (the handle slide move).

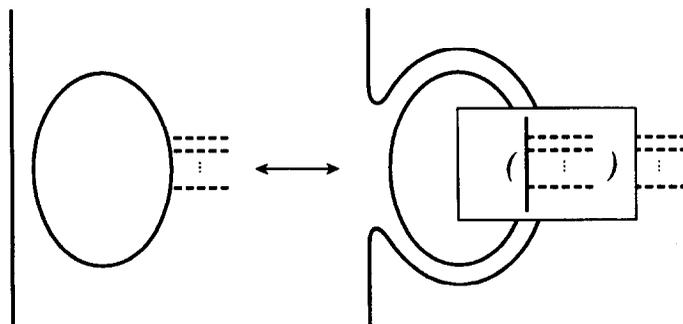


Fig. 6. The chord KII move: the change of $\check{Z}(L)$ under KII moves.

We say that $\xi' \in \hat{\mathcal{A}}(X)$ is obtained from $\xi \in \hat{\mathcal{A}}(X)$ by a *chord KII move* if there is $x \in \hat{\mathcal{A}}(X')$ such that $f(x) = \xi$ and $f(g(x)) = \xi'$.

So, in a sense, we obtained ξ' by first taking parallel of C_2 (using $\Delta_{(C_2)}$), then taking “connected sum” of C_1 with one of the two components coming from C_2 . But this kind of connected sum is not well-defined: it depends on the site of connection, among other things. This is the reason why we have to lift ξ to an element x in $\hat{\mathcal{A}}(X')$ in order to specify the site of connection.

PROPOSITION 1.3 (Le *et al.* [21]). *If a framed oriented link L' is obtained from L by an orientation preserving KII move, then $\check{Z}(L')$ can be obtained from $\check{Z}(L)$ by a chord KII move.*

In essence, we will use the chord KII move as a generator of an equivalence relation in the space of chord diagrams just as we use Kirby moves as an equivalence relation in the set of framed links to identify the set of homeomorphism classes of closed oriented 3-manifolds.

2. REPLACING SOLID CIRCLES BY DASHED GRAPHS

The aim of this section is to construct the series of maps j_n which remove solid components from chord diagrams. The Kontsevich invariant belongs to the space consisting of both solid and dashed lines. When we consider quantum (\mathfrak{g}, R) invariants of links for a Lie algebra \mathfrak{g} and its representation R , the solid and dashed lines correspond to R and \mathfrak{g} respectively. Furthermore, we know that no particular representation R is specified in quantum \mathfrak{g} invariants of 3-manifolds. It implies that the obstruction in constructing invariants of 3-manifolds from the Kontsevich invariant might be the existence of solid lines.

2.1. Chord diagrams with finite support

We consider vertex-oriented uni-trivalent graphs with exactly m univalent vertices located at m labeled points $0, 1, 2, \dots, m - 1$. Denote by $\mathcal{A}(m)$ the vector space over \mathbb{C} spanned by such graphs (without loop components) subject to the AS and IHX relations. Further we denote by $\mathcal{A}(m)_{\text{tree}}$ the vector subspace of $\mathcal{A}(m)$ spanned by connected and simply connected graphs; note that the AS and IHX relations are closed in the subspace. Also note that any uni-trivalent graph with exactly one univalent vertex is equal to 0 (see, for example, [31]).

The symmetric group \mathcal{S}_m acts naturally on $\mathcal{A}(m)$, by permuting the external vertices.

For an element τ in the symmetric group \mathcal{S}_{m-2} acting on the set $\{1, 2, \dots, m - 2\}$, let $T_\tau \in \mathcal{A}(m)$ be the graph shown in Fig. 7.

LEMMA 2.1. *We can take the set of $T_\tau, \tau \in \mathcal{S}_{m-2}$, as a basis of the space $\mathcal{A}(m)_{\text{tree}}$; in particular the space has dimension $(m - 2)!$.*

Proof. Let D be any graph in $\mathcal{A}(m)_{\text{tree}}$. Note that there is a unique path in D connecting the two vertices 0 and $m - 1$ because D is a tree. We color the path red.

We can express D as a linear sum of T_τ 's by induction on the number of trivalent vertices on the red path as follows. We choose a trivalent vertex next to the red path, and apply the IHX relation regarding the segment connecting the vertex and the red path as the character “I” in “IHX”. Then the number of trivalent vertices on the red path increases. Hence we can show that the space $\mathcal{A}(m)_{\text{tree}}$ is spanned by the set of T_τ .

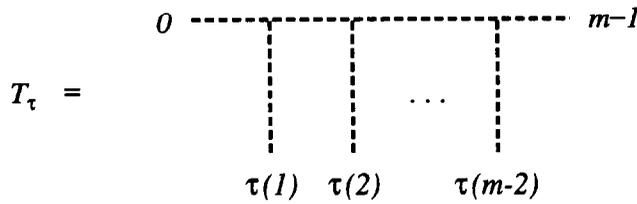


Fig. 7. The definition of T_τ .

In order to complete the proof of this lemma. It is sufficient to prove that T_τ 's are linearly independent. Suppose that T_τ could be expressed as a linear sum of other T_σ 's. We can “substitute” the Lie algebra $sl(m, \mathbb{C})$ to dashed lines (see [6]) to make a linear map from $\mathcal{A}(m)_{\text{tree}}$ to \mathbb{C} . For $j < k$ let E_{jk} be the element in $sl(m, \mathbb{C})$ which has (j, k) entry 1 and the other entries 0. We have the relation $[E_{ij}, E_{jk}] = E_{ik}$. If we substitute E_{12} to the univalent vertex 0 and $E_{k+1, k+2}$ to the vertex $\tau(k)$ for $k = 1, 2, \dots, m - 2$, then T_σ always vanishes unless $\sigma = \tau$. But T_τ does not vanish when we substitute the dual of E_{1m} to the vertex $m - 1$, where we mean the dual with respect to the Killing form. This is a contradiction, completing the proof. \square

2.2. Replacing solid circles with dashed graphs

We would like to replace solid circles with dashed graphs. A natural approach is the following. Suppose C is a solid circle (of some chord diagram) with m external vertices on it. Numerate the vertices, beginning at any vertex and following the orientation of C , by $0, 1, 2, \dots, m - 1$. Now remove the solid circle C , and glue the external vertices to the corresponding vertices of a fixed element T_m in $\mathcal{A}(m)$. Do this with all solid circles of the chord diagram; and we get a map j which transfers chord diagrams on l solid loops to chord diagrams without solid components. We always suppose that $T_0 = T_1 = 0$.

This map j is well-defined if and only if the elements $T_m, m = 2, 3, \dots$ satisfy the following conditions (*) and (**):

$$(012 \dots m - 1)(T_m) = T_m, \tag{*}$$

where $(012 \dots m - 1)$ is the cyclic permutation of $\{0, 1, \dots, m - 1\}$. This equation says that T_m is invariant under cyclic permutation.

$$T_m - (k \ k + 1)(T_m) = T_{m-1} \star_k Y, \tag{**}$$

where $(k \ k + 1)$ is the permutation which interchanges k and $k + 1$ ($0 \leq k \leq m - 2$), and $T_{m-1} \star_k Y$ denotes the element obtained from T_{m-1} by attaching a Y-shaped graph to the vertex k and then re-numerating the vertices so that the remaining two vertices of Y are k and $k + 1$; see Fig. 8.

The second condition makes the map j compatible with the STU relation. The compatibility with the AS and IHX relation is obvious.

Actually in the image of j there may be some chord diagrams containing dashed loops. Let $\mathcal{A}(X)$ be the vector space spanned by chord diagrams with support X , subject to the STU, AS, and IHX relation, like in the case of $\mathcal{A}(X)$, but now the chord diagrams are allowed to contain dashed loops. So, in general, j is a linear map from $\mathcal{A}(\coprod^l S^1)$ to $\mathcal{A}(\phi)$.

There are many solutions of (*) and (**). But if we restrict ourselves to $\mathcal{A}(m)_{\text{tree}}$, then there is a unique solution.

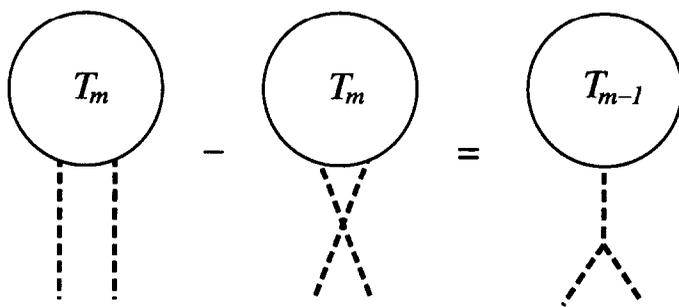


Fig. 8. Pictorial expression of the equation (**).

PROPOSITION 2.2. *Up to constants, there exists a unique sequence $T_m \in \mathcal{A}(m)_{\text{tree}}$, $m = 2, 3, \dots$ satisfying (*) and (**). The solution is given by $T_0 = T_1 = 0$ and*

$$T_m = \sum_{\tau \in \mathcal{A}_{m-2}} \frac{(-1)^{r(\tau)}}{(m-1)^{\binom{m-2}{r(\tau)}}} T_\tau$$

for $m \geq 2$, where $r(\tau)$ is the number of $k \in \{1, 2, \dots, m-2\}$ satisfying $\tau(k) > \tau(k+1)$ and T_τ is given in the previous subsection.

The proof that this sequence of T_m is a solution of (*) and (**) is given in the next subsection. The proof of the uniqueness (which is not used in the subsequent) is not difficult and left for the dedicated reader as an exercise. (Hint: the uniqueness follows from the fact that there is no non-trivial element in $\mathcal{A}(\phi)_{\text{tree}}$ invariant under \mathcal{S}_m .) See Fig. 9 for the first terms of T_m .

There is a simple way to produce new solutions of (*) and (**) from known ones. Consider the space $\prod_{m=2}^\infty \mathcal{A}(m)$. In addition to the usual grading, it has another grading by the number of external vertices. We call this grading the e-grading. There is a shuffle product in $\prod_{m=2}^\infty \mathcal{A}(m)$ defined as follows. Suppose D is a graph in $\mathcal{A}(m)$, D' a graph in $\mathcal{A}(m')$. In the set of external vertices of D there is an order given by $0 < 1 < 2 < \dots < m-1$. Consider the disjoint union of D and D' , and a bijection from the external vertices of D and D' to the set $\{0, 1, 2, \dots, m+m'-1\}$ which preserves the order of external vertices of D and D' . There are $\binom{m+m'}{m}$ such bijections, each gives a graph in $\mathcal{A}(m+m')$. Summing up all such possible graphs we get the shuffle product of D and D' , denoted by $D \bullet D'$.

PROPOSITION 2.3. *Suppose both T_m and T'_m , $m = 2, 3, \dots$, satisfy (*) and (**). Let $T = \sum_{m=2}^\infty T_m$, $T' = \sum_{m=2}^\infty T'_m$. Then $(T \bullet T')_m$, $m = 2, 3, \dots$, also satisfy (*) and (**), where $(T \bullet T')_m$ is the part of e-grading m of $T \bullet T'$.*

Proof. The condition (*) for $(T \bullet T')_m$ is trivial. One needs to show (**). We have $(T \bullet T')_m = \sum_{i=2}^m T_i \bullet T'_{m-i}$. For each term of $(T \bullet T')_m$, consider the difference in the left hand side of (**). If the two adjacent vertices are from T and T' , then the difference vanishes. If they are from the same, say T , then the difference is equal to a term in $(T \bullet T')_{m-1}$ by the hypothesis that T satisfies (**). Hence we obtained the required formula. □

From now on let T_m be as in Proposition 2.2, and $T = \sum_{m=2}^\infty T_m$. Let $T^{\bullet n}$ be the n th power of T in the shuffle product. Denote by T_m^n the part of $(T^{\bullet n}/n!)$ of e-grade m . Then for

$$\begin{aligned}
 T_2 &= 0 \text{-----} 1 \\
 T_3 &= \frac{1}{2} \begin{array}{c} 0 \text{-----} 2 \\ | \\ 1 \end{array} \\
 T_4 &= \frac{1}{3} \begin{array}{c} 0 \text{-----} 3 \\ | \quad | \\ 1 \quad 2 \end{array} - \frac{1}{6} \begin{array}{c} 0 \text{-----} 3 \\ | \quad | \\ 2 \quad 1 \end{array} \\
 &= \frac{1}{6} \begin{array}{c} 0 \quad \quad 3 \\ \diagdown \quad \diagup \\ \text{-----} \\ \diagup \quad \diagdown \\ 1 \quad \quad 2 \end{array} + \frac{1}{6} \begin{array}{c} 0 \quad \quad 3 \\ \diagdown \quad \diagup \\ \text{-----} \\ \diagup \quad \diagdown \\ 1 \quad \quad 2 \end{array}
 \end{aligned}$$

Fig. 9. Simple cases of T_m .

each positive integer n , the sequence $T_m^n, m = 2, 3, \dots$, satisfies (*), (**), and hence defines a map $j_n: \mathcal{A}(\coprod^l S^1) \rightarrow \mathcal{A}(\phi)$ (see Fig. 10).

Note that if $m < 2n$, then, by definition, $T_m^n = 0$. The first non-trivial element $T_{2n}^n \in \mathcal{A}(2n)$ is the following. Partition $2n$ points $\{0, 1, \dots, 2n - 1\}$ into n pairs (there are $(2n - 1)!!$ ways to do this), and then connect the two points of each pair by a dashed line, we get an element of $\mathcal{A}(2n)$. Summing up all such possible elements, we get T_{2n}^n .

For $m > 2n$, T_m^n is more complicated and always contains internal vertices. We show the picture of T_5^2 in Fig. 11. It follows from the definition that j_n decreases the degree of a chord diagram by ln , where l is the number of solid circles of the chord diagram.

Exercise. Suppose $D \in \mathcal{A}(S^1)$. Show that

$$j_n(D) = \frac{1}{n!} j_1[\Delta^{(n)}(D)],$$

where $\Delta^{(1)} = id, \Delta^{(2)} = \Delta, \Delta^{(n+1)} = \Delta_1 \circ \Delta^{(n)}$; and Δ_1 is $\Delta_{(C)}$ with C being the first component. Write down a similar formula for the case when the support of D contains many loops.

2.3. Chord diagrams behaving like a solid circle

In this subsection, our aim is to show that the T_m satisfy (*) and (**) by establishing some symmetry properties of T_m .

LEMMA 2.4. *The elements T_m, T_{m-1} satisfy (**), if $1 \leq k \leq m - 3$.*

Proof. Since we can take the set of T_τ as a basis of $\mathcal{A}(m)_{\text{tree}}$, we can express each side of (**) as a linear sum of T_τ 's. It is sufficient to show that the coefficients of T_τ in both sides are equal for each τ .

If $|\tau^{-1}(k) - \tau^{-1}(k + 1)| \geq 2$, then the coefficient of the left-hand side is equal to zero, since $r(\tau) = r((k \ k + 1) \circ \tau)$ holds in this case, where we mean by $(k \ k + 1)$ the interchange

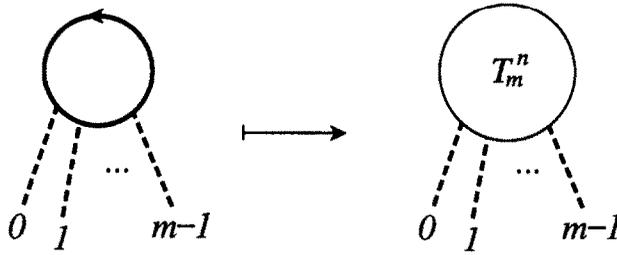


Fig. 10. The definition of j_n .

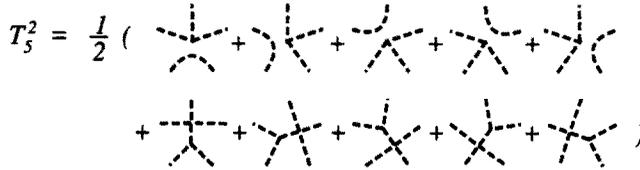


Fig. 11. The picture of T_5^2 .

of k and $k + 1$. On the other hand, the coefficient of the right-hand side is equal to zero too, since the right-hand side is equal to a linear sum of T_{τ} for τ satisfying $|\tau^{-1}(k) - \tau^{-1}(k + 1)| = \pm 1$; we can see it by applying the IHX relation in the right-hand side. Therefore the coefficients of T_{τ} in both sides are equal in this case.

If $\tau^{-1}(k) - \tau^{-1}(k + 1) = -1$, then the coefficient of the left-hand side is equal to $t_{m,\tau} - t_{m,(k\ k+1) \circ \tau}$ where we put

$$t_{m,\tau} = \frac{(-1)^{r(\tau)}}{(m-1)\binom{m-2}{r(\tau)}}.$$

In this case we have $r((k\ k+1) \circ \tau) = r(\tau) + 1$ by the definition of $r(\cdot)$. Hence, we have

$$\begin{aligned} t_{m,\tau} - t_{m,(k\ k+1) \circ \tau} &= \frac{(-1)^{r(\tau)}}{(m-1)\binom{m-2}{r(\tau)}} - \frac{(-1)^{r(\tau)+1}}{(m-1)\binom{m-2}{r(\tau)+1}} \\ &= \frac{(-1)^{r(\tau)}(m-r-2)}{(m-1)(m-2)\binom{m-3}{r(\tau)}} - \frac{(-1)^{r(\tau)+1}(r+1)}{(m-1)(m-2)\binom{m-3}{r(\tau)}} = \frac{(-1)^{r(\tau)}}{(m-2)\binom{m-3}{r(\tau)}}. \end{aligned}$$

On the other hand, the contribution of the right-hand side to T_{τ} comes from $T_{\hat{\tau}} \in \mathcal{A}(m-1)$, where we define $\hat{\tau} \in \mathcal{S}_{m-1}$ by putting $\hat{\tau}(j) = \tau(j + \varepsilon) - \varepsilon'$; here we put $\varepsilon = 1$ if $j > \tau^{-1}(k)$, 0 otherwise, and $\varepsilon' = 1$ if $\tau(j + \varepsilon) > k$, 0 otherwise. Note that we define $\hat{\tau}$ by “gluing” two arrows from k and $k + 1$ to $\tau(k)$ and $\tau(k + 1)$, respectively. The coefficient of $T_{\hat{\tau}}$ is equal to $t_{m-1,\hat{\tau}}$; in this case we have $r(\hat{\tau}) = r(\tau)$ by the definition of $r(\cdot)$. Therefore, the coefficients of both sides are equal, completing this case.

If $\tau^{-1}(k) - \tau^{-1}(k + 1) = 1$, we can show that the coefficients are equal in a similar way as above, completing the proof. □

The symmetric group consisting of permutations of $\{0, 1, 2, \dots, m - 1\}$ acts naturally on $\mathcal{A}(m)$ by permuting the external vertices. Let α be the permutation which replaces $0, 1, \dots, m - 1$ with $m - 1, m - 2, \dots, 0$, respectively, and β be the one which replaces $0, 1, 2, \dots, m - 2, m - 1$ with $m - 2, m - 3, m - 4, \dots, 0, m - 1$, respectively. If the points

0, 1, 2, ..., m - 1 are located at vertices of a regular m-gon, then α and β are two mirror reflections which generate the dihedral group.

LEMMA 2.5. *The mirror reflection α maps T_m to $(-1)^m T_m$:*

$$\alpha(T_m) = (-1)^m T_m.$$

Remark. The sign comes from the fact that there are $(m - 2)$ internal vertices, and $(-1)^{m-2} = (-1)^m$.

Proof. It is easy to see that α maps T_τ to $(-1)^{m-2} T_{\tau'}$ where

$$\tau' = \begin{pmatrix} 1 & 2 & \dots & m-2 \\ m-2 & m-1 & \dots & 1 \end{pmatrix} \circ \tau \circ \begin{pmatrix} 1 & 2 & \dots & m-2 \\ m-2 & m-1 & \dots & 1 \end{pmatrix}$$

and the sign is derived from the number of trivalent vertices; we use the AS relation $(m - 2)$ times. We have $r(\tau) = r(\tau')$, from the definition of $r(\cdot)$. Therefore $\alpha(T_m) = (-1)^m T_m$, completing the proof. □

LEMMA 2.6. *The mirror reflection β maps T_m to $(-1)^m T_m$:*

$$\beta(T_m) = (-1)^m T_m.$$

Proof. Consider the linear map $i: \mathcal{A}(m)_{\text{tree}} \rightarrow \mathcal{A}(m + 1)_{\text{tree}}$ which maps a graph D to the graph obtained from D by adding a Y-shaped graph at the univalent vertex $m - 1$ as shown in Fig. 12. Using the bases of $\mathcal{A}(m)_{\text{tree}}$ and $\mathcal{A}(m + 1)_{\text{tree}}$ specified in Lemma 2.1, one sees at once that i is an injection.

We denote by \mathcal{S}'_{m-1} the symmetric group acting on the set $\{0, 1, \dots, m - 2\}$. For an element $\sigma \in \mathcal{S}'_{m-1}$, let $S_\sigma \in \mathcal{A}(m + 1)_{\text{tree}}$ be the chord diagram shown in Fig. 13. By Lemma 2.1 we can take the set of S_σ as a basis of $\mathcal{A}(m + 1)_{\text{tree}}$.

In order to complete the proof, it is sufficient to show that $\tilde{\beta}(i(T_m)) = (-1)^m i(T_m)$, where $\tilde{\beta}$ is the permutation of $\{0, 1, \dots, m\}$ fixing $m - 1, m$ and replacing $0, 1, \dots, m - 2$ with $m - 2, m - 3, \dots, 0$. The permutation $\tilde{\beta}$ maps S_σ to $S_{\sigma'}$ where

$$\sigma' = \tilde{\beta} \circ \sigma = \begin{pmatrix} 0 & 1 & \dots & m-2 \\ m-2 & m-1 & \dots & 0 \end{pmatrix} \circ \sigma,$$

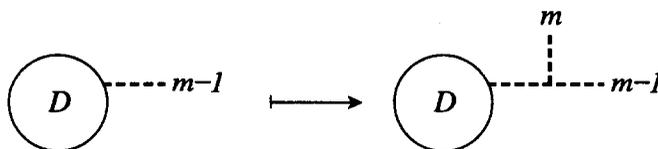


Fig. 12. The map of $\mathcal{A}(m)_{\text{tree}}$ to $\mathcal{A}(m + 1)_{\text{tree}}$.

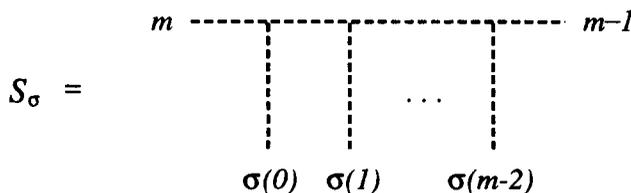


Fig. 13. The definition of S_σ .

note that we have no change of sign in this case because we fix the ends $m - 1$ and m of the “red path”. We have $r(\sigma') = m - 2 - r(\sigma)$ from the definition of $r(\cdot)$. By Lemma 2.7 below, $\tilde{\beta}$ maps $i(T_m)$ to $(-1)^{m_i(T_m)}$, completing the proof. \square

LEMMA 2.7. *We have*

$$i(T_m) = \sum_{\sigma \in \mathcal{S}'_{m-1}} \frac{(-1)^{r(\sigma)}}{(m-1)^{\binom{m-2}{r(\sigma)}}} S_\sigma.$$

Proof. We put $S_{m+1} = i(T_m)$. Since the set of S_σ is a basis of $\mathcal{A}(m+1)_{\text{tree}}$, we have $S_{m+1} = \sum_\sigma s_\sigma S_\sigma$ with some scalars s_σ . We will show the following formula:

$$s_\sigma = \frac{(-1)^{r(\sigma)}}{(m-1)^{\binom{m-2}{r(\sigma)}}} \tag{2.1}$$

in two steps by induction on m .

Step 1. σ is a cyclic permutation $(0, 1, 2, \dots, k)$ for an integer k . We consider all possible $\tau \in \mathcal{S}_{m-2}$ such that $i(T_\tau)$ can contribute non-trivially to S_σ . We expand the left picture in Fig. 14 using the IHX relation to obtain the right picture. We use the IHX relation replacing “I” with the difference of “H” and “X”. Note that the univalent vertex 0 interchanges k times with another vertex, and we must use “X” k times in the expansion. Hence, the possibilities of τ distinguished by $\tau(1) = k$ or $\tau(1) = k + 1$ are as follows:

$$\tau = \begin{pmatrix} 1 & 2 & \dots & l_k - 1 & l_k & k_k + 1 & \dots & l_1 - 1 & l_1 & l_1 + 1 & \dots & m - 2 \\ k + 1 & k + 2 & \dots & l_k + k - 1 & k & l_k + k & \dots & l_1 & 1 & l_1 + 1 & \dots & m - 2 \end{pmatrix}$$

or $\begin{pmatrix} 1 & 2 & \dots & l_{k-1} - 1 & l_{k-1} & l_{k-1} + 1 & \dots & l_1 - 1 & l_1 & l_1 + 1 & \dots & m - 2 \\ k & k + 1 & \dots & l_{k-1} + k - 2 & k - 1 & l_{k-1} + k - 1 & \dots & l_1 & 1 & l_1 + 1 & \dots & m - 2 \end{pmatrix}$

In the first type we can freely choose $\{l_k, l_{k-1}, \dots, l_1\}$ from $\{2, 3, \dots, m - 2\}$. Hence, there are $\binom{m-3}{k}$ possibilities of τ , and $r(\tau) = k$ for this type. Similarly, we have $\binom{m-3}{k-1}$ possibilities of τ , and $r(\tau) = k - 1$ for the second type. Therefore we have

$$s_\sigma = (-1)^k \binom{m-3}{k} \frac{(-1)^k}{(m-1)^{\binom{m-2}{k}}} + (-1)^k \binom{m-3}{k-1} \frac{(-1)^{k-1}}{(m-1)^{\binom{m-2}{k-1}}},$$

where the term $(-1)^k$ is derived from the number of usage of “X”. It’s easy to verify that this formula is the same as (2.1).

Step 2. In this step we will show that if (2.1) holds for σ then it also holds for $(k \ k + 1) \circ \sigma$ for any $k = 1, 2, \dots, m - 3$, where we mean by $(k \ k + 1)$ the interchange of k and $k + 1$. We have the formula shown in Fig. 15, where the first and third equalities are derived from the definition of S_\star and the second from Lemma 2.4.

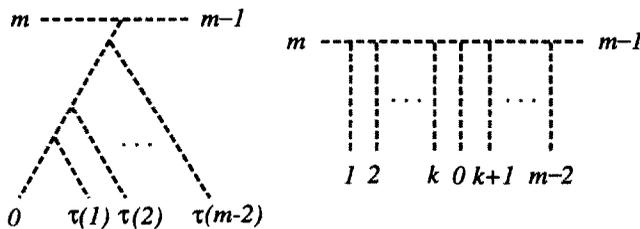


Fig. 14. Expanding T_τ .

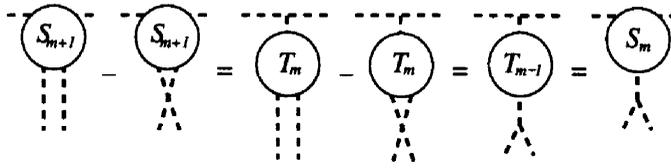


Fig. 15. S_\star satisfies a relation similar to (**)

Hence, S_{m+1} satisfies a relation similar to (**); this means

$$s_\sigma - s_{(kk+1) \circ \sigma} = \begin{cases} s_{\hat{\sigma}} & \text{if } \sigma^{-1}(k) - \sigma^{-1}(k+1) = -1 \\ -s_{\hat{\sigma}} & \text{if } \sigma^{-1}(k) - \sigma^{-1}(k+1) = 1 \\ 0 & \text{if } |\sigma^{-1}(k) - \sigma^{-1}(k+1)| \geq 2. \end{cases}$$

Here we define $\hat{\sigma}$ by putting $\hat{\sigma}(j) = \sigma(j + \varepsilon) - \varepsilon'$, where $\varepsilon = 1$ if $j > \sigma^{-1}(k)$, 0 otherwise; and $\varepsilon' = 1$ if $\sigma(j + \varepsilon) > k$, 0 otherwise. Note that we can obtain the chord diagram $S_{\hat{\sigma}}$ from S_σ by gluing two adjacent dashed edges which have univalent vertices k and $k + 1$, respectively, and we can define $\hat{\sigma}$ only when $\sigma^{-1}(k) - \sigma^{-1}(k + 1) = \pm 1$. We can use (2.1) for $\hat{\sigma}$ by the induction hypothesis. Now (2.1) for σ follows from the identity in Fig. 15 and the induction hypothesis; we thus obtain the required claim of Step 2.

Since we can obtain any $\sigma \in \mathcal{S}_{m-1}$ from cyclic permutations $(0, 1, 2, \dots, k)$ by composed with interchanges $(jj + 1), j = 1, 2, \dots, (m - 3)$ we can show (2.1) for any σ , completing the proof. □

Now we prove that the T_m satisfy (*) and (**). The cyclic permutation of $\{0, 1, \dots, m - 1\}$ is equal to $\alpha\beta$. Hence under the cyclic permutation, T_m moves to $(-1)^{2m}T_m = T_m$. This proves (*). Lemma 2.4, together with (*), prove (**).

3. A SERIES OF INVARIANTS OF 3-MANIFOLDS

In this section we will construct a series of topological invariants $\Omega_m(M)$ of an oriented closed 3-manifold M . We will first show the invariance of $\check{Z}(L)$ under orientation change and Kirby move II using the equivalence relation P_\star defined below. Since the relation P_{n+1} vanishes in low degrees in the image of j_n , we will obtain a series of invariants in low degrees of $\mathcal{A}(\phi)$ through the maps j_n .

3.1. Invariance under orientation change and Kirby move II

Recall that $\mathcal{A}(X)$ is the vector space spanned by chord diagrams subject to the AS, IHX and STU relations, like $\mathcal{A}(X)$, but now the dashed graphs are allowed to have components which are dashed loops. We denote by $\mathring{\mathcal{A}}(X)$ its completion. For each positive integer n , we define an equivalence relation P_n in $\mathring{\mathcal{A}}(X)$ and $\mathring{\mathcal{A}}(X)^\wedge$ as follows. Let P_2 be the equivalence relation shown in Fig. 16; the left-hand side of the first formula in Fig. 16 is the sum over all pairings of 4 points. Similarly, we define the equivalence relation P_n such that the sum over all pairings of $2n$ points is equivalent to zero, or $T_{2n}^n = 0$.

The following lemma will help us to reduce the number of external vertices on a solid component.

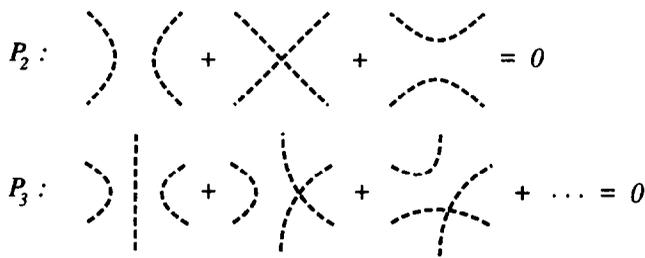


Fig. 16. The relations P_2 and P_3 .

LEMMA 3.1. Let D be a chord diagram in $\mathcal{A}(X)$, and C a component of X .

1. Modulo the equivalence relation P_{n+1} , the chord diagram D is equivalent to a linear sum of chord diagrams each of which has at most $2n$ univalent vertices on C .

2. Further, the chord diagram D becomes equivalent to a linear sum of chord diagrams each of which has either n isolated dashed chords on C or at most $2n - 1$ univalent vertices on C . Here we mean by an isolated chord a dashed arc with no trivalent vertices and two adjacent univalent vertices on C .

Proof. We will prove the case $n = 2$ before the general case. It is sufficient to show that, if the diagram D has k univalent vertices on C with $k > 4$, then it is equivalent to a linear sum of chord diagrams each of which has less than k univalent vertices; we will call them lower terms. We use the relation P_3 as in Fig. 17, where the second equality is derived from the STU relation; we can use the STU relation to interchange two univalent vertices modulo a lower term.

So it suffices to consider the case when C has an isolated chord. Iterating this procedure, we see that D is a sum of chord diagrams, each has at least two isolated chords on C . Then using the relation P_3 again as in Fig. 18, we can represent D as a sum of lower terms. Thus completes the proof of part 1 for the case $n = 2$.

In order to prove part 2, it is sufficient to show that any chord diagram with 4 univalent vertices on C is equivalent to a chord diagram with two isolated chords on C modulo lower terms. We use the relation in Fig. 17 again, to make one isolated chord. We further use the relation P_3 as in Fig. 19. Then we can replace the diagram with a diagram with two isolated chords on C , completing the proof of part 2 for the case $n = 2$.

In general case, to prove part 1, it is sufficient to show that, if the diagram D has m univalent vertices on C with $m > 2n$, then it is equivalent to a linear sum of chord diagrams with at most $m - 1$ univalent vertices. We use relation P_{n+1} as in Fig. 20 for $0 \leq k \leq n$; we use it for $k = n$ at the beginning to make an isolated chord, and use it for $k = n - 1, n - 2, \dots, 1$ to increase isolated chords, and finally use it for $k = 0$ to replace the diagram with lower terms. Note that we can do so assuming $m > 2n$. This completes the proof of part 1.

In order to prove part 2, it is sufficient to show that any chord diagram with $2n$ univalent vertices on C is equivalent to a diagram with n isolated chords on C modulo lower terms. We use the relation in Fig. 20 as above to make $n - 1$ isolated chords on C . We further use the relation for $k = 1$, and we obtain a chord diagram with n isolated chords on C . This completes the proof of part 2. □

Let $L_{<2n}$ be the equivalence relation such that any chord diagram including a solid component with less than $2n$ dashed univalent vertices is equivalent to zero.

$$\begin{aligned}
 \text{Circle } P_3 &= \text{Three vertical lines} + \text{Crossing} + \dots \\
 &= 6 \text{ Three vertical lines} + 3 \text{ Crossing 1} + 3 \text{ Crossing 2} + 3 \text{ Crossing 3} + (\text{lower terms}) = 0
 \end{aligned}$$

Fig. 17. Using P_3 to make isolated chords.

$$\text{Circle } P_3 = 15 \text{ Three vertical lines with three loops} + (\text{lower terms}) = 0$$

Fig. 18. Using P_3 to decrease univalent vertices.

$$\text{Circle } P_3 = 3 \text{ Two crossings with two loops} + 12 \text{ Two vertical lines with one loop} + (\text{lower terms}) = 0$$

Fig. 19. Using P_3 to make two isolated chords.

$$\begin{aligned}
 \text{Circle } P_{n+1} &= a_{k+1} \text{ } (k+1 \text{ vertical lines, } n-k \text{ isolated chords}) + a_{k-1} \text{ } (k-1 \text{ vertical lines, } n-k+1 \text{ isolated chords}) + \dots + a_1 \text{ } (1 \text{ vertical line, } n-k/2 \text{ isolated chords}) \\
 &\quad + (\text{lower terms}) = 0
 \end{aligned}$$

Fig. 20. Using P_{n+1} to decrease univalent vertices; this picture is for even k .

PROPOSITION 3.2. Let D' be obtained from $D \in \mathcal{A}(\coprod^l S^1)$ by a chord KII move (see Section 1.5). Then D and D' has the same equivalence class in $\mathcal{A}(\coprod^l S^1)/L_{<2n}, P_{n+1}$.

Proof. It is sufficient to show that for every $x \in \mathcal{A}(X')$, both x and $g(x)$ represent the same equivalence class in $\mathcal{A}(X')/L_{<2n}, P_{n+1}$. Here X' is the union of 2 strings C_1, C_2 and $l - 1$ loops, and g is defined as in Section 1.5.

One important remark is that the map $\Delta_{(C_2)}$ is well-defined on $\mathcal{A}(X')/L_{<2n}, P_{n+1}$. It follows then from the definition of g that if x_1 and x_2 in $\mathcal{A}(X')$ represent the same equivalence class in $\mathcal{A}(X')/L_{<2n}, P_{n+1}$, and $x'_1 = g(x_1), x'_2 = g(x_2)$, then x'_1, x'_2 represent the same equivalence class in $\mathcal{A}(X')/L_{<2n}, P_{n+1}$. In other words, g is well-defined in $\mathcal{A}(X')/L_{<2n}, P_{n+1}$; and we want to show that it is the identity map on $\mathcal{A}(X')/L_{<2n}, P_{n+1}$.

By Lemma 3.1 and the relation $L_{<2n}$, we may assume that x has exactly $2n$ external vertices on C_2 . Then $g(x)$ is equal to x plus some other terms, each has less than $2n$ external

vertices on C_2 . Therefore, by the relation L_{2n} , x and $g(x)$ represent the same equivalence class in $\mathcal{A}(X')/L_{<2n}, P_{n+1}$. □

3.2. Moving $\check{Z}(L)$ into a set with algebra structure

In order to get the invariance under Kirby move I, we move $\check{Z}(L)$ into a quotient algebra of the algebra $\mathcal{A}(\phi)$, where the product of two 3-valent graphs is their disjoint union. First we prepare the following lemma, which guarantees that we need not consider P_{n+1} in low degrees of $\mathcal{A}(\phi)$.

LEMMA 3.3. *The inclusion map $\mathcal{A}(\phi) \rightarrow \mathring{\mathcal{A}}(\phi)$ induces an isomorphism from the quotient space $\mathcal{A}(\phi)/D_{>n}$ to the quotient space $\mathring{\mathcal{A}}(\phi)/D_{>n}, P_{n+1}, O_n$. Here we define the equivalence relation $D_{>n}$ such that any chord diagram of degree $> n$ is equivalent to zero, and O_n such that any dashed loop component is equal to $-2n$.*

Proof. We can remove the dashed loop component in $\mathring{\mathcal{A}}(\phi)$ by O_n . Hence, it is sufficient to show that claim that, if any element x of $\mathring{\mathcal{A}}(\phi)$ includes T_{2n+2}^{n+1} (see Section 2.2; recall that the relation P_{n+1} is generated by $T_{2n+2}^{n+1} = 0$), then either it vanishes or its degree is greater than n . Note that T_{2n+2}^{n+1} is symmetric with respect to the action of \mathcal{S}_{2n+2} acting on the set of $2n+2$ ends. We will show the claim for any outside of T_{2n+2}^{n+1} .

Choose an end of T_{2n+2}^{n+1} .

If the end is not connected to any other end in the outside, then $x = 0$, since there is no non-trivial element in $\mathcal{A}(1)$. So, we can find a path connecting the end to one of the other ends in the outside.

If there are no trivalent vertices on the path, then $x = 0$ modulo the relation O_n , as shown in Fig. 21.

If there is one trivalent vertex on the path, then $x = 0$ by the symmetry of T_{2n+2}^{n+1} and the AS relation, as shown in Fig. 22.

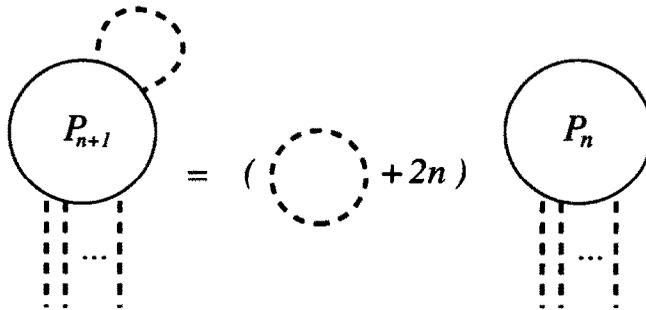


Fig. 21. The relation P_{n+1} vanishes.

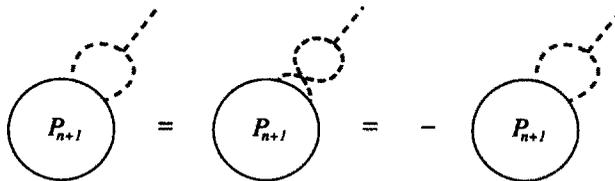


Fig. 22. The relation P_{n+1} vanishes again.

Otherwise, there must be at least two trivalent vertices on the path. Then we associate the nearest trivalent vertex to the end, to obtain an injection of the set of $2n + 2$ ends of T_{2n+2}^{n+1} to the set of trivalent vertices. This means the number of trivalent vertices is greater than $2n$; or the degree is greater than n . By $D_{>n}$, one has $x = 0$. \square

Let us define the following important map ι_n . For a chord diagram D on l circles, $j_n(D)$ is in $\mathcal{A}(\phi)$. Use the relation O_n to remove all the dashed loops and modulo out parts of degree greater than n , then from $j_n(D)$ we get an element $\iota_n(D) \in \mathcal{A}(\phi)/D_{>n}$.

PROPOSITION 3.4. *The image of ι_n in $\mathcal{A}(\phi)/D_{>n}$ does not change under Kirby move II. In other words, if D' is obtained from $D \in \mathcal{A}(\coprod^1 S^1)$ by a chord KII move, then $\iota_n(D') = \iota_n(D)$.*

Proof. Let X be l loops. The map j_n send any chord diagram with less than $2n$ univalent vertices on a solid circle to 0. Hence, we have the following chain of maps:

$$\mathcal{A}(X) \rightarrow \mathcal{A}(X)/L_{<2n}, P_{n+1}, O_n \xrightarrow{\tilde{j}_n} \mathcal{A}(\phi)/P_{n+1}, O_n \xrightarrow{\text{proj}} \mathcal{A}(\phi)/D_{>n}, P_{n+1}, O_n \xrightarrow{\cong} \mathcal{A}(\phi)/D_{>n}$$

Here the first map is the projection, the second is the map induced by j_n , the third is the projection, and the fourth is the isomorphism in Lemma 3.3. Note that the composition of all the maps is exactly ι_n . By Proposition 3.2 the equivalence class $[D]$ in the second set $\mathcal{A}(X)/L_{<2n}, P_{n+1}, O_n$ is invariant under chord KII moves, hence so is the image of ι_n . \square

LEMMA 3.5. *The element $j_n(\check{Z}(L)) \in \mathcal{A}(\phi)$ does not depend on the orientation of L .*

Proof. Let L' be the framed link obtained from L by changing the orientation of a component C . Equation (1.2) says that $\check{Z}(L') = S_{(C)}(\check{Z}(L))$. Hence, it is sufficient to show that $j_n(S_{(C)}(D)) = j_n(D)$, for any chord diagram D with m external vertices on C .

By definition, $S_{(C)}(D)$ is $(-1)^m D$ with the opposite orientation on C . We obtain $j_n(S_{(C)}(D))$ by substituting T_m^n to the solid circle C with the opposite orientation; this is the same as the substitute the mirror image of T_m^n to C without changing the orientation. Note that the mirror image of T_m^n is equal to $(-1)^m T_m^n$, see Lemmas 2.5 and 2.6. So $j_n(S_{(C)}(D)) = (-1)^m (-1)^m j_n(D) = j_n(D)$. \square

Using the map $\hat{\Lambda}$ we will prove the following lemma in the next section.

LEMMA 3.6. *Let U_+ (resp. U_-) be the trivial knot with $+1$ (resp. -1) framing. Then $\iota_n(\check{Z}(U_{\pm}))$ is invertible in $\mathcal{A}(\phi)/D_{>n}$.*

Recall that $D_{>n}$ denotes the equivalent relation such that any chord diagram of degree $> n$ is equivalent to zero.

THEOREM 3.7. *Let L be an oriented framed link, and M the 3-manifold obtained by Dehn surgery on S^3 along L . Then the element*

$$\Omega_n(L) = [\iota_n(\check{Z}(U_+))]^{-\sigma_+} [\iota_n(\check{Z}(U_-))]^{-\sigma_-} \iota_n(\check{Z}(L)) \in \mathcal{A}(\phi)/D_{>n}$$

is a topological invariant of M for any positive integer n , where σ_+ (resp. σ_-) denotes the number of positive (resp. negative) eigenvalues of the linking matrix of L .

Proof. The element $\Omega_n(L)$ is invariant under the orientation change of any component of L by Lemma 3.5 and under the second Kirby move (i.e. the KII move) by Proposition 3.4.

We also have invariance under the first Kirby move, since the change of $i_n(\check{Z}(L))$ under the move cancels with the change of σ_{\pm} . \square

Definition 3.8. We denote the above invariant by $\Omega_n(M)$.

4. A UNIVERSAL PERTURBATIVE INVARIANT OF 3-MANIFOLDS

In this section we unify the series $\Omega_n(M)$ into an invariant $\Omega(M)$ that has the same information (modulo the order of the first homology group) as all the $\Omega_n(M)$ have. We further show that the invariant $\Omega(M)$ is a group-like element in the commutative co-commutative Hopf algebra $\mathcal{A}(\phi)$. In fact, this is a consequence of the fact that the Kontsevich invariant of a link is a group-like element.

4.1. A group-like property of the series $\Omega_n(M)$

We denote by $\hat{\Delta}_{n_1, n_2}$ the map $\mathcal{A}(\phi)/D_{> n_1 + n_2} \rightarrow \mathcal{A}(\phi)/D_{> n_1} \otimes \mathcal{A}(\phi)/D_{> n_2}$ naturally induced by $\hat{\Delta}$; note that this map is well defined since the degree is preserved by $\hat{\Delta}$, where we regard the sum of degrees as the degree in the tensor product.

PROPOSITION 4.1. *One has*

$$\hat{\Delta}_{n_1, n_2}(\Omega_{n_1 + n_2}(M)) = \Omega_{n_1}(M) \otimes \Omega_{n_2}(M).$$

Proof. Noting that $\hat{\Delta}$ is an algebra homomorphism in $\mathcal{A}(\phi)$, this proposition is a direct consequence of Theorem 1.2 and the following lemma. \square

LEMMA 4.2. *Let n, n_1 and n_2 be positive integers satisfying $n = n_1 + n_2$. Then the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{A}(\coprod^l S^1) \hat{\gamma} / L_{< 2n}, PO_n & \xrightarrow{\tilde{j}_n} & \mathcal{A}(\phi) \hat{\gamma} / PO_n \\ \hat{\Delta}_{n_1, n_2} \downarrow & & \downarrow \hat{\Delta} \\ \mathcal{A}(\coprod^l S^1) \hat{\gamma} / L_{< 2n_1}, PO_{n_1} \otimes \mathcal{A}(\coprod^l S^1) \hat{\gamma} / L_{< 2n_2}, PO_{n_2} & \xrightarrow{\tilde{j}_{n_1} \otimes \tilde{j}_{n_2}} & \mathcal{A}(\phi) \hat{\gamma} / PO_{n_1} \otimes \mathcal{A}(\phi) \hat{\gamma} / PO_{n_2} \end{array} \tag{4.1}$$

where we mean by PO_k the equivalence relation generated by P_{k+1} and O_k , and \tilde{j}_k is induced by j_k .

Proof. By Lemma 3.1 we can assume that an element in $\mathcal{A}(\coprod^l S^1) \hat{\gamma} / L_{< 2n}, PO_n$ is a disjoint union of dashed trivalent graphs and l copies of a solid circle with n dashed isolated chords; in fact, the space consists of linear sums of such elements. Since j_{\star} is trivial for dashed trivalent graphs, (4.1) is commutative for them. Hence, it is sufficient to show that (4.1) is commutative for a solid circle with n dashed isolated chords, which we denote by D_n .

The image of D_n by the map j_n is equal to the chord diagram obtained by attaching n isolated chords to T_{2n}^n by the definition of j_n . By the same argument as in Fig. 21, T_{2n}^n with one dashed isolated chord is equal to T_{2n-2}^{n-1} time a sum of $2n - 2$ and a dashed circle; the sum is equal to -2 modulo the relation O_n . Repeating this argument, we can show that the $\tilde{j}_n(D_n)$ is equal to $(-2)(-4) \cdots (-2n) = (-2)^n n!$. Therefore, the clockwise image of D_n in the diagram is equal to $(-2)^n n!$.

The image of D_n by the map $\hat{\Delta}$ is equal to the sum of $\binom{n}{k} D_k \otimes D_{n-k}$ by the definition of $\hat{\Delta}$. Note that it vanishes modulo $L_{< 2n_1}$ or $L_{< 2n_2}$ unless $k = n_1$. Hence, the image by Δ_{n_1, n_2} is

equal to $\binom{n}{n_1} D_{n_1} \otimes D_{n_2}$. By the same argument as above, the image of it by the map $\tilde{j}_{n_1} \otimes \tilde{j}_{n_2}$ is equal to $\binom{n}{n_1} (-2)^{n_1} n_1! (-2)^{n_2} n_2! = (-2)^n n!$ using $n = n_1 + n_2$; this coincides with the above value. \square

Proof of Lemma 3.6. Applying Theorem 1.2 to $\check{Z}(U_{\pm})$, $n - 1$ times, we have $\hat{\Delta}^{(n-1)}(\check{Z}(U_{\pm})) = (\check{Z}(U_{\pm}))^{\otimes n}$, where we define the map $\hat{\Delta}^{(k)}: \mathcal{A}(X) \rightarrow \mathcal{A}(X)^{\otimes(k+1)}$ by $\hat{\Delta}^{(k)} = (\hat{\Delta}(\otimes 1) \circ \hat{\Delta}^{(k-1)})$ recursively. Further, putting $X = \phi$, the map $\hat{\Delta}^{(k)}$ naturally induces a map $\hat{\Delta}_{1,1,\dots,1}^{(k)}$ from $\mathcal{A}(\phi)/D_{>k+1}$ to $(\mathcal{A}(\phi)/D_{>1})^{\otimes(k+1)}$. Applying Lemma 4.2 $n - 1$ times, we obtain the first equality of the following formula;

$$\hat{\Delta}_{1,1,\dots,1}^{(n-1)}([\iota_n(\check{Z}(U_{\pm}))]) = (\iota_1(\check{Z}(U_{\pm})))^{\otimes n} = \left(\mp 1 + \frac{\theta}{16}\right)^{\otimes n} \in \left(\mathcal{A}(\phi)/D_{>1}\right)^{\otimes n}.$$

Here the second equality is derived from Lemma 4.3 below. Therefore the constant term of $[\iota_n(\check{Z}(U_{\pm}))]$ does not vanish, which means that it is invertible. \square

LEMMA 4.3. (Le et al. [21]). *Let θ be the graph depicted in Fig. 25 (see Section 7). One has*

$$\iota_1(\check{Z}(U_{\pm})) = \mp 1 + \frac{\theta}{16} \in \mathcal{A}(\phi)/D_{>1}.$$

This has been proved in [21]. A dedicated reader can check this identity by first computing $\check{Z}(U_{\pm})$ up to degree 4, then applying ι_1 .

Remark 4.4. In fact, the above values of $\iota_1(\check{Z}(U_{\pm}))$ are -2 times the values in [21] because of different normalizations of the map ι_1 .

4.2. A power series invariant $\Omega(M)$

We define a map $\varepsilon: \mathcal{A}(\phi) \rightarrow \mathbb{C}$ to be the projection to the degree 0 part of a linear sum of chord diagrams; recall that we regard the empty diagram as 1 whose degree is 0. By the definitions of $\hat{\Delta}_{1,n-1}$ are ε , we immediately obtain the following lemma.

LEMMA 4.5. *Let $p_{n,n-1}$ be the projection from $\mathcal{A}(\phi)/D_{>n}$ to $\mathcal{A}(\phi)/D_{>n-1}$. Then one has*

$$(\varepsilon \otimes \text{id}) \circ \hat{\Delta}_{1,n-1} = p_{n,n-1}$$

LEMMA 4.6. *Let $(\Omega_1, \Omega_2, \dots)$ be a series of $\Omega_n \in \mathcal{A}(\phi)/D_{>n}$ which satisfies*

$$\hat{\Delta}_{n_1, n_2}(\Omega_{n_1+n_2}) = \Omega_{n_1} \otimes \Omega_{n_2} \tag{4.2}$$

for any positive integers n_1 and n_2 .

1. Then the formula $\Omega_n^{(d)} = \mu^{n-d} \Omega_d^{(d)}$ holds for $d < n$, where we denote by $\alpha^{(d)}$ the degree d part of α and we put μ to be $\Omega_1^{(0)}$.
2. Furthermore, for $\Omega = 1 + \sum_{n=1}^{\infty} \Omega_n^{(n)} \in \hat{\mathcal{A}}(\phi)$, one has $\hat{\Delta}(\Omega) = \Omega \otimes \Omega$; recall that we denote by $\hat{\mathcal{A}}(\phi)$ the completion of $\mathcal{A}(\phi)$ with respect to the degree.

Proof. We apply the map in Lemma 4.5 to Ω_n . Then we have the left-hand side as

$$(\varepsilon \otimes \text{id}) \circ \hat{\Delta}_{1,n-1}(\Omega_n) = (\varepsilon \otimes \text{id})(\Omega_1 \otimes \Omega_{n-1}) = \mu \Omega_{n-1}.$$

From this we obtain part 1.

We put Ω_0 to be 1, then the series $(\Omega_0, \Omega_1, \dots)$ satisfies (4.2) for any non-negative integers n_1 and n_2 . It is sufficient to show the required formula of part 2 for each degree n , i.e. to show that $\hat{\Delta}(\Omega_n^{(n)}) = \sum_{k_1+k_2=n} \Omega^{k_1} \otimes \Omega^{k_2}$. It suffices to show the identity for the image of it under the map $p_{\infty, n_1} \otimes p_{\infty, n_2}$ for each pair n_1 and n_2 with $n_1 + n_2 = n$, where we denote by $p_{\infty, k}$ the projection of $\hat{\mathcal{A}}(\phi)$ to $\mathcal{A}(\phi)/D_{>k}$. The image becomes $\hat{\Delta}_{n_1, n_2}(\Omega_n^{(n)}) = \Omega_{n_1}^{(n_1)} \otimes \Omega_{n_2}^{(n_2)}$ which is a special case of (4.2). This completes the proof. \square

We have the following lemma in [20] where only the invariant $\Omega_1(M)$ was discussed. It is a good exercise to directly prove this lemma.

LEMMA 4.7. *The degree zero part of $\Omega_1(M)$ is equal to $|H_1(M, \mathbb{Z})|$ if M is a rational homology 3-sphere, 0 otherwise. Here we mean by $|\cdot|$ the cardinality of the set.*

We now apply Lemma 4.6 to our series of invariants $\Omega_n(M)$ to reduce the series to $\Omega(M) \in \hat{\mathcal{A}}(\phi)$ and a scalar μ . In our case μ is either the order of the first homology group or zero by Lemma 4.7. Then we obtain the following definition and proposition from Lemma 4.6.

Definition 4.8. We define a topological invariant $\Omega(M)$ of an oriented closed 3-manifold M by $\Omega(M) = 1 + \sum_{n=1}^{\infty} \Omega_n(M)^{(n)} \in \hat{\mathcal{A}}(\phi)$.

PROPOSITION 4.9. *The invariant $\Omega(M)$ satisfies $\hat{\Delta}(\Omega(M)) = \Omega(M) \otimes \Omega(M)$.*

4.3. Logarithm of $\Omega(M)$

We denote by $\mathcal{A}(\phi)_{\text{conn}}$ the vector subspace of $\mathcal{A}(\phi)$ spanned by the set of non-empty vertex-oriented trivalent *connected* dashed graphs. We put $\hat{\mathcal{A}}(\phi)_{\text{conn}}$ to be the completion of it with respect to the degree, which becomes a vector subspace of $\hat{\mathcal{A}}(\phi)$.

It is well known as a property of Hopf algebras (see for example [1]), that a non-zero element $\alpha \in \hat{\mathcal{A}}(\phi)$ is group-like if and only if there exists a primitive element $\omega \in \hat{\mathcal{A}}(\phi)$ satisfying $\alpha = \exp(\omega) = 1 + \omega + (1/2)\omega^2 + \dots$, where we call α *group-like* if it satisfies $\hat{\Delta}(\alpha) = \alpha \otimes \alpha$, and call α *primitive* if it satisfies $\hat{\Delta}(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$. In our case $\alpha \in \hat{\mathcal{A}}(\phi)$ is primitive if and only if α belongs to $\hat{\mathcal{A}}(\phi)_{\text{conn}}$. Note that ω is uniquely determined for given α , since a primitive element always has a positive degree.

Since $\Omega(M)$ is group-like by Proposition 4.9, we have the following definition.

Definition 4.10. We define $\omega(M) \in \hat{\mathcal{A}}(\phi)_{\text{conn}}$ by $\exp(\omega(M)) = \Omega(M)$.

5. PROPERTIES OF $\omega(M)$

We will establish some properties of $\omega(M)$ in this section.

5.1. Formulas for connected sum and opposite orientation

PROPOSITION 5.1. *Let M be the connected sum of two oriented closed 3-manifolds M_1 and M_2 , then $\omega(M)$ is given by*

$$\omega(M) = \sum_{d=1}^{\infty} (\mu_2^d \omega(M_1)^{(d)} + \mu_1^d \omega(M_2)^{(d)}), \tag{5.1}$$

where μ_i ($i = 1, 2$) is the cardinality of $H_1(M_i, \mathbb{Z})$ if M_i is a rational homology 3-sphere, 0 otherwise.

Proof. Let L be the framed link obtained by taking split union of two framed links L_1 and L_2 . By the definition of $\check{Z}(L)$, we have $\check{Z}(L)$ as the disjoint union of $\check{Z}(L_1)$ and $\check{Z}(L_2)$.

Note that, if M_1 and M_2 are obtained by Dehn surgery along L_1 and L_2 , respectively, then M is obtained from L . From the definition of $\Omega_n(M)$ we have $\Omega_n(M) = \Omega_n(M_1)\Omega_n(M_2)$. By Lemmas 4.6 and 4.7 we have

$$\begin{aligned} \Omega_n(M)^{(n)} &= \sum_{d_1+d_2=n} \Omega_n(M_1)^{(d_1)}\Omega_n(M_2)^{(d_2)} \\ &= \sum_{d_1+d_2=n} \mu_1^{d_2}\Omega_{d_1}(M_1)^{(d_1)}\mu_2^{d_1}\Omega_{d_2}(M_2)^{(d_2)}. \end{aligned}$$

Hence

$$\Omega(M) = \sum_{d_1, d_2=0} \mu_1^{d_2}\Omega(M_1)^{(d_1)}\mu_2^{d_1}\Omega(M_2)^{(d_2)}. \tag{5.2}$$

If both of M_1 and M_2 are rational homology 3-spheres, then (5.2) implies $\sum_d \Omega(M_i)^{(d)}/\mu_i^d$ is multiplicative with respect to connected sum. Hence $\sum_d \omega(M_i)^{(d)}/\mu_i^d$ is additive, and we get (5.1).

If either of M_1 and M_2 , say M_1 , is not a rational homology 3-sphere, then we have $\mu_1 = 0$. Hence $\Omega(M) = \sum_d \Omega(M_2)^{(d)}/\mu_2^d$ by (5.2) with $d_1 = 0$, where we regard 0^0 as 1. Therefore we obtain (5.1), completing the proof. \square

PROPOSITION 5.2. *Let $-M$ be the 3-manifold M with the opposite orientation. Suppose b_1 is the first Betti number. Then the following formula holds:*

$$\omega(-M) = \sum_{d=1}^{\infty} (-1)^{d(b_1+1)}\omega(M)^{(d)}.$$

Proof. We define a map $\hat{S}: \mathcal{A}(X) \rightarrow \mathcal{A}(X)$ by putting $\hat{S}(D) = (-1)^d D$ for a chord diagram D where d is the degree of D . For the mirror image \bar{L} of a framed link L , we have $\check{Z}(\bar{L}) = \hat{S}(\check{Z}(L))$; we can show the formula by checking it for elementary tangles. Suppose L has l components. Then i_n decreases the degree by nl , hence $i_n(\check{Z}(\bar{L})) = (-1)^{nl}\hat{S}[i_n(\check{Z}(L))]$. Similarly, since $\bar{U}_+ = U_-$, one has $i_n(\check{Z}(U_-)) = (-1)^n\hat{S}[i_n(\check{Z}(U_+))]$. Since $-M$ is obtained by Dehn surgery along \bar{L} , we have $\Omega_n(-M) = (-1)^{(nl-n\sigma_- - n\sigma_+)}\hat{S}(\Omega_n(M))$. Taking the n -degree part, noting that $(l - \sigma_- - \sigma_+) = b_1$, one has

$$\Omega(-M)^{(n)} = (-1)^{n(b_1+1)}(\Omega(M))^{(n)}.$$

The proposition now follows from this identity. \square

5.2. The first term in $\omega(M)$

The degree 1 subspace of $\mathcal{A}(\phi)$ is one-dimensional and generated by θ , see Fig. 25. The following was obtained in [21].

PROPOSITION 5.3. *Let M be an oriented closed 3-manifold. Then the coefficient of θ in $\omega(M)$ is equal to $(-1)^{b_1(M)}\lambda_{\text{Lescop}}(M)/2$, where $b_1(M)$ is the first Betti number of M and $\lambda_{\text{Lescop}}(M)$ is Lescop’s generalization of the Casson–Walker invariant as described in [22].*

Note that $\lambda_{\text{Lescop}}(M) = |H_1(M; \mathbb{Z})| \lambda_w(M)/2$ if M is a rational homology 3-sphere, where λ_w is the Casson-Walker invariant in [32]. Also $\lambda_{\text{Lescop}}(P) = 1$, where P is the Poincare homology 3-sphere, obtained by Dehn surgery along the right-hand trefoil with framing 1.

Remark 5.4. We can “substitute” a Lie algebra into dashed lines, see [6]. When we substitute sl_N, so_N and sp_N to θ , we obtain values $2N(N^2 - 1), N(N - 1)(N - 2)/2$ and $2N(N + 1)(2N + 1)$ respectively. When we substitute sl_2 into the first term (i.e. θ and its coefficient) of $\omega(M)$, we have the value 12 times the coefficient. Hence, if one expects the existence of “G-Casson invariant $\lambda^G(M)$ ” and expects that it should recover from the first term in $\omega(M)$ by substituting the Lie algebra of G into dashed lines, it must satisfy

$$\begin{aligned} \lambda^{su(N)}(M) &= \frac{N(N^2 - 1)}{6} \lambda(M), \\ \lambda^{so(N)}(M) &= \frac{N(N - 1)(N - 2)}{24} \lambda(M), \\ \lambda^{sp(N)}(M) &= \frac{N(N + 1)(2N + 1)}{6} \lambda(M). \end{aligned}$$

In [24] we have the same formula for $\lambda^{su(N)}(M)$ of each lens space M , which is obtained by expanding quantum $PSU(N)$ invariant of lens spaces computed in [28] into a power series in $q - 1$. As for the second term in $\omega(M)$ for some lens spaces, see Section 7.3.

6. EXTENSION TO INVARIANTS OF LINKS IN 3-MANIFOLDS

6.1. Extension of $\Omega(M)$ to invariants of links in 3-manifolds

Let $L \cup L'$ be an oriented framed link in S^3 , with l and l' components respectively. Let M be the 3-manifold obtained by Dehn surgery on S^3 along L' . We denote the image, under the surgery, of L in M also by the same L . Note that, any pair (M, L) of a closed 3-manifold M and an oriented framed link L in M can be obtained by this way.

We consider the linear map

$$i'_n : \mathcal{A} \left(\coprod^{l+l'} S^1 \right) \rightarrow \mathcal{A} \left(\coprod^l S^1 \right) / D_{>n}$$

which first replaces the latter l' solid circles with T_m^n 's in the same way as the definition of j_n in Section 2.2, then replaces every dashed loop by $-2n$, and finally divide out the part of degree greater than n .

In the same way as Proposition 3.4, Lemma 3.5 and Theorem 3.7, we obtain the following propositions and theorem.

PROPOSITION 6.1. *The element $[i'_n(\check{Z}(L \cup L'))] \in \mathcal{A}(\coprod^l S^1) / D_{>n}$ is invariant*

- (a) *under KII move which slides a handle of any component of $L \cup L'$ over any component of L' , and*
- (b) *under orientation change of any component of L' .*

THEOREM 6.2. *The element*

$$\Omega(M, L) = [i_n(\check{Z}(U_+))]^{-\sigma'} [i_n(\check{Z}(U_-))]^{-\sigma'} [i'_n(\check{Z}(L \cup L'))] \in \mathcal{A} \left(\coprod^l S^1 \right) / D_{>n}$$

is a topological invariant of the pair (M, L) for any positive integer n , where σ'_+ (resp. σ'_-) denotes the number of positive (resp. negative) eigenvalues of the linking matrix of L' . Here, in the formula, we use the product $\mathcal{A}(\phi) \otimes \mathcal{A}(\coprod^l S^1) \rightarrow \mathcal{A}(\coprod^l S^1)$ defined by taking disjoint union of chord diagrams.

PROPOSITION 6.3. *One has*

$$\hat{\Delta}_{n_1, n_2}(\Omega_{n_1+n_2}(M, L)) = \Omega_{n_1}(M, L) \otimes \Omega_{n_2}(M, L)$$

where we denote by $\hat{\Delta}_{n_1, n_2}$ the map $\mathcal{A}(\coprod^{n_1+n_2} S^1)/D_{>n_1+n_2} \rightarrow \mathcal{A}(\coprod^{n_1} S^1)/D_{>n_1} \otimes \mathcal{A}(\coprod^{n_2} S^1)/D_{>n_2}$ naturally induced by $\hat{\Delta}$.

Let D_0 be the chord diagram consisting of l solid circles and no dashed graphs. Note that the degree 0 part of $\mathcal{A}(\coprod^l S^1)$ is spanned by D_0 . Hence we can put $\Omega_1(M, L)^{(0)} = \mu D_0$ with a scalar μ .

LEMMA 6.4. (1) *The above μ is equal to $|H_1(M, \mathbb{Z})|$ if M is a rational homology 3-sphere, 0 otherwise.*

(2) $\Omega_n(M, L)^{(d)} = \mu^{n-d} \Omega_d(M, L)^{(d)}$

(3) *Put $\Omega(M, L)$ to be $D_0 + \sum_{n=1}^{\infty} \Omega_n(M, L)^{(n)} \in \hat{\mathcal{A}}(\coprod^l S^1)$, then*

$$\hat{\Delta}(\Omega(M, L)) = \Omega(M, L) \otimes \Omega(M, L).$$

Proof. Let $\hat{\varepsilon}'$ be the map $\hat{\mathcal{A}}(\coprod^{l+l'} S^1) \rightarrow \hat{\mathcal{A}}(\coprod^l S^1)$ defined as follows. If the latter l' solid circles of a chord diagram D has no dashed univalent vertices, then $\hat{\varepsilon}'(D)$ is the chord diagram obtained by removing the latter l' solid circles from D , and $\hat{\varepsilon}'(D) = 0$ otherwise. Then, by using part 3 of Proposition 1.1 repeatedly, we obtain

$$\hat{\varepsilon}'(\check{Z}(L \cup L')) = \check{Z}(L),$$

where $L \cup L'$ is a link with $l + l'$ -components; they are ordered so that the first l ones are in L and the latter l' ones in L' .

Starting with the above formula, following the construction of $\Omega_n(M, L)$, we see that

$$\hat{\varepsilon}(\Omega_n(M, L)) = \Omega_n(M),$$

where $\hat{\varepsilon}$ is the map $\hat{\mathcal{A}}(\coprod^l S^1) \rightarrow \hat{\mathcal{A}}(\phi)$ defined as follows. If D is a disjoint union of a dashed trivalent graph and l solid circles, the $\hat{\varepsilon}(D)$ is the dashed trivalent graph, and $\hat{\varepsilon}(D) = 0$ otherwise. Hence

$$|H_1(M, \mathbb{Z})| = \Omega_1(M)^{(0)} = \hat{\varepsilon}(\Omega_1(M, L)^{(0)}) = \mu,$$

proving part 1. We obtain parts 2 and 3 in the same way as Lemmas 4.6 and 4.7. \square

The above invariant $\Omega(M, L)$ is an extension of $\Omega(M)$ in the sense of the following proposition.

PROPOSITION 6.5. *One has*

$$\Omega(S^3, L) = \check{Z}(L)$$

$$\Omega(M, \phi) = \Omega(M)$$

$$\hat{\varepsilon}(\Omega(M, L)) = \Omega(M).$$

Proof. We obtain the first and the second formulas by the definition of $\Omega(M, L)$. Further, we obtain the third required formula in the same way as in the proof of Lemma 6.4(1). □

6.2. Formula for Dehn surgery along a link in a 3-manifold

In this subsection, we suppose that M is a rational homology 3-sphere. Under this assumption, we define

$$\hat{\Omega}(M) = \sum_{d=0}^{\infty} |H_1(M, \mathbb{Z})|^{-d} \Omega(M)^{(d)},$$

$$\hat{\Omega}(M, L) = \sum_{d=0}^{\infty} |H_1(M, \mathbb{Z})|^{-d} \Omega(M, L)^{(d)},$$

recall that $\alpha^{(d)}$ denotes the degree d part of α .

Then from Proposition 5.1 one has $\hat{\Omega}(M_1 \# M_2) = \hat{\Omega}(M_1) \hat{\Omega}(M_2)$.

We define the linking number of two knots K_1 and K_2 in a rational homology 3-sphere M as follows. Since m times K_1 is null homologous in M for some integer m , we can find an oriented surface F in M whose boundary covers K_1 m times. We define the linking number of K_1 and K_2 to be $1/m$ times the intersection number of F and K_2 ; it is a rational number in general. Further, we identify the framing of a framed knot in M with a rational number as follows; we express the knot by an annulus in M , and we define the number to be the linking number of two boundaries of the annulus. In such a way, we can define the linking matrix of a framed link L in a rational homology 3-sphere M .

THEOREM 6.6. *Let M be a rational homology 3-sphere, L a framed link in M , and M_L the 3-manifold, not necessarily a rational homology 3-sphere, obtained by Dehn surgery on M along L . Then, we have*

$$\Omega(M_L) = \sum_{n=0}^{\infty} |H_1(M, \mathbb{Z})|^n ([\iota_n \check{Z}(U_+)]^{-\sigma_+} [\iota_n \check{Z}(U_-)]^{-\sigma_-} [\iota_n \hat{\Omega}(M, L)])^{(n)},$$

where σ_+ (resp. σ_-) denotes the number of positive (resp. negative) eigenvalues of the linking matrix of L defined in the above sense.

Remark 6.7. Putting $M = S^3$, we recover the definition of $\Omega(M)$ from the above formula (see the formula in Theorem 3.7).

Proof of Theorem 6.6. Let $L \cup L'$ be a framed link in S^3 , and M a 3-manifold obtained by Dehn surgery on S^3 along L' . There is a remain of L in M ; we denote it also by L . Note that the pair (M, L) in the theorem can be constructed as above. The linking matrix of L in M is, in general, different from the linking matrix of L in S^3 ; and σ_+, σ_- are the number of positive and negative eigenvalues of the linking matrix of L in M .

Let l and l' be the number of components of L and L' respectively. Let ι'_n be the map from $\mathcal{A}(\coprod^{l+l'} S^1)$ to $\mathcal{A}(\coprod^l S^1)$ defined in the previous subsection, and ι_n the standard map from $\mathcal{A}(\coprod^l S^1)$ to $\mathcal{A}(\phi)$. By the definition of $\Omega(M)$, we have

$$\Omega(M, L)^{(n)} = ([\iota_n(\check{Z}(U_+)]^{-\sigma_+} [\iota_n(\check{Z}(U_-)]^{-\sigma_-} [\iota'_n \check{Z}(L \cup L')])^{(n)},$$

where σ'_+ (resp. σ'_-) is the number of positive (resp. negative) eigenvalues of the linking matrix of L' . On the other hand, we have

$$\Omega(M_L)^{(n)} = ([\iota_n(\check{Z}(U_+)]^{-\sigma_+ - \sigma'_+} [\iota_n(\check{Z}(U_-)]^{-\sigma_- - \sigma'_-} [\iota'_n \check{Z}(L \cup L')])^{(n)}.$$

Note that the numbers of positive and negative eigenvalues of the linking matrix of $L \cup L'$ (in S^3) are $\sigma_+ + \sigma'_+$ and $\sigma_- + \sigma'_-$, because using elementary operations (with rational coefficients) on the linking matrix of $L \cup L'$ (in S^3), not touching the part of L' , we can obtain the block sum of the linking matrix (with rational entries) of L in M and the linking matrix of L' in S^3 .

For every positive integer d we have

$$\begin{aligned} & ([i_n(\check{Z}(U_+))^{-\sigma'_+} [i_n(\check{Z}(U_-))^{-\sigma'_-} [i_n i'_n \check{Z}(L \cup L')]])^{(d)} \\ &= i_n(([i_n(\check{Z}(U_+))^{-\sigma'_+} [i_n(\check{Z}(U_-))^{-\sigma'_-} [i'_n \check{Z}(L \cup L')]])^{(d+n)})) \\ &= i_n \Omega_n(M, L)^{(d+n)} \\ &= i_n |H_1(M, \mathbb{Z})|^{-d-n(l-1)} \Omega_{d+nl}(M, L)^{(d+n)} \\ &= |H_1(M, \mathbb{Z})|^n i_n \hat{\Omega}(M, L)^{(d+n)}. \end{aligned}$$

Here, for the first equality, we use the fact that i_n decreases the degree of chord diagrams by n per solid circle; and we obtain the third equality from the relation

$$\Omega_{m-k}(M, L)^{(m)} = |H_1(M, \mathbb{Z})|^{-k} \Omega_m(M, L)^{(m)},$$

which can be proved in the same way as Lemma 4.6; note that, only in this formula, we must regard $\Omega_{m-k}(M, L)^{(m)}$ not as an element in $\mathcal{A}(\coprod^l S^1)/D_{>(m-k)}$, but as an element in $\mathcal{A}(\coprod^l S^1)/P_{(m-k)+1}, O_{m-k}$.

Hence, noting that i_n and multiplication with an element in $\mathcal{A}(\phi)$ commute, we have

$$\Omega(M_L)^{(n)} = |H_1(M, \mathbb{Z})|^n ([i_n \check{Z}(U_+)]^{-\sigma'_+} [i_n \check{Z}(U_-)]^{-\sigma'_-} [i_n \hat{\Omega}(M, L)])^{(n)},$$

and we obtain the required formula. □

6.3. Quantum sl_N invariant of links

In this subsection we suppose that M is a rational homology 3-sphere. Under this assumption, we can define $\hat{\Omega}(M, L)$ as in the previous subsection. We will define a version of the Homfly polynomial for links in rational homology 3-spheres.

If L' is obtained from a framed link L in S^3 by increasing the framing of a component C by one full twist, then one has (see Theorem 3 of [18])

$$\check{Z}(L') = e^{\Theta/2} \#_C \check{Z}(L),$$

where the right-hand side is the connected sum of $\check{Z}(K)$ and $e^{\Theta/2}$ along C ; and Θ is the chord diagram in $\mathcal{A}(S^1)$ with exactly one isolated chord. From the above formula and from the definition of Ω we have

PROPOSITION 6.8. *If L' is obtained from a framed link L in a closed 3-manifold M by increasing the framing of a component C by one full twist, then*

$$\Omega(M, L') = e^{\Theta/2} \#_C \Omega(M, L).$$

Consider the bilinear symmetric invariant form on sl_N defined by

$$\langle x, y \rangle = \text{tr}[\rho(x)\rho(y)],$$

where ρ is the fundamental representation of sl_N . Note that this form is not equal to, though it is proportional to, the Killing form; the corresponding invariant quadratic element is $(N - 1/N)$ times the identity in the fundamental representation. This bilinear form defines

a weight system on $\mathcal{A}(\coprod^1 S^1)$, i.e. it defines a scalar $\Gamma(sl_N, \rho)$ for a chord diagram Γ ; for the definition of the scalar, see, for example, [5, 18]. We define a map $\Phi: \mathcal{A}(\coprod^1 S^1) \rightarrow \mathbb{Q}[[\hbar]]$ by $\Phi(\Gamma) = \Gamma(sl_N, \rho)\hbar^{d(\Gamma)}$ where $d(\Gamma)$ denotes the degree of Γ . Let $\hat{\Omega}(M)$ be as in the previous subsection.

THEOREM 6.9. *Let M be a rational homology 3-sphere.*

1. *One has the following skein relation:*

$$e^{h/2N}\Phi(\hat{\Omega}(M, L_+)) - e^{-h/2N}\Phi(\hat{\Omega}(M, L_-)) = (e^{h/2} - e^{-h/2})\Phi(\hat{\Omega}(M, L_0)),$$

where L_+, L_-, L_0 are framed links identical everywhere except for a ball in which they are as in Fig. 23. We suppose that the framing vector of the part in Fig. 23 is always perpendicular to the page and pointing to the reader.

2. *If L' is obtained from L by increasing the framing of a component by one full twist, then*

$$\Phi(\hat{\Omega}(M, L')) = e^{(N-1/N)h/2}\Phi(\hat{\Omega}(M, L)).$$

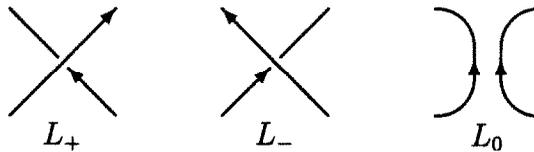


Fig. 23. The skein triple.

Proof. The proof of part 1 is an easy generalization of that of the main theorem in [17]. Part 2 follows from Proposition 6.8 and an easy computation. \square

Let $|L.L|$ be the sum of all the framing numbers of components of L ; for the definition of the framing number in a rational homology 3-sphere, see the previous subsection. Let us define

$$\tau_N(M, L) = e^{(1/N-N)|L.L|h/2}\Phi(\hat{\Omega}(M, L)).$$

Then $\tau_N(M, L)$ does not depend on the framing of L , and

$$e^{Nh/2}\tau_N(M, L_+) - e^{-Nh/2}\tau_N(M, L_-) = (e^{h/2} - e^{-h/2})\tau_N(M, L_0).$$

This means, $\tau_N(M, L)$ is a generalization of the HOMFLY polynomial for oriented links in any rational homology sphere. In [9], Kalfagianni and Lin constructed a generalization of the HOMFLY polynomial for links in a special class of rational homology 3-spheres (the rational homology sphere there must be irreducible, and must be either an atoroidal manifold or a Seifert fibered space).

Whether $\tau_N(M, L)$ is a convergent formal power series is an interesting question. In the case $M = S^3$, all the series $\tau_N(M, L)$ are convergent because they are polynomials in $e^{\pm h/2}$. For the case $L = \phi$, see also Conjecture 7.3.

7. CALCULATIONS FOR SOME SIMPLE 3-MANIFOLDS

We will calculate $\Omega(M)$ for some simple 3-manifolds up to degree 2, and give a conjecture about the relation between $\Omega(M)$ and quantum invariants.

7.1. *Notations for calculations*

For a series $\Omega_n \in \mathcal{A}(\phi)/D_{>n}$ ($n = 1, 2, \dots$) satisfying the condition of Lemma 4.6, namely, $\hat{\Delta}_{n_1, n_2}(\Omega_{n_1 + n_2}) = \Omega_{n_1} \otimes \Omega_{n_2}$, let

$$\begin{aligned} \mu(\Omega_\star) &= \Omega_1^{(0)} \in \mathbb{C}, \\ \Omega(\Omega_\star) &= 1 + \sum_{n=1}^{\infty} \Omega_n^{(n)} \in \hat{\mathcal{A}}(\phi). \end{aligned}$$

The $\hat{\Delta}(\Omega(\Omega_\star)) = \Omega(\Omega_\star) \otimes \Omega(\Omega_\star)$ (see Lemma 4.6). Furthermore, we define $\omega(\Omega_\star) \in \hat{\mathcal{A}}_{\text{conn}}$ by $\exp(\omega(\Omega_\star)) = \Omega(\Omega_\star)$.

Note that, if we put $\Omega_n = \Omega_n(M)$, then

$$\begin{aligned} |H_1(M, \mathbb{Z})| &= \mu(\Omega_\star) \\ \omega(M) &= \omega(\Omega_\star). \end{aligned}$$

LEMMA 7.1. *Let Ω_\star and Ω'_\star be series satisfying the condition of Lemma 4.6.*

1. *For the inverse of Ω_\star , we have*

$$\begin{aligned} \mu(\Omega_\star^{-1}) &= \mu(\Omega_\star)^{-1}, \\ \mu(\Omega_\star^{-1}) &= - \sum_{d=1}^{\infty} \mu(\Omega_\star)^{-2d} \omega(\Omega_\star)^{(d)}, \end{aligned}$$

2. *For the product of Ω_\star and Ω'_\star , we have*

$$\begin{aligned} \mu(\Omega_\star \Omega'_\star) &= \mu(\Omega_\star) \mu(\Omega'_\star), \\ \mu(\Omega_\star \Omega'_\star) &= \sum_{d=1}^{\infty} (\mu(\Omega'_\star)^d \omega(\Omega_\star)^{(d)} + \mu(\Omega_\star)^d \omega(\Omega'_\star)^{(d)}). \end{aligned}$$

Proof. The formulas are obtained by easy calculations. □

7.2. *Calculations*

Note that $\hat{\mathcal{A}}(S^1)$ is a commutative co-commutative Hopf algebra whose primitive elements are linear combination of chord diagrams with connected dashed graph. For a knot K , since $\hat{Z}(K)$ is a group-like element, $\log(\hat{Z}(K))$ is a linear combination of chord diagrams with connected dashed graph. So, up to degree 4 one has

$$\log \hat{Z}(K) = v_1 D_1 + v_2 D_2 + v_3 D_3 + v_4 D_4 + v'_4 D'_4 + \text{higher terms},$$

where D_d and D'_d are chord diagrams in $\mathcal{A}(S^1)$ of degree d shown in Figure 24, and $v_d = v_d(K)$ and $v'_d = v'_d(K)$ are certain Vassiliev invariants of framed knots of order d . Here we use $\check{Z}(K)$ defined by $\check{Z}(K) = \hat{Z}(K) \# v^{-1}$; this is another normalization of the Kontsevich invariant such that $\check{Z}(K_0) = 1$ for the trivial knot K_0 with framing 0. It is known that $v_1(K) = k/2$, where k is the framing of K , and $v_2(K) = -a_2/2$, where a_2 is the second coefficient of the Conway polynomial of K . From the definition, one can easily show that

$$\log v^2 = \frac{1}{24} D_2 - \frac{1}{23040} D_4 - \frac{1}{2880} D'_4 + \text{higher terms}.$$

Hence up to degree 4 we have

$$\log \check{Z}(K) = v_1 D_1 + \left(v_2 + \frac{1}{24}\right) D_2 + v_3 D_3 + \left(v_4 - \frac{1}{23040}\right) D_4 + \left(v'_4 - \frac{1}{2880}\right) D'_4.$$

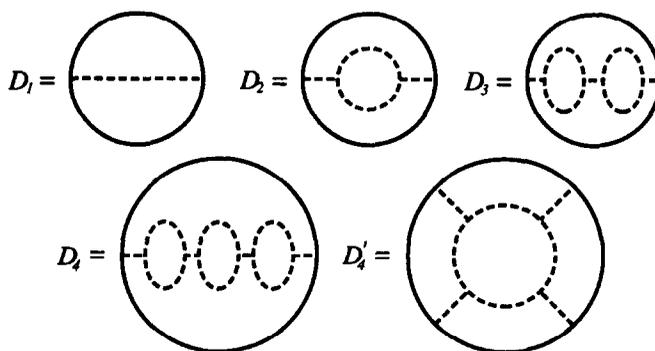


Fig. 24. Diagrams D_1, D_2, \dots, D_4 .

From the definition of i_n , one can compute the following values:

$$\begin{aligned} i_1(D_1) &= -2 \\ i_1(D_1^2) &= \frac{1}{6}\theta \\ i_1(D_2) &= \theta \\ i_2(D_1^4) &= \frac{1}{12}\theta^2 \\ i_2(D_1^2 D_2) &= \frac{1}{6}\theta^2 - \frac{1}{3}\theta_2 \\ i_2(D_2^2) &= \theta^2 + 2\theta_2 \\ i_2(D_1 D_3) &= -2\theta_2 \\ i_2(D_4) &= 0 \\ i_2(D_4') &= \frac{5}{2}\theta_2, \end{aligned}$$

where $\theta, \theta_2 \in \mathcal{A}(\phi)_{\text{conn}}$ are as shown in Fig. 25.

Hence, we have

$$\begin{aligned} \mu([i_\star(\check{Z}(K))]) &= -2v_1 \\ \omega([i_\star(\check{Z}(K))]) &= (\frac{1}{12}v_1^2 + v_2 + \frac{1}{24})\theta \\ &\quad + ((v_2 + \frac{1}{24})^2 - \frac{1}{6}v_1^2(v_2 + \frac{1}{24}) - 2v_1v_3 + \frac{5}{2}(v_4 - \frac{1}{2880}))\theta_2 + \text{higher terms.} \end{aligned}$$

In particular, for U_\pm (where $v_1 = \pm \frac{1}{2}$ and $v_i = v'_i = 0$ for $i > 1$), we have

$$\begin{aligned} \mu([i_\star(\check{Z}(U_\pm))]) &= \mp 1 \\ \omega([i_\star(\check{Z}(U_\pm))]) &= \frac{1}{16}\theta - \frac{1}{1152}\theta_2. \end{aligned}$$

Let $M(K, k)$ be the 3-manifold obtained by the Dehn surgery on S^3 along the knot K with framing $k, k \neq 0$. Let $\delta = k/|k|$. From Lemma 7.1, we get

$$\begin{aligned} \omega(M(K, k)) &= \omega([i_\star(\check{Z}(U_\delta))]^{-1}[i_\star(\check{Z}(K))]) \\ &= (-\frac{1}{24}(2\delta v_1^2 - 3v_1 + \delta) - v_2)\theta \\ &\quad + ((v_2 + \frac{1}{24})^2 - \frac{1}{6}v_1^2(v_2 + \frac{1}{24}) - 2v_1v_3 + \frac{5}{2}(v_4 - \frac{1}{2880}) + \frac{1}{288}v_1^2)\theta_2 \\ &\quad + \text{higher terms.} \end{aligned}$$

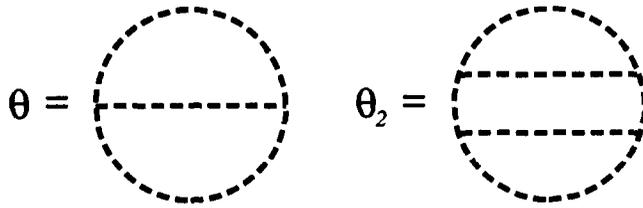


Fig. 25. Diagrams θ and θ_2 .

Note that $v_1 = k/2$ and $v_2 = -a_2/2$, where a_2 is the second coefficient of the Conway polynomial of K . Hence, the coefficient of θ is:

$$\begin{aligned} \text{coeff. of } \theta &= -\frac{1}{48}(\delta k^2 - 3k + 2\delta) + \frac{1}{2}a_2 \\ &= \frac{1}{4} |H_1(M(K, k), \mathbb{Z})| \lambda_w(M(K, k)) = \frac{1}{2} \lambda_{\text{Lescop}}(M(K, k)). \end{aligned}$$

This formula implies Proposition 5.3 in the case when M is a rational homology 3-sphere obtained by Dehn surgery along a knot.

Let $M_{m,k}$ be the 3-manifold obtained by Dehn surgery along the $(2, m)$ -torus knot K with framing k . From the definition of \hat{Z} , one can calculate the values of v_i, v'_i :

$$v_1 = \frac{k}{2}, \quad v_2 = -\frac{m^2 - 1}{16}, \quad v_3 = \frac{m^3 - m}{96}, \quad v_4 = \frac{m^4 - 1}{384}.$$

Hence we have, up to degree 2,

$$\begin{aligned} \omega(M_{m,k}) &= \left(\frac{1}{2} \lambda_{\text{Lescop}}(M_{m,k})\right)\theta \\ &+ \left(\frac{1}{1152} k^2(3m^2 - 4) - \frac{1}{96} k(m^3 - m) + \frac{1}{1152} (12m^4 - 15m^2 + 4)\right)\theta_2. \end{aligned} \quad (7.1)$$

Remark 7.2. Two manifolds $M_{10l+3,1}$ and $M_{8l+3,1} \neq M_{6l+1}$ have the same homology group and Casson invariant. However they have different coefficients of θ_2 . This implies that, as an invariant of 3-manifolds, degree 2 part is independent of degree 0 and 1 parts.

For $S^2 \times S^1$, we have

$$\omega(S^2 \times S^1) = \frac{1}{24}\theta + \frac{1}{1152}\theta_2 + (\text{higher terms}).$$

7.3. Relations between the quantum $PSU(N)$ invariant for some lens spaces

Takata [28] had computed the values of the quantum $PSU(N)$ invariant $\tau_r^{PSU(N)}$ of lens space $L(k, l)$ for odd prime r :

$$\tau_r^{PSU(N)}(L(k, l)) = \left(\frac{k}{r}\right)^{N-1} q^{-\binom{N-1}{2}s(l,k)/2} \frac{[(1/k)^{\vee}]^{N-1} [(2/k)^{\vee}]^{N-2} \dots [(N-1/k)^{\vee}]}{[1]^{N-1} [2]^{N-2} \dots [N-1]}. \quad (7.2)$$

Here $s(l, k)$ is the Dedekind sum, $\left(\frac{\cdot}{\cdot}\right)$ the Legendre symbol, $q = \exp(2\pi\sqrt{-1}/r)$, $[m] = (q^{m/2} - q^{-m/2})/(q^{1/2} - q^{-1/2})$, and we denote ab' by $(a/b)^{\vee}$, where b' is the inverse of b in $\mathbb{Z}/r\mathbb{Z}$. For the definition of the quantum $PSU(N)$ invariant, see [12]. In this case, the value of $\tau_r^{PSU(N)}(L(k, l))$ belongs to $\mathbb{Z}[q]$.

In general, under the following assumption:

$$\tau_r^{PSU(N)}(M) \in \mathbb{Z}[q], \quad (7.3)$$

we can expand $\tau_r^{PSU(N)}(M)$ into a power series in $q - 1$:

$$\tau_r^{PSU(N)}(M) = a_{r,0} + a_{r,1}(q - 1) + a_{r,2}(q - 1)^2 + \dots,$$

with integer coefficient $a_{r,n}$. Note that $a_{r,n}$ (modulo r) is uniquely determined in $n \leq r - 2$, since the ring $\mathbb{Z}[q]$ has only one relation: $\sum_{m=0}^{r-1} q^m = 0$. Further, for a rational homology 3-sphere M and for each non-negative integer n , we expect that

$$\text{there is a rational number } \lambda_n \text{ satisfying } (\lambda_n)^{\vee} \equiv \left(\frac{|H_1(M, \mathbb{Z})|}{r} \right)^{N-1} a_{r,n} \tag{7.4}$$

for any sufficiently large odd prime r . Note that, under the assumption (7.4), λ_n is uniquely determined by $\tau_r^{PSU(N)}(M)$; we denote it by $\lambda_r^{PSU(N)}(M)$, which becomes a topological invariant of M . Under the assumptions (7.3) and (7.4), we define a formal power series $\tau_r^{PSU(N)}(M) \in \mathbb{Q}[[t - 1]]$ in $t - 1$ by

$$\tau_r^{PSU(N)}(M)(t) = \sum_{n=0}^{\infty} \lambda_r^{PSU(N)}(M)(t - 1)^n.$$

We can check that the quantum $PSU(N)$ invariant of each lens space given in the formula (7.2) satisfies the assumptions (7.3) and (7.4), and we have

$$\begin{aligned} \tau_r^{PSU(N)}(L(k, 1)) &= |k|^{-N(N-1)/2} \left(1 + \left(\frac{N(N^2 - 1)}{2} \lambda(L(k, 1)) \right) h \right. \\ &\quad \left. + \left(\frac{1}{2} \left(\frac{N(N^2 - 1)}{2} \lambda(L(k, 1))^2 - \frac{k^2 - 1}{288k^2} N^2(N^2 - 1) \right) h^2 + \text{higher terms} \right) \right). \end{aligned} \tag{7.5}$$

On the other hand, putting $m = 1$ in the formula (7.1), we have

$$\omega(L(k, 1)) = \left(\frac{1}{4} |k| \lambda(L(k, 1)) \right) \theta - \frac{k^2 - 1}{1152} \theta_2 + \text{higher terms}. \tag{7.6}$$

Recall the notation $\Gamma(\mathfrak{sl}_N)$ mentioned in Section 6.3; here we omit the representation ρ because we have no solid lines in this case. By definition, for the dashed θ curve, $\theta(\mathfrak{sl}_N)$ is equal to the product of the dimension of the Lie algebra \mathfrak{sl}_N and the eigenvalue of the Casimir element on the adjoint representation of \mathfrak{sl}_N , i.e. we have $\theta(\mathfrak{sl}_N) = (N^2 - 1) \cdot 2N$. Similarly, we have $\theta_2(\mathfrak{sl}_N) = (N^2 - 1) \cdot 4N^2$. Therefore the degree 0, 1 and 2 parts of the formulas (7.5) and (7.6) satisfy the following conjecture.

- CONJECTURE 7.3. *Suppose that M is a rational homology 3-sphere. Then, we conjecture*
- (1) *the quantum $PSU(N)$ invariant of M satisfies the assumptions (7.3) and (7.4); hence the formal power series $\tau^{PSU(N)}(M)$ exists,*
 - (2) *further, it is determined by $\hat{\Omega}(M)$ as follows:*

$$\tau^{PSU(N)}(M)(e^h) = |H_1(M, \mathbb{Z})|^{-N(N-1)/2} \Phi(\hat{\Omega}(M)) \tag{7.7}$$

where the map Φ is defined in §6.3. Recall that $\hat{\Omega}(M) = \sum_{d=0}^{\infty} |H_1(M, \mathbb{Z})|^{-d} \Omega(M)^{(d)}$.

Remark 7.4. Conjecture 7.3(1) is true for $N = 2$. This is obtained in [24]. Furthermore, in this case Conjecture 7.3(2) is also true in the degree 0 and 1 parts, by results in [23].

Remark 7.5. For any simply connected compact simple Lie group G , Formula (7.7) might be generalized to

$$\tau^{PG}(M)(e^h) = |H_1(M, \mathbb{Z})|^{-\text{rank } G(\text{rank } G - 1)/2} \Phi(\hat{\Omega}(M)),$$

if we could define “ $\tau^{PG}(M)$ ” for the quotient Lie group PG and of G divided by its center. Here we use Φ defined by substituting the Lie algebra of G to dashed lines of chord diagram.

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