CS 1050 Homework 3 Solutions

1. Let S, T, U be sets, $f : S \to T$, $g : T \to U$, such that $g \circ f$ is one-to-one. Then f is one-to-one.

Proof. Let S, T, U be sets, $f : S \to T$, $g : T \to U$, such that $g \circ f$ is one-to-one. For a contradiction, suppose that f is not one-to-one. Then there are $x, y \in S$, $x \neq y$, such that $f(x) = f(y)$. (This is exactly what it means for f not to be one-to-one.) $(g \circ f)(x) = g(f(x))$. Because $f(x) = f(y)$, $g(f(x)) = g(f(y)) = (g \circ f)(y)$. Now $x \neq y$ yet $(g \circ f)(x) = (g \circ f)(y)$. So $g \circ f$ is not one-to-one, a contradiction. □

For the example, set $S := \{1, 2\}$, $T := \{1, 2, 3, 4\}$, $U := \{1, 2, 3\}$. Define $f(1) = 1$ and $f(2) = 2$. Thus $f : S \to T$. Set $g(1) = 1$, $g(2) = 2$, $g(3) = 3$ and $g(4) = 3$. Clearly $g : T \to U$ and g is not one-to-one. The function $g \circ f : S \to U$. The domain of $g \circ f$ is of size two. Now $(g \circ f)(1) = g(f(1)) = g(1) = 1$ and $(g \circ f)(2) = g(f(2)) = g(2) = 2$. So $g \circ f$ *is* one-to-one.

2. Let $f : S \to T$ and $g : T \to U$ be two functions such that $g \circ f$ is onto U. Then g is onto U.

Proof. Let $f : S \to T$ and $g : T \to U$ be any two functions such that $g \circ f$ is onto U. Choose any $z \in U$. We must show that there is a $y \in T$ such that $g(y) = z$. Because $g \circ f$ is onto U, there is an x in S such that $(g \circ f)(x) = z$. Of course $(g \circ f)(x) = g(f(x))$. Define $y := f(x)$. Clearly $g(y) = z$. Because $f : S \to T$, $y \in T$. Since we have found a $y \in T$ such that $g(y) = z$, clearly such a y exists. □

For the example, set $S := \{1\}, T := \{1, 2\}, U := \{3\}$. Set $f(1) = 1$, $g(1) = 3$, $g(2) = 3$. Notice that $(g \circ f)(1) = g(f(1)) = g(1) = 3$. Thus $g \circ f$ is onto U. However, f is not onto T.

3.a Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by $f(x_1, x_2) = (2x_1 + x_2, 3x_1 - x_2, 2x_1 + x_2)$ for all reals x_1, x_2.

Prove that f is one-to-one.

Proof. Let x, y, x', $y' \in \mathbb{R}$ such that $f(x, y) = f(x', y')$. For f to be $1 \to 1$, we must prove that $x = x'$ and $y = y'$.
As $f(x, y) = f(x', y')$,

$2x + y = 2x' + y'$ (first part of 3-tuple)
$3x - y = 3x' - y'$ (second part of 3-tuple)
$2x + y = 2x' + y'$ (third part of 3-tuple)

Solving for x in the first equation gives us $x = \frac{(2x' + y' - y)}{2}$. By substituting this value for x in the second equation, we get (after reducing, where x' cancels out) $5y' - 5y = 0$. Thus, $y = y'$. Substituing this value for y back into the first equation gives us that $2x = 2x'$. Thus, $x = x'$.

3.b Disprove the conjecture that f (given in problem 3.a) is onto.

Proof. Take, for example, the 3-tuple $y = (1, 2, 3)$. y is in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, but there is no $f(x_1, x_2) = y$, since the first element of the three-tuple must be equal to the third element of the three-tuple as defined by f. □

4.a Theorem: $A \cup B \subseteq A \cap B$ implies $A = B$.

Proof. Assume that $A \cup B \subseteq A \cap B$. We will show first that $A \subseteq B$ and then that $B \subseteq A$.

Let x be any element of A. Since $x \in A$, $x \in A \cup B$. Because $A \cup B \subseteq A \cap B$, $x \in A \cap B$. Therefore $x \in B$. Since we have shown that an arbitrary element of A is also in B, $A \subseteq B$.

Let x be any element of B. Since $x \in B$, $x \in A \cup B$. Because $A \cup B \subseteq A \cap B$, $x \in A \cap B$. Therefore $x \in A$. Since we have shown that an arbitrary element of B is also in A, $B \subseteq A$.

We have shown that $A = B$. □

4.b Theorem: $(A \cap \emptyset) \cup B = B$.

Proof. By the domination law, we have $A \cap \emptyset = \emptyset$. So, $(A \cap \emptyset) \cup B = \emptyset \cup B$. And by identity law we have that $\emptyset \cup B = B$. Thus, $(A \cap \emptyset) \cup B = B$. □

5.a Theorem 2: If A, B, and C are subsets of U, then $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$.

Proof. Let $X = A \cup B$. Then by Theorem 1, $(X \cup C)^c = X^c \cap C^c$. Substituting for X, we get $(A \cup B)^c \cap C^c$. Applying Theorem 1 again gives us $A^c \cap B^c \cap C^c$. □