CS 1050 - Proofs
Homework 2
Assigned August 26
Due Thursday, September 2

Recall that in the last homework you proved the following lemma:

Lemma 1 *If a is an integer and a^2 is a multiple of 3, then a is also a multiple of 3.*

Now use this lemma to prove the following theorem:

Theorem 2 *$\sqrt{3}$ is not rational.*

Make sure your proof uses the lemma (since otherwise it is probably wrong).

2. Prove that $\sqrt{5}$ is not rational. You will need to state and prove the relevant lemmas yourself. You will have five lemmas: one for each of the four nonzero remainders when an integer is divided by 5, and one using those four. Then you will need to prove the theorem that $\sqrt{5}$ is not rational. (If this isn't making sense yet, go back to the first homework and look at the lemmas that proved there that were used to prove the lemma in the first question of this homework.)

3. We all know that $\sqrt{4}$ *is* rational. This means that you cannot prove (obviously) that $\sqrt{4}$ is not rational. Try to prove that $\sqrt{4}$ is not rational, using the ideas of the first few problems, and explain why you cannot complete the proof. Do not just state that $\sqrt{4}$ is rational! Show that one of the three lemmas you would need to prove (one for each nonzero remainder when an integer is divided by 4) is false, so can never be proved (at least by this method).

4. The following is a theorem we will use but not prove:

Theorem 3 *Suppose that a, b are integers, and let p be a prime number. Let $z = ab$. Suppose that z is a multiple of p. Then either a or b is a multiple of p.*

For example, let $a = 6$, $b = 3$, $p = 2$. 18 is a multiple of 2, and a is also. Let $a = 21$, $b = 22$. $z = 462 = 21 * 22$ is a multiple of 7, and also 21 is a multiple of 7. The theorem *needs* the assumption that p is prime, for otherwise it would be false. For example, let $a = 4$, $b = 15$. $z = 60$ *is* a multiple of 6, but *neither* a nor b is a multiple of 6! (Because 6 isn't prime, this doesn't contradict the theorem, of course.)

a) Using this theorem, prove the following lemma:

Lemma 4 *Suppose a is an integer and a^2 is a multiple of p, where p is a prime number. Then a is a multiple of p.*

b) Disprove the following conjecture:

Conjecture 1 *If a, p are integers, $p \neq 0$, such that a^2 is a multiple of p, then a is a multiple of p.*

Be careful. To prove the conjecture false, it is not enough simply to *try* to prove that the conjecture is true, and fail; you must prove that that it really is false. Since the conjecture says "If A holds, then B holds," the way to do this is to prove that it is possible that even when A holds, B does *not* hold. Typically, this is done by giving an example for which A holds and B does not.

5. a) Let $S = T = \{1, 2, ..., 10\}$. Define functions from S to T such that:

$f_1 : S \to T$ is 1-1 and onto.

$f_2 : S \to T$ is 1-1 but not onto.

$f_3 : S \to T$ is onto but not 1-1.

$f_4 : S \to T$ is neither 1-1 nor onto.

b) Now let $U = \{1, 2, ..., 10\}$ and let $V = \{1, 2, ..., 5\}$. Give an example of a function $g : U \to V$ that is onto. (Notice that there cannot be a function from U to V that is 1-1. We will not prove this now, but do you see why this is true?)