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LOCAL RULES FOR QUASIPERIODIC TILINGS

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Abstract We discuss local rules theory for quasiperiodic tilings. Two versions of local rules, with or without decorations, are distinguished. Weak local rules are also considered. For the classes of tilings obtained by the canonical projection method, we present necessary conditions and sufficient conditions for the existence of local rules. Every set of quasiperiodic tilings obtained from the canonical strip projection method and based on quadratic irrationalities always admits local rules *after decoration*. In many cases there exist local rules *without decoration*. Examples of pentagonal tilings and 2-dimensional quasiperiodic tilings, obtained by the projection method from 4-dimensional space, are considered in detail. We prove that the existence of (even weak) local rules without decoration implies that the projection plane is based on algebraic irrationalities. The topology of sets of tilings obtained by projection methods is described.

1. Introduction

The aim of the paper is to give a survey of recent results on the theory of local rules for quasiperiodic tilings obtained by the projection method.

One of the most interesting problems in tiling theory is to find sets of building blocks, say polyhedra, and rules which state which building block can be put next to another one, such that every tiling obeying these rules is *aperiodic* and/or quasi-periodic in some sense.

For example, consider two "arrowed" rhombi in Figure 1 (in Section 2) as building blocks. The acute angles of these rhombi are $\pi/5$ and $2\pi/5$, and their edges have the same length. The rule is that only edges with the same kind of arrows can be matched. There are uncountably many tilings of the plane obeying this rule; every such tiling is ***aperiodic*** in the sense that it is different from every nontrivial translate of itself. Moreover these tilings are quasi-periodic in a sense that will be explained later. These tilings are known as Penrose tilings (Penrose, 1978; de Bruijn, 1981).

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A general form of this rule will be called a local rule; some authors call "matching rule" or "local matching rule". A "local rule", in some sense, contains information in a local finite radius. It is far from trivial to decide when a local rule forces a tiling to be aperiodic, or quasiperiodic, since the latter are global properties.

The first local rule enforcing aperiodicity was found in (Berger, 1966), but it contains too many building blocks, and the tilings are just the infinite checker board with some decoration. The best known example is the above mentioned Penrose local rule. Until recently there were known only a few local rules which force quasiperiodicity. We will present here infinitely many such local rules.

We can begin with a set of *aperiodic tilings* and ask whether this set of tilings admits a local rule, i.e., if there is a local rule such that this set of tilings is exactly the set of tilings obeying the local rule. This question also has importance for physics. It seems that only sets of tilings with local rules can serve as model for the *real* quasicrystals discovered in 1984.

There are two methods for generating aperiodic (quasiperiodic) tilings: the substitution method (see, for example, (Grümbaum and Shephard, 1987; Senechal, 1995; Danzer, 1991)) and the projection method (and its modifications, see (de Bruijn, 1981; Kramer and Neri, 1984; Kramer and Schlottmann, 1989; Oguey et al., 1988; Gähler and Rhyner, 1986)). Tilings obtained by the projection method seem closer to periodic tilings. Tilings obtained by the substitution method may have more exotic structures. The Penrose tilings can be obtained by either method.

The main question of this paper is when a set of tilings obtained by the *canonical* projection method admits a local rule (see Section 3). This question has been investigated in many special cases, see, for example, (Baake et al., 1990; Burkov, 1988; Danzer, 1989; Ingersent, 1991; Katz, 1988; Klitzing et al., 1993; Socolar, 1989; Socolar, 1990). We will give a survey of known necessary and sufficient conditions for the existence of local rules. Among other things, we prove that a necessary condition is that the projection plane must be based on *algebraic irrationality*. We also describe in detail the topology of sets of tilings obtained by the projection method.

For local rules for tilings obtained by substitution methods, see (Danzer, 1991; Radin, 1994; Senechal, 1995). There are several sets of tilings obtained from noncanonical projection methods and admitting local rules, see (Baake et al., 1990, 1991; Danzer et al., 1993; Klitzing et al., 1993; Klitzing and Baake, 1994).

We will distinguish between two types of local rules, one without decoration, and one with decoration. The first type is stronger than the second, and local rules of the first type are much rarer that the second. There are many sets of tilings which admit local rules of the second type but do *not* admit any local rules of the first type.

We will introduce basic definitions about local rules and mutual local derivability in Section 2, together with examples of Penrose and Ammann local rules which illustrate the difference between the two types of local rule.

In Section 3 we will first recall the projection method and its equivalent form, the canonical cut method, following (Oguey et al., 1988). Then we describe in detail the space of these tilings and its topology, and formulate a simple criterion when two sets of tilings are mutually locally derivable.

Important results concerning local rules are surveyed in Sections 4–7.

In Section 4, we review known necessary and sufficient conditions for the existence of local rules (both types) for the sets of pentagonal tilings, or generalized Penrose tilings; including a candidate for local rules for the set of *all* pentagonal tilings. The ideas of the proofs are discussed.

In Section 5 two general sufficient conditions for the existence of local rules *after decoration* are given. This provides infinitely many local rules, enforcing quasi-periodicity, in any dimension greater than 1, and includes all previously known local rules (for canonical projection tilings). These local rules always select tilings in a single *local isomorphism class* (see Section 3.3).

In Section 6 we present Levitov's SI condition as a necessary condition for the existence of a local rule without decoration, and formulate a much stronger necessary condition about algebraicity, which is proved in Section 8.

In Section 7 we give necessary and sufficient conditions for the existence of local rules *without decoration* for the case when the superspace has dimension 4. The results are much fuller than in higher dimensional cases. An infinite series of sets of tilings, previously unknown, admitting local rules *without decoration* is given.

In Section 8 we give a proof of the fact that if the set of tilings obtained by the projection method admits a local rule without decoration, or even a *weak* local rule, then the projection plane must have algebraic slope. The proof is based on Tarski's theory of real algebras. Other related notions, weak local rules and r-volumes, are also discussed.

2. Definitions and preliminary facts

For the purpose of this paper, many definitions have stricter meaning than generally accepted.

A ***decorated polyhedron*** is a pair (P, j) where P is a polyhedron in a Euclidean space, and j is an arbitrary element, called the ***decoration*** of this polyhedron. Two decorated polyhedra are ***congruent*** if their decorations are the same and the second is an image of the first under an isometry of the Euclidean space. Two decorated polyhedra are ***t-congruent*** if their decorations are the same and the second is a translate of the first. We always distinguish between two congruent polyhedra.

A ***tiling*** of $\mathbb{R}^k$ is a family of k-dimensional polyhedra which covers $\mathbb{R}^k$ without overlaps such that up congruence there are only a finite number of polyhedra in this family. A polyhedron of this family is called a ***tile***. A tiling is ***face-to-face*** if the intersection of every two polyhedra is a common facet of lower dimension, if not empty. In this paper tilings are always assumed to be face-to-face unless otherwise stated.

A ***decorated tiling*** is a tiling whose tiles are decorated polyhedra such that up to congruence there are only a finite number of tiles. A nondecorated (or plain) tiling can be regarded as the decorated tiling with exactly one decoration.

Definition 2.1 *An **r-map** is an arbitrary collection of **decorated** polyhedra intersecting a ball of radius r, where $r \geq 0$ is a real number.*

Two r-maps are t-congruent if the second is a translate of the first and the corresponding decorations of polyhedra are the same. We are interested only in r-maps whose polyhedra fit together and cover the r-ball.

Let T be a decorated tiling and v a vertex of T. The *r-map* of T at v is the collection of decorated tiles of T intersecting the ball centered at v and of radius r. An r-map of T means any r-map of T at some vertex.

For example, a 0-map of a tiling T at a vertex v is the collection of tiles incident to this vertex. A 0-map is also called a ***vertex configuration***.

Now we can introduce the main definition:

Definition 2.2 (Levitov, 1988) *An **r-rule** is any finite set $\mathcal{A}$ of decorated r-maps. A* decorated tiling *T satisfies the r-rule $\mathcal{A}$ if every r-map of T is t-congruent to an r-map in $\mathcal{A}$.*

Remark 2.1 For the tilings in this paper, it is more convenient to consider t-congruence instead of the usual congruence.

A ***facet-configuration*** is a collection of two decorated polyhedra of the same dimension sharing a common facet of codimension 1.

Definition 2.3 *A **facet-rule** is a finite set of facet-configurations. A tiling T satisfies a facet-rule $\mathcal{A}$ if every facet-configuration of T is t-congruent to a facet-configuration in $\mathcal{A}$.*

We are interested in r-rules such that every tiling satisfying this r-rule is quasi-periodic in some sense.

Definition 2.4 *A set $\mathcal{T}$ of decorated tilings admits a* **local rule** *if there is an r-rule such that $\mathcal{T}$ is exactly the set of all decorated tilings satisfying this r-rule.*

A set $\mathcal{T}$ of decorated tilings admits a **strict local rule** *if there is a facet-rule such that $\mathcal{T}$ is the set of all tilings satisfying this facet-rule.*

Recall that the case with "plain" tilings corresponds to the case when there is only one kind of decoration.

Certainly if a set admits a strict local rule, then it admits a local rule.

An interesting question in tiling theory is: when does a given set of (decorated) tilings admit a local rule? Usually this set of tilings is constructed by some method, say a substitution method (cf. (Grümbaum and Shephard, 1987)) or a projection method.

We can reformulate the question as follows. Let $\mathcal{T}(r)$ be the set of all tilings every r-map of which is t-congruent to an r-map of a tiling in $\mathcal{T}$. Then $\mathcal{T}$ admits a local rule if and only if there is some r such that $\mathcal{T} = \mathcal{T}(r)$.

Obviously $\mathcal{T}(r') \subset \mathcal{T}(r)$ if $r' > r$, and every $\mathcal{T}(r)$ contains $\mathcal{T}$. Hence

$$\overline{\mathcal{T}} := \bigcap_{r \in \mathbb{R}, r>0} \mathcal{T}(r)$$

contains $\mathcal{T}$. We call $\overline{\mathcal{T}}$ the ***closure*** of $\mathcal{T}$.

If $\mathcal{T}$ admits a local rule, then it is closed, i.e., $\mathcal{T} = \overline{\mathcal{T}}$.

2.1. TWO EXAMPLES

1. THE PENROSE TILINGS: The best known examples of quasiperiodic tilings are the Penrose tilings (cf. Penrose, 1978; de Bruijn, 1981).

(a) *Decorated Penrose tilings.* Let us consider the two decorated rhombi in Figure 1. The acute angles of the rhombi are $\pi/5$ and $2\pi/5$. The sides

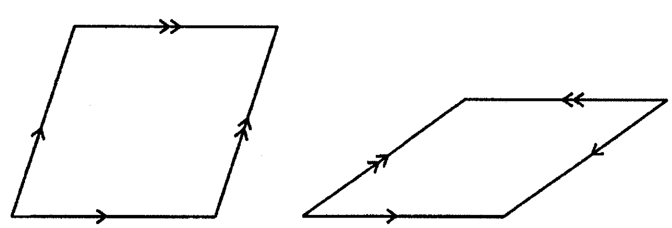

Figure 1. Building blocks of Penrose tilings

have length 1, and are equipped with single or double arrows; this information can be easily converted into the decoration of the rhombi.

Let $\mathcal{A}$ be the facet-rule consisting of all facet-configurations; each is formed by 2 copies such that arrows on the common edge are the same. Here a copy is an image of one of the two decorated rhombi by a translation and a rotation by $m\pi/5, m \in \mathbb{Z}$.

A tiling satisfying this facet-rule is called a ***Penrose tiling***. It is a nontrivial fact that the set of Penrose tilings has uncountably many elements which are pairwise noncongruent, see (Penrose, 1978). Every Penrose tiling is *aperiodic* and *quasiperiodic* in a sense which is explained later. Penrose tilings can be obtained by a substitution method (de Bruijn, 1981).

(b) *Plain Penrose tilings.* A ***plain Penrose tiling*** is a nondecorated tiling obtained from a Penrose tiling by ignoring the decoration (i.e., erasing the arrows). It is known that the set of plain Penrose tilings admits a local rule of radius 2, see (Senechal, 1995). Actually, every tiling, whose 2-maps are t-congruent to those of plain Penrose tilings, can be decorated by arrows so that the resulting decorated tilings satisfying the facet-rule described above.

Plain Penrose tilings can be obtained by the canonical projection method (de Bruijn, see below). Every plain Penrose tiling can be decorated in exactly one way to become a Penrose tiling.

2. Ammann Octagonal Tilings: These are analogs of Penrose tilings.

(a) *Decorated Ammann tilings.* Consider the decorated rhombus and the decorated square in Figure 2; the acute angle of the rhombus is $\pi/4$, and sides of the rhombus and the square have length 1.

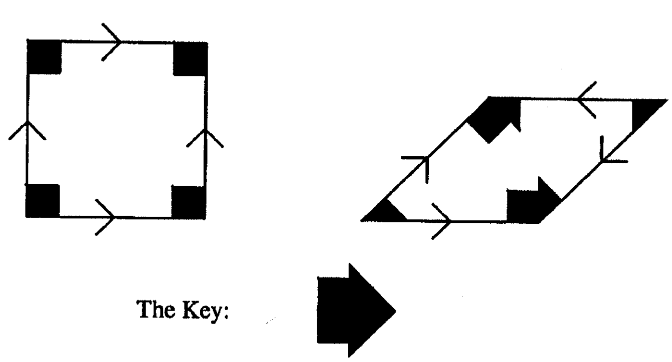

Figure 2. Building blocks of the Ammann tilings and the key

Consider the local rule consisting of 0-maps (vertex configurations) which can be formed by copies of the decorated rhombi and squares with the following constraints: matching at edges and matching at vertices. Here

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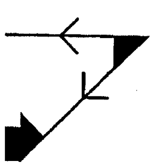

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a copy is an image of the decorated rhombus or square under a combination of translations, mirror reflection with respect to a horizontal line, and rotations by $\pi/4$; matching at an edge means the arrows are the same; matching at a vertex means the marking at the vertex must form the key.

An ***Amman tiling*** is any tiling satisfying this 0-rule.

There are uncountably many pairwise noncongruent Ammann tilings, see (Ammann et al., 1992). All of them are aperiodic and quasi-periodic.

(b) *Plain Ammann tilings*. A ***plain Ammann tiling*** is a nondecorated tiling obtained from an Ammann tiling by ignoring the decoration. These tilings were studied in (Beenker, 1982; Burkov, 1988; Le, 1993; Socolar, 1989).

It is known that the set of all plain Ammann tilings does *not* admit any local rule, see (Burkov, 1988). Plain Ammann tilings can be obtained by the projection method (see below).

An extremely interesting fact is that there are two different Ammann tilings which are the same if the decorations are ignored; in fact, there are infinitely many such pairs. Hence some plain Ammann tilings can be decorated in many different ways (in fact 1, 2, 4 or 8 different ways, see (Le, 1993)); and the set of plain Ammann tilings which can be decorated in more than 1 way is of measure 0.

This is very different from the Penrose case: every plain Penrose tiling can be decorated in exactly one way.

2.2. LOCAL RULES AFTER DECORATING

Many tilings constructed by geometrical methods are not decorated, and since a "decorated local" rule in some sense is more powerful than a nondecorated one, we introduce the following:

Definition 2.5 *Suppose $\mathcal{T}$ is a set of* nondecorated *tilings. We say that $\mathcal{T}$ admits a* ***local rule after decoration*** *if there is a set $\mathcal{T}^c$ of decorated tilings which admits a local rule such that when ignoring the decorations, the two sets $\mathcal{T}$ and $\mathcal{T}^c$ are coincident.*

Note that there may be two different tilings in $\mathcal{T}^c$ which are the same if we ignore the decorations.

For example, as noted above, the set of plain Ammann tilings admits a local rule after decoration; but it does not admit any local rule!

Some authors do not distinguish between local rules and local rules after decoration, and the existence of a local rule *after decoration* is sometimes considered as the existence of local rules.

The existence of a local rule for a set of plain tilings is a much rarer phenomenon than the existence of a local rule after decoration.

2.3. TOPOLOGICAL EQUIVALENCE, MUTUAL LOCAL DERIVABILITY

Let $\mathcal{T}$ be a set of *decorated* tilings of $\mathbb{R}^k$, with a fixed origin. Suppose $\mathcal{T}$ is invariant under translations. Consider the following topology on $\mathcal{T}$.

For a positive number r and a tiling T in $\mathcal{T}$, let $V_{(r,T)}$ be the set of tilings $T' \in \mathcal{T}$ such that there exists a vector α whose length is less than $1/r$ with the property: two tilings T and $T' + \alpha$ are the same inside the ball of radius r centered at the origin.

Considering $V_{(r,T)}$, for $r > 0$, as a basis of neighborhoods of T, we get a topology on $\mathcal{T}$. In fact, this is the topology considered in (Radin, 1991; Robinson, 1992), adapted to the case of decorated tilings.

The group $\mathbb{R}^k$ acts on $\mathcal{T}$ by translations, and $(\mathcal{T}, \mathbb{R}^k)$ is considered as a dynamical system. Two sets of decorated tilings $\mathcal{T}, \mathcal{T}'$ of $\mathbb{R}^k$ are ***topologically conjugate*** if there is a homeomorphism between them which respects the action of $\mathbb{R}^k$.

Suppose that there are invariant measures on $\mathcal{T}$ and $\mathcal{T}'$. We say that $\mathcal{T}$ and $\mathcal{T}'$ are ***metrically conjugate*** if there is a homeomorphism between a set $\mathcal{U} \subset \mathcal{T}$ and a set $\mathcal{U}' \subset \mathcal{T}'$ which respects the action of $\mathbb{R}^k$, where the complements $\mathcal{T} \setminus \mathcal{U}$ and $\mathcal{T}' \setminus \mathcal{U}'$ have measure 0.

Another equivalence relation between sets of tilings is the following. Suppose that there is a fixed $r \geq 0$ and a *t-congruence-equivariant mapping*

$$f : \{\ r\text{-maps of tilings in } \mathcal{T}\} \to \{\text{tiles of tilings in } \mathcal{T}'\}$$

such that if T is a tiling in $\mathcal{T}$, then the images of all r-maps of T fit together to form a tiling in $\mathcal{T}'$. In that case we say that $\mathcal{T}'$ is locally derivable from $\mathcal{T}$, (Baake et al., 1991).

If two sets of tilings are locally derivable from each other, we say that they are ***mutually locally derivable***, or they belong to the same ***mutual local derivability class*** (or MLD class, in short).

Example 2.1 The set of Penrose tilings and the set of plain Penrose tilings are locally derivable. The set of Ammann tilings and the set of plain Ammann tilings are not mutually locally derivable.

Theorem 2.6 *Consider the following statements:*

(a) *$\mathcal{T}$ and $\mathcal{T}'$ are mutually locally derivable.*

(b) *$\mathcal{T}$ and $\mathcal{T}'$ are topologically conjugate.*

(c) *$\mathcal{T}$ admits a local rule if and only if $\mathcal{T}'$ admits a local rule.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

So, admitting a local rule is a property of the whole MLD class.

The implication (a) $\Rightarrow$ (c) is easy, and has already been shown, see (Baake et al., 1991). The part (a) $\Rightarrow$ (b) is also easy, but the part (b) $\Rightarrow$ (c) is more difficult. For a proof, see (Le, 1995c). We conjecture that (a) is equivalent to (b).

3. The strip projection method and the cut method

There are many criticisms of these methods, mainly because the "superspace" does not have an adequate physical interpretation. But still, tilings obtained by these methods form an important class of quasiperiodic tilings.

While there is no widely accepted definition of quasi-periodicity, all tilings obtained by these methods are quasiperiodic by all known definitions. The well-known Penrose and Ammann tilings are in this class of tilings.

The question about existence of local rules for these tilings has not been answered fully, and is far from trivial.

We recall here briefly the strip projection method and the cut method. The cut method has been known for a long time, see (Kramer and Neri, 1984). Its main idea is to obtain quasiperiodic structures by using a section of a periodic structure in higher dimensional space. A mathematical foundation is presented in (Oguey et al., 1988), and we follow this paper. A general method for generating tilings as section of periodic structure in higher dimensional space is given in (Kramer and Schlottmann, 1989). What we are discussing in the next subsections is known as the canonical projection (or cut) method.

3.1. THE STRIP PROJECTION METHOD

In the Euclidean space $\mathbb{R}^n$ with origin $\mathbf{0}$ we fix a standard basis $e_1, ..., e_n$. Let $\mathbb{Z}^n$ be the integer lattice. For vectors $\alpha_1, ..., \alpha_m$ in $\mathbb{R}^n$, the set

$$\operatorname{zon}(\alpha_1, ..., \alpha_m) = \left\{ \sum_{i=1}^{i=m} \lambda_i \alpha_i \mid \lambda_i \in [0,1] \right\}$$

is called the ***zonotope*** generated by $\alpha_1, \ldots, \alpha_m$. The set $\gamma = \operatorname{zon}(e_1, ..., e_n)$ is the *unit cube*.

For a multi-index $I = (i_1, ..., i_q)$ with $1 \le i_1 < i_2 \cdots < i_q \le n$, the set $\operatorname{zon}(e_{i_1}, \ldots, e_{i_q})$ and its translates by vectors from $\mathbb{Z}^n$ are called ***q-facets*** of the lattice $\mathbb{Z}^n$. A 0-facet of $\mathbb{Z}^n$, by definition, is any point in $\mathbb{Z}^n$.

Suppose E is a k-dimensional subspace in $\mathbb{R}^n$. Let $E^\perp$ be its orthogonal complement. Then $\mathbb{R}^n = E \oplus E^\perp$. Denote by p and $p^\perp$, respectively, the corresponding projections onto E and $E^\perp$. Let

$$e_i^\parallel = p(\varepsilon_i) \quad \text{and} \quad e_i^\perp = p^\perp(\varepsilon_i).$$

We assume that E satisfies:

$$\text{Every set of } k \text{ vectors from } e_i^{\parallel},\ i = 1, \ldots, n \text{ is linearly independent.} \tag{3.1}$$

E is considered as a point in the Grassmannian $G_{k,n}$ which is a smooth algebraic subvariety of a projective space. It is easy to see that the set of all k-dimensional subspaces not satisfying (3.1) is a closed subvariety of $G_{k,n}$. Hence the set of all k-dimensional subspaces satisfying (3.1) is open and dense (even in the Zariski topology) in $G_{k,n}$.

Condition (3.1) is not essential, but without it one needs to make modifications to many statements and proofs bellow. For a way to deal with it, see the remark in Section 8.7. The condition is equivalent to the following: every $n-k$ vectors from $e_i^{\perp}, i = 1, \ldots, n$ are linearly independent.

For every α in $\mathbb{R}^n$ consider the tube $\gamma + E + \alpha$ obtained by shifting the unit cube γ along an affine plane parallel to E. A point α is ***E*-regular** if the boundary of the tube $\gamma + E + \alpha$ does not contain any integer point.

Theorem 3.1 (cf. Oguey et al., 1988) *Suppose α is E-regular. Then the union of all k-facets of $\mathbb{Z}^n$ lying inside the tube $\gamma+E+\alpha$ is a continuous k-dimensional surface. This surface projects (along $E^{\perp}$) homeomorphically onto E and contains all q-facets of $\mathbb{Z}^n$ falling inside the tube $\gamma+E+\alpha$, for $0 \le q \le k$.*

The surface in the theorem has an obvious polyhedral structure. By projecting this polyhedral structure along $E^{\perp}$ onto E we get a tiling T_α of E. The tiles of T_α are the projections of k-dimensional facets of the lattice $\mathbb{Z}^n$; and there are $\binom{n}{k}$ of them, up to translations.

It's enough to consider shift vectors α in $E^{\perp}$. A point $\alpha \in E^{\perp}$ is ***irregular*** if it is not regular. Denote Ir the set of all irregular points. This set is of measure 0 and plays a fundamental role (see Section 3.5).

Let $\mathcal{T}_E$ be the set of all tilings T_α, with regular α, and their translates. One would like to know whether $\mathcal{T}_E$ admits a local rule. Unfortunately $\mathcal{T}_E$ is *never* closed unless E is rational (i.e., spanned by vectors with rational coordinates), in which case the tilings are periodic. So the question should be formulated as follows: when does the closure $\overline{\mathcal{T}_E}$ admit a local rule?

Tilings in $\overline{\mathcal{T}_E}$ are called ***quasiperiodic tilings*** associated with E.

We will say that E admits a local rule, if $\overline{\mathcal{T}_E}$ admits a local rule.

3.2. THE CANONICAL CUT METHOD

Let's consider another construction for the same tilings, known as the cut method.

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Again E is a k-dimensional subspace of $\mathbb{R}^n$. A set X in $\mathbb{R}^n$ is called an ***E-prism*** (or simply prism) if

$$X = p(X) + p^\perp(X).$$

If X is a prism then the intersection of X with a k-plane $E+\alpha$ for $\alpha \in \mathbb{R}^n$, if not empty, is always t-congruent to the base $p(X)$.

We first construct a periodic tiling of $\mathbb{R}^n$ consisting of prisms. There is a standard way to construct such tiling.

For each $I = (i_1, \ldots, i_k)$ with $1 \le i_1 < \cdots < i_k \le n$, let

$$C_I = P_I + P_I^\perp,$$

where P_I is the zonotope generated by

$$p(\varepsilon_{i_1}), \ldots, p(\varepsilon_{i_k}),$$

$P_I^\perp$ is the zonotope generated by

$$-p^\perp(\varepsilon_{j_1}), \ldots, -p^\perp(\varepsilon_{j_{n-k}}).$$

Here

$$\{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\}.$$

(The minus signs are very important.)

Proposition 3.2 (cf. Oguey et al.,1988) *The family*

$$\mathcal{O} = \{C_I + \xi, \xi \in \mathbb{Z}^n\}$$

is a tiling of $\mathbb{R}^n$ which is not face-to-face.

Two important properties of $\mathcal{O}$ are

(a) it is periodic: $\mathcal{O}$ is invariant under action of $\mathbb{Z}^n$, and

(b) its tiles are E-prisms.

For a prism X we define

$$\partial^{\|}(X) = p(X) + \partial(p^\perp(X)),$$
$$\partial^\perp(X) = \partial(p(X)) + p^\perp(X,$$

where ∂Y is the boundary of the set Y in E or in $E^\perp$. The sets $\partial^{\|}$(X) and $\partial^\perp$(X) are called respectively the *parallel* and the *perpendicular* boundaries of prism X.

Denote by $B^{\|}$ the union of parallel boundaries of all prisms in $\mathcal{O}$. If α is a point of $\mathbb{R}^n$ such that $E+\alpha$ does not meet $B^{\|}$ then the intersections of the k-plane $E+\alpha$ with all members of $\mathcal{O}$ form a tiling of $E+\alpha$ and hence a tiling of E by projecting onto E. The reason why we have to choose regular α is that when α is not regular, all the intersections of $\alpha+E$ with tiles of $\mathcal{O}$ cover $E+\alpha$ with overlaps. The equivalence between the cut method and the projection method is now stated as follows.

Theorem 3.3 (Oguey et al., 1988) *The point $\alpha \in \mathbb{R}^n$ is regular if and only if $E+\alpha$ does not meet the parallel boundary $B^{\parallel}$. If α is regular then the intersections of $E+\alpha$ with prisms in $\mathcal{O}$ define a tiling on $E+\alpha$ and by projecting onto E we get exactly the tiling T_α obtained by the projection method.*

It follows that the set Ir of irregular points in $E^\perp$ is $p^\perp\left(B^{\parallel}\right)$.

3.3. SUBSETS OF $\overline{\mathcal{T}_E}$, SINGLE LI CLASSES

Definition 3.4 *Two tilings belongs to the same local isomorphism class (or* LI *class) if every r-map of the first is t-congruent to an r-map of the second and vice-versa, for every $r > 0$.*

In general, the tilings in $\overline{\mathcal{T}_E}$ do not all belong to the same LI class.

Let $Z(E)$ be the smallest rational subspace (i.e., a subspace that can be spanned by vectors with rational coordinates) containing E. Then

$$\Delta(E) = Z(E)^\perp \equiv \mathbb{R}^n/Z(E)$$

is a rational subspace. Note that $\Delta(E)$ is the maximal rational subspace of $E^\perp$.

Proposition 3.5 *If α,β are regular and $\alpha-\beta$ belongs to $Z(E)$ then T_α and T_β belong to the same* LI *class.*

This is a convenient reformulation of Proposition 1 in (Levitov, 1988)).

For $t \in \Delta(E)$ denote by $\mathcal{T}_{E;t}$ the set of all tilings T_α with regular α in $t+Z(E)$. By this Proposition, every two tilings in $\mathcal{T}_{E;t}$ belong to the same LI class. The closure $\overline{\mathcal{T}_{E;t}}$ is a single LI class, i.e., every two tilings in it belong to the same LI class, and if T is a tiling belonging to the same LI class as a tiling in $\overline{\mathcal{T}_{E;t}}$, then T is in $\overline{\mathcal{T}_{E;t}}$.

Although $\mathcal{T}_E = \cup\mathcal{T}_{E;t}$, in general, $\overline{\mathcal{T}_E}$ is bigger than $\cup\overline{\mathcal{T}_{E;t}}$. When $E^\perp \cap \mathbb{Z}^n = 0$, $\overline{\mathcal{T}_E}$ is a single LI class.

3.4. EQUIVALENCE

The main question is, when does $\overline{\mathcal{T}_E}$, or $\overline{\mathcal{T}_{E;t}}$, admit local rules (after decoration)? Since the property of admitting a local rule is a property of a mutual local derivability class, first we would like to know when two sets $\overline{\mathcal{T}_E}$ and $\overline{\mathcal{T}_F}$, corresponding to two different k-dimensional subspaces, are mutually locally derivable.

Theorem 3.6 (1) *Let E and F be two k-dimensional subspaces. The following are equivalent.*

(*a*) *Two sets of tilings $\overline{\mathcal{T}_E}$ and $\overline{\mathcal{T}_F}$ are mutually locally derivable.*

(b) $\overline{\mathcal{T}_E}$ *and* $\overline{\mathcal{T}_F}$ *are topologically conjugate.*

(c) There is a nondegenerate linear mapping from $E^\perp$ *to* $F^\perp$ *transforming* Ir_E *to* Ir_F*.*

(2) *Suppose that* $t, t' \in \Delta(E)$*. Then two sets* $\overline{\mathcal{T}_{E;t}}$ *and* $\overline{\mathcal{T}_{E;t'}}$ *are mutually locally derivable if and only if there is a nondegenerate linear mapping from* $E^\perp \cap (t + Z(E))$ *to* $E^\perp \cap (t' + Z(E))$ *which transforms* $Ir \cap (t + Z(E))$ *to* $Ir \cap (t' + Z(E))$. (Compare Baake et al., 1991.)

That (a) is equivalent to (b) can be proved using the topology of $\overline{\mathcal{T}_E}$ described in the next section. The other statements are more difficult.

3.5. IRREGULAR POINTS, TILINGS IN $\overline{\mathcal{T}_E} - \mathcal{T}_E$

As noted above, the set Ir of irregular points in $E^\perp$ is $p^\perp(B)$. For

$$J = (j_1, \dots, j_{n-k-1}), \qquad 1 \le j_1 < j_2 < \dots < j_{n-k-1} \le n,$$

let h_J be the $(n-k-1)$-plane spanned by

$$e^\perp_{j_1}, \dots, e^\perp_{j_{n-k-1}}.$$

It is of codimension 1 in $E^\perp$. Then Ir is exactly the union of $\binom{n}{n-k-1}$ families of parallel $(n-k-1)$-planes, each of the form

$$h_J + p^\perp(\mathbb{Z}^n),$$

see (Oguey et al., 1988). Each family is dense in $E^\perp$ but the union of its members has measure 0.

We now describe the tilings in $\overline{\mathcal{T}_E}$ which are not in $\mathcal{T}_E$.

If $\alpha \in E^\perp$ is irregular, then there are several hyperplanes in Ir passing through α; they divide $E^\perp$ into many parts. By a *corner* at α we mean one of these parts, without points on the boundary (i.e., each corner is an open subset of $E^\perp$). For a regular α, the whole $E^\perp$ will be considered as the only corner at α.

For a pair (α, x) where $\alpha \in E^\perp$ and x is a corner at α, let us consider the following tiling $T_{(\alpha,x)}$.

Choose a sequence of *regular* points $\alpha_1, \alpha_2, \dots$ which lie inside x and converge to α. Since the set of regular points is a dense set one can always choose such a sequence. Using the cut method one can see that for every $r > 0$, there is a number N such that all tilings T_{α_i}, with $i > N$, are coincident inside the ball of radius r centered at 0 (see (Le et al., 1993)). Hence the sequence of tilings T_{α_i} defines a limit tiling, denoted $T_{\alpha,x}$. This tiling depends on α and the corner x at α, but not on the particular sequence α_i. If α is regular, then $x = E^\perp$ and $T_{(\alpha,x)} = T_\alpha$.

Theorem 3.7 *Every tiling in $\overline{\mathcal{T}_E}$ is a translate of a tiling of the form $T_{(\alpha,x)}$.*

This follows easily from the analysis of limits of tilings in $\mathcal{T}_E$; see (Le, 1995c) for details. If x, y are two different corners at α, then $T_{(\alpha,x)}$ and $T_{(\alpha,y)}$ are different.

Thus, when α is regular, it defines exactly one quasiperiodic tiling, but when α is irregular, it defines several, finitely many, quasiperiodic tilings. Irregular points corresponding to the irregular grids in (de Bruijn, 1981)

Example 3.1 Suppose only one hyperplane in Ir passes through α. Then there are two corners, each is a half-space, denoted by x_1 and x_2. Two tilings $T_{(\alpha,x_1)}$ and $T_{(\alpha,x_2)}$ are identical everywhere, except for a d-neighborhood of a hyperplane in E, where d is a positive number. Hence they are identical in a half-space. In the Penrose tilings case, the difference between the two tilings is exactly the Conway worm.

3.6. TOPOLOGY OF $\overline{\mathcal{T}_E}$

Let $\mathcal{X}$ be the set of all pairs (α, x) where $\alpha \in E^\perp$ and x is a corner at α. There is a natural mapping $f : \mathcal{X} \to E^\perp$ which sends (α, x) to α. This mapping is finitely-many-to-one, and is one-to-one for all regular α.

For a point (α, x) of $\mathcal{X}$ let

$$V_{(\alpha,x)} = \{(\beta, y) \in \mathcal{X} \quad | \quad \beta \in \bar{x} \quad \text{and} \quad x \cap y \neq \emptyset\}.$$

Here $\bar{x}$ is the closure of x in the usual topology of $E^\perp$.

We endow $\mathcal{X}$ with the topology in which all the sets $V_{(\alpha,x)}$ form a basis. Consider the product $\mathcal{X} \times E$ with the product topology. The above mapping f can be extended to a mapping, also denoted by f, from $\mathcal{X} \times E$ to $E^\perp \times E = \mathbb{R}^n$. The mapping is one-to-one almost everywhere.

The lattice $\mathbb{Z}^n$ acts on $\mathbb{Z}^n$ by translation, and it is easy to see that this action can be lifted to an action on $\mathcal{X} \times E$. The action is free.

Theorem 3.8 *The factor space $(\mathcal{X} \times E)/\mathbb{Z}^n$ is homeomorphic to $\overline{\mathcal{T}_E}$.*

The homeomorphism is defined by: a pair $((\alpha, x), v) \in \mathcal{X} \times E$ is mapped to the tiling $T_{\alpha,x} + v$. This theorem follows easily from the above description of tilings in $\overline{\mathcal{T}_E}$.

By adding a vector from E (which is isomorphic to $\mathbb{R}^k$) to vectors in $\mathbb{R}^n$, we get an action of $\mathbb{R}^k$ on $\mathbb{R}^n$. This action can also be lifted to an action on $\mathcal{X} \times E$. Both actions then induce actions on $(\mathcal{X} \times E)/\mathbb{Z}^n$ and $\mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$, the n-dimensional torus.

The above mapping f induce to a mapping $g : \overline{\mathcal{T}_E} \to \mathbb{T}^n$. The dynamical system $(\overline{\mathcal{T}_E}, \mathbb{R}^k)$ is exactly $((\mathcal{X} \times E)/\mathbb{Z}^n, \mathbb{R}^k)$. The mapping g, though not a

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4. Pentagonal T

4.1. SETTINGS

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homeomorphism, is continuous, respects the action of $\mathbb{R}^k$ and is one-to-one outside a subset of measure 0 (or, $(\overline{\mathcal{T}_E}, \mathbb{R}^k)$ and $(\mathcal{T}^n, \mathbb{R}^k)$ are metrically conjugate). Hence if we consider the dynamical system $(\overline{\mathcal{T}_E}, \mathbb{R}^k)$ up to metrical conjugacy, then it is rather trivial.

Remark 3.1 In case $\Delta(E) \neq 0$, then for each $t \in \Delta(E)$ we have an LI class $\overline{\mathcal{T}_{E;t}}$. Note that for each $t \in \Delta(E)$, the set of irregular points in $t + p^{\perp}(Z(E))$ is the union of a finite families of parallel hyperplanes: the set of irregular points is the intersection of Ir and $t + p^{\perp}(Z(E))$. In a similar way as in Sections 3.5 and 3.1 one can describe the set $\overline{\mathcal{T}_{E;t}}$ together with its topology using this collection of hyperplanes.

4. Pentagonal Tilings

4.1. SETTINGS

In this section we consider a class of tilings known as the generalized Penrose tilings. Here $\dim E = 2$ and $\dim \mathbb{R}^n = 5$. Following is a way to fix the 2-dimensional subspace E.

Let the cyclic group

$$C_5 = \langle g \mid g^5 = 1 \rangle$$

act on $\mathbb{R}^5$ by circular permutation of the basis: $g(e_i) = (e_{i+1})$ (indices are taken modulo 5).

It is easy to see that $\mathbb{R}^5$ decomposes into three invariant subspaces E,$\bar{E}$, and Δ. Here Δ is the 1-dimensional subspace spanned by

$$\delta = (e_0 + e_1 + e_2 + e_3 + e_4)/5.$$

Note that Δ is rational; and on Δ the group $\mathbb{Z}_5$ acts trivially.

E is a 2-dimensional subspace spanned by two vectors

$$v_1 = (4, \sqrt{5}-1, -\sqrt{5}-1, -\sqrt{5}-1, \sqrt{5}-1),$$
$$v_2 = (\sqrt{5}-1, 4, \sqrt{5}-1, -\sqrt{5}-1, -\sqrt{5}-1).$$

$\bar{E}$ is spanned by the two vectors $\bar{v}_1$ and $\bar{v}_2$ (conjugate, replacing $\sqrt{5}$ by $-\sqrt{5}$).

On E g acts as rotation by $2\pi/5$ and on $\bar{E}$ g acts as rotation by $4\pi/5$.

We have

$$E^{\perp} = \bar{E} \oplus \Delta \quad \text{and} \quad \mathbb{R}^5 = E \oplus \bar{E} \oplus \Delta = E \oplus E^{\perp}.$$

Note that $Z(E) = E + \bar{E}$, and hence $Z(E)^{\perp} = \Delta$.

A tiling of $\overline{\mathcal{T}_E}$ is called a ***pentagonal quasiperiodic tiling***, because every such tiling and its rotation by $2\pi/5$ belong to the same LI class. These tilings are also known as ***generalized Penrose tilings***. Up to rotations and translations there are two tiles as in plain Penrose tilings.

For a real number $t \in \mathbb{R}$ let $\bar{E}_t = \bar{E} + t\delta$, and $\mathcal{T}_t = \{T_\alpha, \alpha \in \bar{E}_t\}$.

From the results of Section 3.3 we know that $\overline{\mathcal{T}}_t$ is a single LI class.

Theorem 4.1 (de Bruijn, 1981) *The set of plain Penrose tilings is* $\overline{\mathcal{T}}_0$.

The sets $\overline{\mathcal{T}}_t$ have many interesting properties, see (Kleman and Pavlovitch, 1987). Note that if $t - t' = 1$ then $\overline{\mathcal{T}}_t = \overline{\mathcal{T}}_{t'}$. One interesting property is that the vertex-configuration densities depend continuously on t. It is not difficult to show that all the sets $\overline{\mathcal{T}}_t$ are metrically conjugate.

4.2. LOCAL RULES WITHOUT DECORATION

Now the question is for what $t \in \mathbb{R}$ does the set $\overline{\mathcal{T}}_t$ admit local rules? For local rules without decoration the answer is complete.

Theorem 4.2 (Ingersent and Steinhardt, 1991) *If $\overline{\mathcal{T}}_t$ admits a local rule then $t = p + q\tau$, where $p, q \in \mathbb{Z}$, and τ is the golden ratio,*

$$\tau = (1 + \sqrt{5})/2.$$

The converse is true:

Theorem 4.3 *If $t = p + q\tau$ then $\overline{\mathcal{T}}_t$ admits a local rule. Moreover, the radius of the local rule can be chosen $< q(5 + \sqrt{5})/2 + 3 < 4q + 3$.*

One proof of Theorem 4.3 is the following. First we observe that

Theorem 4.4 *Two sets $\overline{\mathcal{T}}_t$ and $\overline{\mathcal{T}}_{t'}$ are mutually locally derivable if and only if $t - t'$ is of the form $m + n\tau$, with integers m, n.*

Proof This follows from Theorem 3.6, since when $t - t' = m + n\tau$, it can be checked easily (using the description of sets of irregular points in Section 3.5) that the two corresponding sets of irregular points are t-congruent. □

We know that to admit a local rule is a property of the whole MLD class. Since the case $t = 0$ admits a local rule, all the cases $t = m + n\tau$ admit local rules. This prove the first part of Theorem 4.3. The radius of the local rule can be obtained by a closer examination of the mutual local derivability corresponding.

Actually, in (Ingersent, 1991) it was noticed that when $t = n\tau$ then the sets $\overline{\mathcal{T}}_t$ and $\overline{\mathcal{T}}_0$ are "equivalent", which, in our terminology, means they are in the same MLD class. So this is a special case of Theorem 4.4.

Note that this proof of Theorem 4.3 is based on results of de Bruijn saying that the case $t = 0$ admits a local rule. Another proof, which does not need de Bruijn's result, is sketched in the next subsection. In fact, while proving Theorem 4.3 using the second method, we recover the Penrose local rule.

For a concrete mapping realizing the mutual local derivability between $t = 0$ and $t = \tau$ cases, see (Le, 1995b).

Since the case $t = 0$ has inflation/deflation properties, see (de Bruijn, 1981; Grümbaum and Shephard, 1987; Penrose, 1978) it follows that the set $\overline{T_t}$ of pentagonal tilings with $t = p + q\tau$ also has inflation/deflation properties.

4.3. LOCAL RULES AFTER DECORATION

We have the following result (see (Le, 1995a)).

Theorem 4.5 *If $t = (p + q\sqrt{5})/m$, where p, q, m are integers (i.e., $t \in \mathbb{Q}[\sqrt{5}]$), then $\overline{T_t}$ admits a local rule after decoration. Moreover, the local rule after decoration can be chosen to be a facet-rule (strict local rule).*

The proof of this theorem is a construction of local rules; and in the case $t = 0$ we get exactly the Penrose rule! The proof does not involve inflation-deflation theory, and can be generalized to other cases of higher dimensions (see next subsection).

In (Le, 1995a) we first proved Theorem 4.5, and then deduced from it Theorem 4.3 by showing that when $t = p + q\tau$ there is a number r such that the decoration of a tile (in a decorated tiling satisfying the decorated local rule) is uniquely determined by the plain r-map of a vertex of the tile.

If $t \in \mathbb{Q}[\sqrt{5}]$ but is not of the form $p + q\tau$, then $\overline{T_t}$ does not admit any local rule, but DOES admit a local rule after decoration. And there are infinitely many tilings in $\overline{T_t}$ which can be decorated in more than one way.

Example 4.1 Consider the case $t = 1/2$. This case is of special interest since tilings in $\overline{T_t}$ have 10-fold symmetry, as in the $t = 0$ case. By the above results, $\overline{T_t}$ does not admit any local rules, but admits a local rule after decoration. In (Le, 1995a) we found the following local rule (with decoration). We have 2 decorated rhombi in Figure 3 and their rotations by $m\pi/5$; and a 0-map in the local rule is any 0-map formed by translates of these rhombi with matching at edges and matching at vertices. Matching at an edge means the arrows of a common edge must be the same; matching at a vertex means only marked vertices can meet, and the marking form the key of Figure 3.

Every decorated tiling satisfying this local rule is, ignoring the decoration, a tiling in $\overline{T_t}$, and every tiling in $\overline{T_t}$ can be decorated (in 1, 2, or 4 ways) to become a tiling satisfying this local rule. The set of tilings in $\overline{T_t}$ which can be decorated in more than 1 way has measure 0.

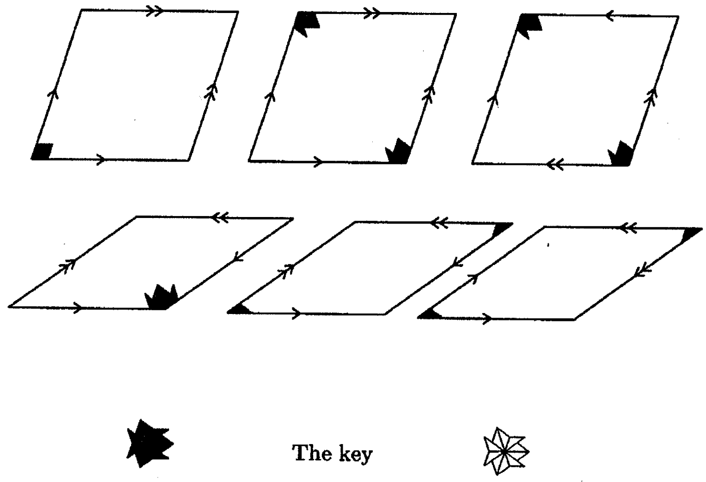

Figure 3. A local rule for pentagonal tilings with $t = 1/2$

4.4. LOCAL RULES FOR ALL PENTAGONAL TILINGS

The set $\overline{\mathcal{T}_E}$ contains tilings of different local isomorphism classes. It is conjectured that this set admits a local rule. Moreover, it is conjectured that the Kleman–Pavlovitch rule in (Kleman and Pavlovitch, 1987) is a local rule for this set. We have a weaker result.

Let us consider the rule described in Figure 4. We have 6 decorated rhombi, together with their rotations by $m\pi/5$. The rule is a 0-rule whose 0-maps are those formed by these decorated rhombi with matching at edges and matching at vertices. Here matching at an edge means the arrows of a common edge must be the same; matching at a vertex means only marked vertices can meet.

Note that this is a strengthened version of the Kleman–Pavlovitch rule: if we erase the marking at vertices, then we get the Kleman–Pavlovitch rule. Denote by $\mathcal{T}^*$ the set of all tilings satisfying this local rule.

Theorem 4.6 (a) *Every pentagonal tiling can be decorated in a unique way to become a tiling in* $\mathcal{T}^*$.

(b) *The set* $\mathcal{T}^*$ *and the set* $\overline{\mathcal{T}_E}$ *of all pentagonal tilings are metrically conjugate, i.e., every tiling in* $\mathcal{T}^*$, *except for a subset of* $\mathcal{T}^*$ *of measure 0, is a pentagonal tiling, ignoring the decoration.*

(c) *If* T *is in* $\mathcal{T}^*$, *but not in* $\overline{\mathcal{T}_E}$, *then there is a tiling* T' *in* $\overline{\mathcal{T}_E}$ *such that* $T = T'$ *except for* 2*-neighborhoods of 5 straight lines.*

For a proof see (Le, 1995c).

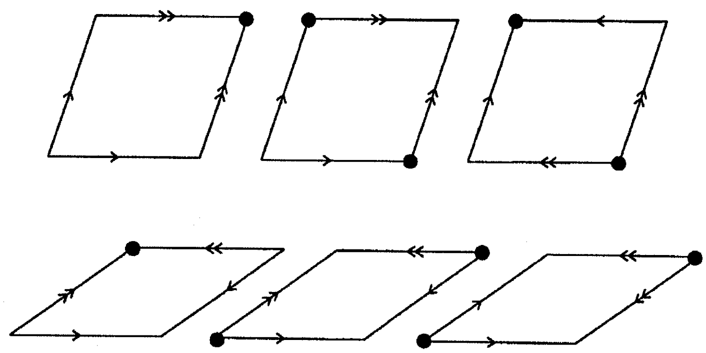

Figure 4. The 0-rules are defined by matching at vertex and at edge

We cannot exclude the case in (c), neither can we find any example of a tiling in $\mathcal{T}^*$ but not in $\overline{\mathcal{T}_E}$.

We conjecture that this rule is really a local rule (after decoration) for the set of all pentagonal tilings. Since the decoration is local, i.e., can be determined by inspection inside a disk of finite radius, it follows that if this is a local rule after decoration, then local rules without decoration exist.

5. Sufficient conditions

Here, we give some sufficient conditions for the existence of local rules *after decoration*. First, we introduce some definitions for subspaces of $\mathbb{R}^n$.

A k dimensional subspace E of $\mathbb{R}^n$, equipped with a basis

(a) is ***totally irrational*** if it does not contain any rational point except 0.

(b) has ***quadratic slope*** if there is a positive integer D such that E is spanned by k vectors with coordinates in $\mathbb{Z}[\sqrt{D}]$.

First we consider the case $n = 2k$. A sufficient condition for the existence of local rules *after decoration* is the following:

Theorem 5.1 (see Le et al., 1992, 1993; Le and Piunikhin, 1995) *If $n = 2k > 3$ and E is totally irrational and has* quadratic slope *then $\overline{\mathcal{T}_E}$ admits a LR* after decoration. *This means there is a set $\overline{\mathcal{T}_E}^c$ of decorated tilings admitting a local rule such that when the decorations are ignored we have $\overline{\mathcal{T}_E} = \overline{\mathcal{T}_E}^c$. In addition, all the decorated tilings in $\overline{\mathcal{T}_E}^c$ are quasiperiodic in the sense that they are sections of a* decorated *periodic tiling of $\mathbb{R}^n$.*

Note that when E is totally irrational and has quadratic slope, $E^\perp$ is also totally irrational. Hence $\overline{\mathcal{T}_E}$ is a single LI class.

The Ammann octagonal tilings ($n = 4 = 2k$) and the 3-dimensional icosahedral tilings ($n = 6 = 2k$) have quadratic slope, hence, by this theorem, they admit local rules after decoration.

Theorem 5.1 is valid only in case $n = 2k$. When $n > 2k$, as in the case of pentagonal quasiperiodic tilings, there is a similar result.

Suppose that $n \geq 2k > 3$, and that E is totally irrational and has quadratic slope. The space E is spanned by k vectors $v_1, \dots, v_k$ with coordinates in $\mathbb{Q}[\sqrt{D}]$. Let $\bar{E}$ be the space spanned by $\bar{v}_1, \dots, \bar{v}_k$ (conjugate, replacing $\sqrt{D}$ by $-\sqrt{D}$).

Proposition 5.2 *The subspace $E + \bar{E}$ is of dimension $2k$, and*

$$Z(E) = E \oplus \bar{E}.$$

The proof is easy. Each $t \in \Delta(E) = (E \oplus \bar{E})^{\perp}$ defines a single LI class $\overline{\mathcal{T}_{E;t}}$ (see Section 3.3). The dimension of $\Delta(E)$ is $n - 2k$.

Theorem 5.3 *If $t \in \Delta(E)$ has coordinates in $\mathbb{Q}[\sqrt{D}]$ then $\overline{\mathcal{T}_{E;t}}$ admits a local rule after decoration.*

Moreover, if E and t satisfies some additional conditions then $\overline{\mathcal{T}_{E;t}}$ admits a local rule (without decoration). The additional conditions are rather complicated. In the case of 5-fold symmetry ($n = 5$ and $k = 2$), E and t, where t is a multiple of τ, satisfies these additional conditions, hence $\overline{\mathcal{T}_{E;t}}$ admits local rules (without decoration). Some sufficient conditions for the existence of local rules *without decoration* for the case $n = 4, k = 2$ will be given later.

The theorem is a generalization of Theorem 4.5; actually, the proof of Theorem 4.1 in (Le, 1995a) can be easily generalized to a proof of this theorem (see also (Le and Piunikhin, 1995)).

Theorems 5.1 and 5.3 provide us with infinitely many set of quasiperiodic *decorated* tilings admitting local rules; not only in 2- and 3- dimensional cases, but in any dimensional greater than 1.

The proofs of both theorems are reduced to classification of some kind of homotopy classes, and are technically complicated. For details see (Le et al., 1992; Le et al., 1993; Le and Piunikhin, 1995; Le, 1995a).

6. Necessary conditions

To our knowledge all the necessary conditions are formulated only for the nondecorated version of local rules. The reader should not confuse the decorated and nondecorated versions.

6.1. THE SI CONDITION

For any tuple of $(n-k-1)$ indices $J = (j_1, \dots, j_{n-k-1})$ with $1 \le j_1 < \dots < j_{n-k-1} \le n$, let H_J be the $(n-k-1)$-dimensional subspace spanned by $e_{j_1}, \dots, e_{j_{n-k-1}}$. Of course H_J is a rational subspace.

Definition 6.1 *E satisfies the second intersection* (SI) *condition if for every multi-index $J = (j_1, \dots, j_{n-k-1})$, the space $H_J + E$ (whose dimension is $n-1$) contains a rational $(n-k)$-dimensional subspace.*

Remark 6.1 Note that $H_J + E$ always contains a rational $(n-k-1)$-dimensional subspace: H_J.

Originally, Levitov introduced the following definition.

For each $i \in \{1, 2, \dots, n\}$ let L_i be the $(n-1)$-dimensional subspace spanned by $(n-1)$ vectors from $e_1, \dots, e_n$, without e_i. Then the family

$$\mathcal{L}_i = L_i + \mathbb{Z}^n = L_i + me_i, \qquad m \in \mathbb{Z}$$

is a family of equidistant $(n-1)$-planes of $\mathbb{R}^n$. The intersection of $\mathcal{L}_i$ with E is a family of equidistant parallel planes of codimension 1, called the i-th grid of E. In general every k planes of codimension 1 in E have exactly one intersection point. It follows that every k grids have at least one intersection point different from $\mathbf{0}$.

We say that E satisfies the ***Levitov SI condition*** if every $(k+1)$ grids have intersection points different from $\mathbf{0}$.

Suppose E is spanned by k vectors with coordinates :

$$\begin{aligned} v_1 &= (v_{11}, v_{12}, \dots, v_{1n}), \\ v_2 &= (v_{21}, v_{22}, \dots, v_{2n}), \\ &\vdots \\ v_k &= (v_{k1}, v_{k2}, \dots, v_{kn}), \end{aligned}$$

For $I = (i_1, \dots, i_k)$, let d_I be the determinant of the matrix consisting of k columns $i_1, \dots, i_k$.

Proposition 6.2 *The following conditions are equivalent:*

(a) *E satisfies the* SI *condition.*

(b) *E satisfies the Levitov* SI *condition.*

(c) *For every $(k+1)$ indices $\{i_1, i_2, \dots, i_{k+1}\}$ from $\{1, 2, \dots, n\}$, the $(k+1)$ numbers d_{I-j}, $j = 1, \dots, k+1$ are linearly dependent over $\mathbb{Q}$.*

This is an easy exercise in linear algebra.

Theorem 6.3 *If $\overline{\mathcal{T}_E}$ admits a local rule then E satisfies the* SI *condition.*

Levitov gave a proof for the case $k = 2$, $n > 3$, see (Levitov, 1988). The proof has a gap: the first of several lemmas used in the proof asserts that if two tilings in $\overline{\mathcal{T}_E}$ are coincident on a half-plane, then they are the same. This is not true. In $\overline{\mathcal{T}_E}$ there are infinitely many pairs of different tilings which are coincident on a half-plane. For example, there are many pairs of Penrose tilings which are differ by exactly an infinite Conway worm, see Section 3.5.

A geometric proof of the theorem is given in (Le, 1992).

The SI condition is a nice one, but it is not very strong. Among all k-dimensional subspaces of $\mathbb{R}^n$, there are uncountably many that satisfies the SI condition, while there are at most a countable number of them whose corresponding sets of tilings admit local rules.

6.2. ALGEBRAICITY IS NECESSARY

A k-dimensional subspace E of $\mathbb{R}^n$ is said to have ***algebraic slope*** if E is spanned by k vectors whose coordinates are algebraic numbers.

Theorem 6.4 *If $\overline{\mathcal{T}_E}$ admits a local rule then E has algebraic slope.*

A proof is given in Section 8.

7. The case $n = 4$, $k = 2$

We consider local rules *without decorations.*

7.1. QUADRATICITY IS NECESSARY

We first recall the well-known Plücker embedding. See, for example (Griffiths and Harris, 1978). A 2-dimensional subspace E in $\mathbb{R}^4$ is determined by two linear equations:

$$a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 = 0,$$
$$b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_4 = 0.$$

Here a_i, b_i are real numbers; and λ_i's are the coordinates. Let

$$A_{ij} = \det\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix},$$

with $1 \le i < j \le 4$. Then

$$A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = 0. \qquad (7.1)$$

Conversely, every six numbers A_{ij}, not all zero, satisfying this equation, define a 2-dim subspace of $\mathbb{R}^4$. Two collections A_{ij} and A'_{ij} define the same

2-dim subspace if and only if they are proportional. In other words, the Grassmanian $G_{2,4}$ is a quadric in the projective space $\mathbb{R}P^5$, defined by equation (7.1).

It is easy to see that E has quadratic slope if and only if, after multiplying by some scalar, all the A_{ij} are in $\mathbb{Q}[\sqrt{D}]$ for some natural number D. The set $\mathcal{Q}$ of all quadratic 2-dim subspaces is countable.

Using the definition of the SI condition, one can prove the following

Proposition 7.1 *The set of* 2*-dim subspaces which satisfy the* SI *condition is the union of* $\mathcal{Q}$ *and* $\mathcal{C}$*, where* $\mathcal{C}$ *is the union of infinitely* (*countably*) *many algebraic curves in* $G_{2,4}$.

Note that while the cardinal of $\mathcal{Q}$ is countable, the cardinal of $\mathcal{C}$ is the continuum. It is interesting that the intersection of $\mathcal{Q}$ and $\mathcal{C}$ is not empty.

Levitov conjectured that only those E in $\mathcal{Q}$ can admit local rules. We have an even stronger result:

Theorem 7.2 *If E admits a local rule, then E is in $\mathcal{Q}$, but not in $\mathcal{C}$.*

(For a proof see (Le, 1992).) This means, if E admits a local rule, then E has quadratic slope and in addition, E must satisfy some additional condition (not belonging to $\mathcal{C}$). The latter condition can be made explicit as follows.

Theorem 7.3 *If E admits a local rule, then E has quadratic slope and*

$$(A_{13}A_{24} : A_{14}A_{23}) \notin \mathbb{Q}. \tag{7.2}$$

(The ratio of any two of the three terms in equation (7.1) is irrational).

This is a reformulation of conditions given in (Le, 1992), see (Le, 1995c). The set of all 2-dimensional subspaces having quadratic slope but not satisfying (7.2) is the set of roots of a polynomial with rational coefficients in $\mathcal{Q}$. Hence most 2-dim subspaces having quadratic slope satisfy (7.2).

The quadraticity of E follows from the fact that $G_{2,4}$ is a quadric! In fact, in (Le, 1992), we show that if local rules exist, then E is in the intersection of a projective line of rational coefficients with the quadric $G_{2,4}$.

A stronger result is

Theorem 7.4 *E admits a* weak *local rule if and only if E has quadratic slope and satisfies* (7.2).

For definition of weak local rules see Section 8. Proofs of most results in this section are in (Le, 1995c).

Compared with the previous theorems (about the existence of local rules after decoration), we see that if E has quadratic slope but does not satisfy (7.2), then E does not admit (even weak) local rules, but does admit a local rule after decoration. Hence local rules after decoration, in some sense, are even weaker than weak local rules. In such a case there are tilings which can be decorated in different (but a finite number of) ways.

Example 7.1 The plain Ammann tilings can be obtained using the strip projection method (see (Beenker, 1982)) with

$$\dim(E) = 2 \quad \text{and} \quad \dim(\mathbb{R}^n) = 4.$$

The projective coordinates of E are the following:

$$A_{12} = 3, \quad A_{13} = -\sqrt{2}, \quad A_{14} = -1,$$
$$A_{23} = 1, \quad A_{24} = -\sqrt{2}, \quad A_{34} = 1.$$

E has quadratic slope but does not satisfy (7.2): $A_{12}A_{34} : A_{23}A_{14} = 3/2$, a rational number. By Theorems 5.1 and 7.3, E does not admit local rules (first proved by Burkov, 1988), and E does admit a local rule after decoration, see (Ammann et al., 1992; Socolar, 1989).

The proof of the existence of local rules, using the method in (Le et al., 1992) leads to a concrete local rule, which is in fact the same as Ammann rules (see (Le, 1993)).

7.2. SUFFICIENT CONDITIONS

The question now is which 2-dimensional subspaces admit a local rule without decoration. If E has quadratic slope and satisfies (7.2), then it is very close to admitting a local rule.

Theorem 7.5 *Suppose that E has quadratic slope and satisfies* (7.2). *Suppose, in addition, that*

(a) *the $\mathbb{Z}$-module spanned by A_{13}, A_{23}, A_{34} contains* $A_{23}A_{13}/A_{12}$, $A_{23}A_{14}/A_{12}$, $A_{23}A_{34}/A_{24}$ *and* $A_{23}A_{14}/A_{24}$; *and*

(b) *the $\mathbb{Z}$-module spanned by A_{12}, A_{23}, A_{24} contains* $A_{23}A_{14}/A_{13}$, $A_{23}A_{12}/A_{13}$, $A_{23}A_{14}/A_{34}$ *and* $A_{23}A_{24}/A_{34}$.

Then $\overline{T_E}$ admits a local rule (*without decoration*).

The proof is a refined version of the proof in (Le et al., 1992), see (Le, 1995c).

For example, in the following situation, E satisfies all the condition listed in Theorem 7.5.

Let

$$\lambda = \frac{\sqrt{d^2 \pm 4} - d}{2},$$

where d is a positive integer, and in the version with the minus sign, d is greater than 2. (Note that $1/\lambda$ is a Pisot number.)

Let E be the 2-dimensional subspace of $\mathbb{R}^4$ with the following projective coordinates:

$$A_{12} = 1, \quad A_{13} = \lambda^{m+1}, \quad A_{14} = \lambda^{m-l}(\lambda^2 - 1),$$
$$A_{23} = \lambda^l, \quad A_{24} = \lambda, \quad A_{34} = \lambda^m,$$

where m, l are arbitrary integers. It can be checked easily that E satisfies all the condition listed in Theorem 7.5, hence

Theorem 7.6 *The set* $\overline{\mathcal{T}_E}$ *admits a local rule* (*without decoration*).

So we have infinitely many sets of 2-dimensional quasiperiodic tilings which admit local rules *without* decoration. These sets of tilings are pairwise nontopologically conjugate; any two of them are not mutually locally derivable. We don't know if these sets of tilings have inflation/deflation theory.

8. Algebraicity is necessary

We are going to prove Theorem 6.4. We will discuss the notion of weak local rules and r-volumes.

8.1. LIFTING A TILING OF E; WEAK LOCAL RULE

A ***k*-polyhedral surface** is any union of k-facets of $\mathbb{Z}^n$ which projects (by π) homeomorphically onto E.

Every k-polyhedral surface defines a tiling T of E, by projecting the polyhedral structure onto E. We say that the k-polyhedral surface is a ***lift*** of T. (This kind of lifting is different from the lifting in (Le, 1995a; Le et al., 1993)).

Suppose T is a tiling of E *whose tiles are translates of tiles in* $\overline{\mathcal{T}_E}$, with one vertex at the origin.

Proposition 8.1 *There is at most one lift of* T *passing through the origin.*

Proof One sees that if a vertex of T has been lifted, then all neighboring vertices, that can be connected to this vertex by an edge of the tiling, can be lifted uniquely. □

This proposition says that if there are two lifts of a tiling, then the two lifts are t-congruent.

For example, for regular α, T_α has a lift which lies in the tube $\gamma + E + \alpha$. In particular, the projection of a lift onto $E^\perp$ is a bounded set.

Definition 8.2 *We say that* E *admits a weak local rule if there is a number* r *such that every tiling in* $\overline{\mathcal{T}_E}(r)$*, after a translation, has a lift whose projection onto* $E^\perp$ *is bounded. If such an* r *exists, we say that the* r*-rule is a weak local rule for* $\overline{\mathcal{T}_E}$.

Recall that $\overline{\mathcal{T}_E}(r)$ is the set of tilings every r-map of which is t-congruent to an r-map of a tiling in $\overline{\mathcal{T}_E}$.

Since the lift is unique up to translations, if E admits a local rule, then it admits a weak local rule. Also if such r exists, as in the definition, then every tiling satisfying the r-rule is ***aperiodic*** (we assume that E is totally irrational). This means weak local rules always guarantee aperiodicity.

Theorem 6.4 is a consequence of the following.

Theorem 8.3 *If E is totally irrational and does not have algebraic slope, then E does not admit any weak local rules.*

The rest of this section is devoted to a proof of this theorem. The main idea is to show that for every given $r > 0$, there is another k-plane F very close to E, such that r-maps of tilings in $\overline{\mathcal{T}_F}$ are t-congruent to r-maps of tilings in $\overline{\mathcal{T}_E}$. We show this by using the notion of r-volumes.

8.2. APPROXIMATION OF E

From now on we fix a k-dimensional subspace E which is totally irrational and *does not* have algebraic slope.

Fix a number r. Replacing r by a larger number if necessary, one may assume that there is no projection of integer points on the boundary of the ball V_r in E of radius r and centered at the origin,

$$\pi(\mathbb{Z}^n) \cap \partial(V_r) = \emptyset.$$

Since $\pi(\mathbb{Z}^n)$ is countable, such an r, larger than the original one, can always be chosen.

Let $K = p^{\perp}(\gamma)$, where γ is the unit cube. Let $\mathring{K}$ be the interior of K in $E^{\perp}$. The following is trivial.

Lemma 8.4 *For $x \in E^{\perp}$, $\mathring{K}$ and $\mathring{K} - x$ have nonempty intersection if and only if $x \in \mathring{K} - \mathring{K}$.*

Lemma 8.5 *There is no integer point in $\mathring{K} - \mathring{K}$, except for the origin.*

This lemma has been proved in (Oguey et al., 1988).

Let $\mathcal{M}$ be the set of all integer points in $(\mathring{K} - \mathring{K}) + V_r$. It is a finite set.

Consider the tube $\gamma + E$ and its translations by some integer vectors

$$\gamma + E + \eta_i, \qquad \eta_i \in \mathbb{Z}^n,\ i = 1, \ldots, m.$$

These tubes may or may not have intersection.

Theorem 8.6 *There exists a sequence $E_1, E_2, \ldots,$ of k-dimensional subspaces of $\mathbb{R}^n$ such that*

(a) *E_s converge to E as $s \to \infty$, and $E_s \neq E$,*

(b) *For any integer points* $\eta_1, \ldots, \eta_m$ *in*

$$\mathcal{M} = \mathbb{Z}^n \cap (\mathring{K} - \mathring{K} + V_r),$$

the intersection

$$\bigcap_{i=1}^{m} (E + \gamma + \eta_i)$$

has dimension n if and only if the intersection

$$\bigcap_{i=1}^{m} (E_s + \gamma + \eta_i)$$

has dimension n, for every $s = 1, 2, \ldots$.

(c) *Any integer point in the boundary of* $(K - K) + V_r$ *must be also in the boundary of each* $(\gamma - \gamma) + E_s$.

This follows from the fact that E does not have algebraic slope. The proof is based on Tarski's theory of real algebra and is given in Section 8.7.

For a subset P of $E^\perp$, let us define $\varphi(P)$ as the subset of E consisting of projections of all integer points in the tube $P + E$,

$$\varphi(P) = p[(E + P) \cap \mathbb{Z}^n].$$

For example, if α is E-regular, then the set of vertices of T_α is $\varphi(\alpha + K)$. If P does not contain any projection point of $\mathbb{Z}^n$ then $\varphi(P) = \emptyset$.

There is a natural way to subdivide $\alpha + K$ into smaller polyhedra, called r-volumes. Points of $\mathbb{Z}^n$ which project into the same r-volumes project on E into vertices whose r-maps are t-congruent. The notion of r-volumes was introduced in (Ingersent and Steinhardt, 1991), and is presented in the next section.

8.3. R-VOLUMES

Suppose $\alpha \in E^\perp$ is E-regular. We know that v is a vertex of T_α if and only if $v \in \varphi(\alpha + K)$, i.e., v is the projection of an integer point ξ in the tube $E + \alpha + K$. By Lemma 8.5 the integer point ξ is unique.
We consider the question when the edge connecting v and $v + e_i^{\|}$ is an edge of T_α. This happens if and only if both ξ and $\xi + \varepsilon_i$ belong to the tube $E + \alpha + K$. This means $p^\perp(\xi)$ and $p^\perp(\xi + \varepsilon_i)$ belong to $\alpha + K$. Hence one has

Proposition 8.7 *A point v is a vertex of* T_α*, and at the same time the segment connecting v and* $v + e_i^{\|}$ *is an edge of* T_α *if and only if v is in* $\varphi(\alpha + Q)$*, where* $Q = K \cap (K - e_i^\perp)$.

Lemma 8.8 *If v and $v + x$ are vertices of T_α, then x is the projection of a unique integer vector ξ whose projection onto $E^\perp$ lies in $\mathring{K} - \mathring{K}$.*

Proof Let v be the projection of integer point ξ_1, and $v + x$ be the projection of integer point ξ_2 with both ξ_1 and ξ_2 in the tube $\alpha + K + E$. Then $\xi = \xi_2 - \xi_1$. Both $p^\perp(\xi_1)$ and $p^\perp(\xi_2)$ are in $\alpha + \mathring{K}$. Hence their difference is in $(\alpha + \mathring{K}) - (\alpha + \mathring{K}) = \mathring{K} - \mathring{K}$. Uniqueness of ξ follows from Lemma 8.5. □

Recall that $\mathcal{M}$ is the set of integer points in $(\mathring{K} - \mathring{K}) + V_r$. The projection of $\mathcal{M}$ onto $E^\perp$ is in $\mathring{K} - \mathring{K}$. Consider all translations of K of the form $K - x$ with $x \in p^\perp(\mathcal{M})$. The superposition of these translates of K divides K into smaller convex polyhedra

$$K = \bigcup_{i=1}^{N} K^{(i)},$$

called the *r-volumes*. The number N, as well as the division, depends on the radius r. The following theorem explains the significance of r-volumes.

Theorem 8.9 *Suppose α, β are E-regular. Then*

(a) *The boundary of $\alpha + K^{(i)}$ does not contains any projection of integer points. In other words $\varphi(\partial(\alpha + K^{(i)})) = \emptyset$.*

(b) *If v belongs to $\varphi(\alpha + K^{(i)})$ and w belongs to $\varphi(\beta + K^{(i)})$, with the same index i, then the r-maps at v of T_α and at w of T_β are t-congruent. Hence each r-volume defines an r-map.*

(c) *Two different r-volumes define different r-maps.*

Proof (a) The boundary of $\alpha + K$ does not intersect $p^\perp(\mathbb{Z}^n)$. The latter, being a $\mathbb{Z}$-module, is invariant under translations from $p^\perp(\mathbb{Z}^n)$. Now the r-volumes are formed by K and its translations by vectors from the $\mathbb{Z}$-module. Hence the boundary of an r-volume does not intersect $p^\perp(\mathbb{Z}^n)$.

(b) and (c) follow from Proposition 8.7 and Lemma 8.8. □

Here is a way to recover an r-map from an r-volume. Let the center of the r-map be the origin. Then the set of vertices of the r-map is the set of points of the form $\varphi(x)$, $x \in p^\perp(\mathcal{M})$, such that $x + K^{(i)}$ lies in K.

8.4. PLANES CLOSE TO E

We will consider k-dimensional subspaces F which are very close to E. This new F defines a new set of tilings $\overline{\mathcal{T}_F}$. The tiles are different from tiles of $\overline{\mathcal{T}_E}$. We can transform tilings in $\overline{\mathcal{T}_F}$ to tilings whose tiles are translates of tiles in $\overline{\mathcal{T}_E}$ as follows.

Suppose α is F-regular, then the tube $F + \alpha + \gamma$ contains a unique k-polyhedral surface, and projecting along $F^\perp$ onto F gives a tiling in $\overline{\mathcal{T}_F}$.

Instead of projection along $F^{\perp}$ onto F we consider the projection along $E^{\perp}$ onto E. Since E satisfies condition (3.1), when F is sufficiently close to E, the k-polyhedral surface projects (along $E^{\perp}$) one-to-one onto E, giving a a tiling, denoted by $U(\alpha, F)$. The tiles of $U(\alpha, F)$ are the same as the tiles of the tilings in $\overline{\mathcal{T}_E}$, since they are projections of k-facets along the same $E^{\perp}$ and onto the same E.

A lift of $U(\alpha, F)$ is in the tube $F + \alpha + \gamma$, and hence the projection of the lift onto $E^{\perp}$ (along E) is not bounded. This implies that $U(\alpha, F)$ cannot belong to $\overline{\mathcal{T}_E}$.

Let $\mathcal{U}(F)$ be the set of all tilings $U(\alpha, F)$, with α regular with respect to F. Then the closure $\overline{\mathcal{U}(F)}$ and $\overline{\mathcal{T}_F}$ are mutually locally derivable. In some sense, tilings in $\overline{\mathcal{U}(F)}$ and tilings in $\overline{\mathcal{T}_F}$ have the same order, only the shapes of tiles are slightly different.

If every r-map of $\mathcal{U}(F)$ is t-congruent to an r-map of $\overline{\mathcal{T}_E}$, then $\overline{\mathcal{T}_E}(r) \supset \overline{\mathcal{U}(F)}$, hence the r-rule cannot be a weak local rule for E.

This will be used in the proof of absence of weak local rules.

We suppose that the E_s of Theorem 8.6 are sufficiently close to E so that one can construct the set of tilings $\overline{\mathcal{U}(E_s)}$ as above described.

Denote by $\pi_s^{\perp}$ the projection along E_s onto $E^{\perp}$. Let $K_s = \pi_s^{\perp}(\gamma)$.

For a set P in $E^{\perp}$ we define

$$\varphi_s(P) = p[(P + E_s) \cap \mathbb{Z}^n],$$

the set of projections of integer point in the tube $P + E_s$. Then the set of vertices of $U(\alpha, E_s)$ is $\varphi_s(\alpha + K_s)$. For $\overline{\mathcal{U}(E_s)}$ one can construct similar r-volumes as follows.

Lemma 8.10 *For sufficiently large s, the set of integer points in $\mathring{K}_s - \mathring{K}_s + V_r$ is $\mathcal{M}$, the same as the set of integer points in $\mathring{K} - \mathring{K} + V_r$.*

Proof Consider the cell $(K - K) + V_r$. By definition, all points of $\mathcal{M}$ are in the interior of this cell. Since $\mathcal{M}$ is finite, the minimum distance from points in $\mathcal{M}$ to the boundary of the cell is a positive number, hence if s is sufficiently large, i.e., if E_s is sufficiently close to E, then the cell $(K_s - K_s) + V_r$ contains $\mathcal{M}$ in its interior.

If $(K_s - K_s) + V_r$ contains a new integer point ξ, then ξ must be on the boundary of $(K - K) + V_r$. But by c) of Theorem 8.6, ξ is also on the boundary of $\gamma - \gamma + E_s$, which contains the cell $(K_s - K_s) + V_r$. This contradicts the fact that ξ is an interior point of this cell. □

By restricting to a subsequence, we can assume that all E_s satisfy the conclusion of this lemma.

Consider the superposition of all translates of K_s of the form $K_s - x$, where $x \in \pi_s^{\perp}(\mathcal{M})$. Together they divide K_s into smaller convex polyhedra

$$K_s = \bigcup_{i=1}^{N_s} K_s^{(i)}.$$

Each $K_s^{(i)}$ is called an r-volume of $\overline{\mathcal{T}}(E_s)$. The following result is similar to Theorem 3 and can be proved in the same way.

Theorem 8.11 *Fix s. Suppose α, β are E_s-regular. Then*

(a) *The boundary of $\alpha + K_s^{(i)}$ does not contains any projection of integer points. In other words $\varphi_s(\partial(\alpha + K_s^{(i)})) = \emptyset$.*

(b) *If v belongs to $\varphi_s(\alpha + K_s^{(i)})$ and w belongs to $\varphi_s(\beta + K_s^{(i)})$, with the same index i, then the r-maps at v of $U(\alpha, E_s)$ and at w of $U(\beta, E_s)$ are t-congruent. Hence each r-volume defines an r-map.*

(c) *Two different r-volumes define different r-maps.*

We will see that the division of K_s into r-volumes is the same as that of K.

8.5. PROOF OF ALGEBRAICITY

For a fixed s, we say that $K_s - \pi_s^{\perp}(x)$ is *corresponding to* $K - p^{\perp}(x)$, where $x \in \mathcal{M}$. Consider the following condition:

(*) m polyhedra $K - x_i, i = 1, \dots, m$ with $x_i \in p^{\perp}(\mathcal{M})$ have nonempty *interior* intersection if and only if their corresponding polyhedra have nonempty *interior* intersection.

If (*) is true for every finite collection $x_1, \dots, x_m$ in $\mathcal{M}$, then from the construction of r-volumes, one sees that $N_s = N$, for every s, and the division of K into r-volumes is the same as the division of K_s into r-volumes. This means, after a permutation, one has that $K^{(i)}$ and $K_s^{(i)}$ define exactly the same r-map. This follows from the construction of r-volumes, and the way to recover r-maps from r-volumes.

So if (*) is fulfilled for every s, then every r-map of $\overline{\mathcal{U}(E_s)}$ is t-congruent to an r-map of $\overline{\mathcal{T}_E}$; hence

$$\overline{\mathcal{T}_E}(r) \supset \overline{\mathcal{U}(E_s)}.$$

This implies that r-rule is not a weak local rule for E.

Note that $K_s + \pi_s^{\perp}(\eta_i)$ is the projection (by $\pi_s^{\perp}$) of the tube $E_s + \gamma + \eta_i$, hence m polyhedra $K_s + \pi_s^{\perp}(\eta_i), i = 1, \dots, m$, have nonempty interior intersection if and only if the intersection

$$\bigcap_{i=1}^{m} (\gamma + E_s + \eta_i) \tag{8.1}$$

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8.6. TARSKI'S THEO

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By Theorem 8.6, for every finite number of vectors $\eta_i, i = 1, \ldots, m$ in $\mathcal{M}$, the dimension of the tube (8.1) is n if and only if the tube (8.1), with E_s replaced by E, has dimension n; which, in turn, happens if and only if m polyhedra $K + p^{\perp}(\eta_i)$ have nonempty interior intersection. Hence (*) is true.

Thus we have proved that for every r, r-rule is not a weak local rule for E. This means E does not admit any weak local rule. Theorem 8.3 is proved, and hence so is Theorem 6.4.

It remains to prove Theorem 8.6. We first present Tarski's theory on real algebra.

8.6. TARSKI'S THEORY

A ***rational polynomial condition*** on variables $z_1, \ldots, z_l$ is one of the following equation or inequalities:

$$f = 0, \quad f > 0, \quad f < 0, \quad f \leq 0, \quad f \geq 0.$$

Here f is a polynomial in z_i with *rational coefficients*.

The set of solutions $(z_1, \ldots, z_l)$, $z_i \in \mathbb{R}$, of a finite system of polynomial conditions is called a ***quasi-algebraic set***. We emphasize that all the z_i's must be *real* numbers.

A ***generalized quasi-algebraic set*** is the union of any finite number of quasi-algebraic sets.

A point in $\mathbb{R}^l$ is ***algebraic*** if all its coordinates are algebraic numbers.

Theorem 8.12 (Tarski) (see, for example, (Jacobson, 1963).) *Let Σ be a finite set of rational polynomial conditions on $(q+m)$ variables $z_1, \ldots, z_q, y_1, \ldots, y_m$.*

(a) *The set of $(y_1, \ldots, y_m) \in \mathbb{R}^m$ such that the system Σ has a solution is a generalized quasi-algebraic set.*

(b) *Σ has a solution in $\mathbb{R}^{q+m}$ if and only if it has an algebraic solution.*

The following fact is well-known.

Theorem 8.13 *In every generalized quasi-algebraic set, the set of algebraic points is dense in the usual topology.*

Proof (We thank S. Schanuel for supplying this proof). It suffices to consider a quasi-algebraic set Y, which is defined by a finite system Σ of rational polynomial conditions. Suppose $(z_1^0, \ldots, z_q^0)$ is a point in Y. Choose rational numbers a_i, b_i such that $a_i < z_i^0 < b_i$, and $|a_i - b_i| < \varepsilon$, where $\varepsilon > 0$ is any given number. Adding the condition $a_i < z_i < b_i$ to Σ, by part (b) of Theorem 8.12, one see that there is an algebraic point satisfying this enhanced system. The distance between the algebraic point and the given point is as small as desired. □

The following follows easily from the definition.

Proposition 8.14 *Suppose that X, Y are generalized quasi-algebraic sets. Then $X \cap Y$, $X \cup Y$, and $X \setminus Y$ are generalized quasi-algebraic.*

8.7. PROOF OF THEOREM 8.6

We have fixed coordinates in $\mathbb{R}^n$. Then there is an $(n-k) \times n$ matrix $\tilde{A}_E$ of rank $(n-k)$ such that E is the set of all vectors satisfying the linear equation $\tilde{A}_E(v) = 0$. This matrix $\tilde{A}_E$ is not unique, say , one can replace one row by the sum of itself and a multiple of another row. Using elementary operations on rows, the fact that rank of $\tilde{A}_E$ is k, and a permutation of basis vectors if needed, one may assume that the first $n-k$ columns of $\tilde{A}_E$ form the unit matrix. Denote by A_E the $(n-k) \times k$ matrix formed by the last k columns.

In general, if A is an $(n-k) \times k$ matrix, let $\tilde{A}$ be the $(n-k) \times n$ matrix obtained from A by adding the unit $(n-k) \times (n-k)$ matrix to the left of A.

The set of all $k \times (n-k)$ matrices is isomorphic to $\mathbb{R}^{k(n-k)}$, which has the same dimension as $G_{k,n}$. And every such matrix A defines a k-dimensional subspace consisting of vectors v such that $\tilde{A}(v) = 0$.

The corresponding $A \to k$-dimensional subspace of $\mathbb{R}^n$ is a homeomorphism between $\mathbb{R}^{k(n-k)}$ and a neighborhood W of E in $G_{k,n}$. We use the matrix A as coordinates in this neighborhood W. It is easy to see that E has algebraic slope if and only if all the entries of matrix A_E are algebraic numbers.

Proposition 8.15 *For fixed integer vectors $\eta_1, \dots, \eta_m \in \mathbb{Z}^n$, the set of k-dimensional subspaces F in W such that the dimension of $\cap_{i=1}^m (F + \gamma + \eta_i)$ is n, is generalized quasi-algebraic.*

Proof The intersection $\cap_{i=1}^m (F + \gamma + \eta_i)$ has dimension n if and only if the intersection

$$\bigcap_{i=1}^{m} (F + \mathring{\gamma} + \eta_i) \tag{8.2}$$

is not empty, where $\mathring{\gamma}$ is the interior of the unit cube γ.
A vector v is in the intersection (8.2) if and only if v is in each tube $\mathring{\gamma} + F + \eta_i$. Or, equivalently, there are vectors α_i in γ such that $v - \alpha_i \in F$, i.e.,

$$\tilde{A}_F(v - \alpha_i) = 0, \tag{8.3}$$

for every $i = 1, \ldots, m$.

Consider equations (8.3) as equations on coordinates of the v, α_i and A_F. They are rational polynomial conditions. Add to this system the conditions that coordinates of the α_i's are strictly between 0 and 1, so that the α_i's are in the interior of unit cube. We have a system of rational polynomial conditions. And the set of F in question is the set of F such that this system has a solution. It follows from Theorem 8.12, part (a), that this set is generalized quasi-algebraic. □

Corollary 8.16 *For fixed integer vectors $\eta_1, \ldots, \eta_m \in \mathbb{Z}^n$, the set of $F \in W$ such that the dimension of $\cap_{i=1}^m (F+\gamma+\eta_i)$ is* not *equal to n is generalized quasi-algebraic.*

The set in the proposition is the complement to the set of the previous proposition. Hence this follows from Proposition 8.14.

Proposition 8.17 *Fix an integer point η. The set of all F in W such that η is on the boundary of $(\gamma - \gamma) + F$ is a generalized quasi-algebraic set.*

Proof Let Z_1 be the set of all $F \in W$ such that η is in $F + (\gamma - \gamma)$, Z_2 be the set of all $F \in W$ such that η is in $F + (\hat{\gamma} - \hat{\gamma})$.

The point η is in $(\gamma - \gamma) + F$ if and only if there is a vector $\alpha \in (\gamma - \gamma)$ such that $\eta - \alpha$ is in F, i.e., $\tilde{A}_F(\eta - \alpha) = 0$. This is a finite set of rational polynomial conditions on coordinates of F and α. Add to this system the conditions that all the coordinates of α are between -1 and 1 (so that α is in $\gamma - \gamma$). Applying Tarski's theorem we see that Z_1 is a generalized quasi-algebraic set.

In a similar way one can prove that Z_2 is a generalized quasi-algebraic set. Now note that the set in question is $Z_1 \setminus Z_2$, and hence is also a generalized quasi-algebraic set. □

Proof of Theorem 8.6 Let $\kappa = \{\eta_1, \ldots, \eta_m\}$ be a finite set of vectors in $\mathcal{M}$. If the dimension of $\cap_{i=1}^m (E + \gamma + \eta_i)$ is n, let Y_κ be set of all F in W such that the dimension of $\cap_{i=1}^m (F + \gamma + \eta_i)$ is equal to n. If the dimension of $\cap_{i=1}^m (E + \gamma + \eta_i)$ is *not* n, let Y_κ be the set of all F in W such that dimension of $\cap_{i=1}^m (F + \gamma + \eta_i)$ is *not* n.

An integer point on the boundary of $(K - K) + V_r$ must be in $\partial(K - K) + V_r$, since there are no projections of integer points on the boundary of V_r, by the assumption about r. So every integer point on the boundary of $(K - K) + V_r$ is on the boundary of $\gamma - \gamma + E$. For each integer point η on the boundary of $(K - K) + V_r$ let Z_η be the set of all F in W such that η is on the boundary of $\gamma - \gamma + F$.

By Propositions 8.13–8.16, Y_κ and Z_η are generalized quasi-algebraic, hence so is the intersection Y of all Y_κ, Z_η, when κ runs through the set of all subsets of $\mathcal{M}$, and η runs through the set of integer points on the boundary of $K - K + V_r$.

It is important that E is in this intersection Y. Since the set of algebraic points is dense in every generalized quasi-algebraic set, by Theorem 8.13, there is a sequence of algebraic points $E_1, E_2, \dots$ in the intersection which converges to E. The fact that E is not algebraic means that $E_s \neq E$ for every s. All E_s are in every Y_κ, hence for every $\kappa = \{\eta_1, \dots, \eta_m\}$, the dimension of $\cap_{i=1}^m (F + \gamma + \eta_i)$ is equal to n if and only if the dimension of $\cap_{i=1}^m (F + \gamma + \eta_i)$ is equal to n. This proves (b).

All E_s are in Z_η, for every integer point η on the boundary of $K - K + V_r$, hence η is on the boundary of $\gamma - \gamma + E_s$, by the definition of Z_η. This proves (c). Theorem 8.6 is proved. □

Remark 8.1 When E does not satisfy (3.1), the only obstruction to the proof is that the projection along $E^\perp$ onto E, when restricted to the k-polyhedral surface in the tube $F + \gamma + \alpha$, may not be one-to-one for every F in a small neighborhood of E. But this can be overcome easily, since the set of all F in W such that the mentioned projection is one-to-one is a generalized quasi-algebraic set. In all statements about the existence of E_s, one needs to add the restriction that E_s is in this generalized quasi-algebraic set. The remainder of the proof remains the same.

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Abstract Sequences
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