CS 1050 Homework 8 Solutions

1. We have to prove that there is a strategy for player 1 such that if atleast one of the two piles have an even number of raisins then player 1 will win. We will prove the statement by strong induction.

Proof by strong induction on n where $2n$ is the size of the smaller even pile of raisins. If there is only one even pile then $2n$ is the size of that pile. We need to show that player 1 will win for all $n \geq 1$.

Base Case: $n = 1$. This means that there is a pile of size 2. Let us call this pile A and the other pile B. Player 1 eats up pile B and divides A into two piles A_1 and A_2 of size 1 each. Clearly player 2 has to eat up one of those piles and is left with a pile of size 1 to divide, which is impossible. So player 2 loses and player 1 wins. So the base case is true.

Induction Hypothesis: Let the statement be true for all n such that $1 \leq n \leq k$.

Induction Step: We need to show that player 1 wins if the smaller even pile has $2(k+1)$ raisins. Let us call this pile A and the other one B. Player 1 eats up pile B and divides A into 2 piles A_1 and A_2 such that A_1 has 1 raisin and A_2 has $2k+1$ raisins. Now look at the choices player 2 has.

Case 1. Player 2 eats up A_2. In this case, player 2 will not be able to divide A_1 into 2 piles because A_1 has only 1 raisin. Therefore player 2 loses and player 1 wins.

Case 2. Player 2 eats up A_1. Now player divides A_2, into 2 piles A_{21} and A_{22}. Clearly both A_{21} and A_{22} cannot be odd because A_2 is an odd pile and the sum of two odd numbers is even. Similarly, both A_{21} and A_{21} cannot be even because the sum of two even numbers is even and A_2 is odd. Therefore exactly one of the two piles is even, say A_{21}. Also, since A_{22} has atleast one raisin, the size of A_{21} is atmost $2k$. So, the size of the smaller even pile is atmost $2k$. So by our induction hypothesis player 1 can win this game.

We see that no matter what player 2 does, player 1 can win if the smaller even pile has $2(k+1)$ raisins. So the statement is true for $k+1$, so by mathematical induction it is true for all $n \geq 1$. Hence proved.

2. Let $P(n)$ be the formula, $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$. We want to prove

$P(n)$ for all $n \geq 1$. We prove it by induction on n.

Base Case: LHS of $P(1)$ is $\frac{1}{2}$ and the RHS is $1 - \frac{1}{2} = \frac{1}{2}$, which are equal. So, $P(1)$ is true.

Induction Hypothesis: Let $P(k)$ be true for some $k \geq 1$.

Induction Step: Now, LHS of $P(k+1)$ is, $\quad \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}}$

$= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}$ (from induction hypothesis)

$= 1 - \frac{2-1}{2^{k+1}}$

$= 1 - \frac{1}{2^{k+1}}$, which is the RHS of $P(k+1)$.

So $P(k+1)$ is true, and so by mathematical induction $P(n)$ is true for all $n \geq 1$.

3. Let $P(n)$ be the formula, $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$. We want to prove $P(n)$ for all $n \geq 1$. We prove it by induction on n.

Base Case: LHS of $P(1)$ is $\frac{1}{1\cdot 2} = \frac{1}{2}$, and the RHS is $1 - \frac{1}{2} = \frac{1}{2}$, which are equal. So, $P(1)$ is true.

Induction Hypothesis: Let $P(k)$ be true for some $k \geq 1$.

Induction Step: LHS of $P(k+1)$ is,

$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$ (from induction hypothesis)

$= 1 - \frac{k+2-1}{(k+1)(k+2)}$

$= 1 - \frac{k+1}{(k+1)(k+2)}$

$= 1 - \frac{1}{(k+1)+1}$, which is the RHS of $P(k+1)$.

So, $P(k+1)$ is true and so by mathematical indcution $P(n)$ is true for all $n \geq 1$. Hence proved.

4a. $a_0 = 2,\ a_1 = 6,\ a_2 = 18,\ a_3 = 54.$

b. $a_n = 2 \cdot 3^n$

c. Given that $a_0 = 2$ and $a_n = 3a_{n-1}$ for all $n \geq 1$, we need to prove $P(n)$ for all $n \geq 0$ where $P(n)$ is : $a_n = 2 \cdot 3^n$. We prove it by induction on n.

Base Case. LHS of $P(0)$ is a_0 which is given to be 2, and RHS is $2 \cdot 3^0 = 2$, which are equal. Therefore $P(0)$ is true.

Induction Hypothesis. Let $P(k)$ be true for some $k \geq 0$.

Induction Step. LHS of $P(k+1)$ is a_{k+1}

$= 3a_k$ (since $k+1 \geq 1$ and we are given $a_n = 3a_{n-1}$ for all $n \geq 1$)

$= 3 \cdot 2 \cdot 3^k$ (from induction hypothesis)

$= 2 \cdot 3^{k+1}$, which is the RHS of $P(k+1)$.

So $P(k+1)$ is true, and by mathematical induction $P(n)$ is true for all integers $n \geq 0$. Hence proved.

5. We have that $a_n = 2a_{n-1} - a_{n-2}$ for all $n \in \mathbb{Z},\ n > 1$ - Eqn 1.

a. If $a_0 = 0$ and $a_1 = 1$ then we have to prove $P(n)$ for all $n \geq 0$, where $P(n)$ is: $a_n = n$. We will prove it by strong induction on n.

Base Case. We have two base cases here, $P(0)$ and $P(1)$. $P(0)$ is $a_0 = 0$ and $P(1)$ is $a_1 = 1$ which are true as they are given.

Induction Hypothesis. Let $P(m)$ be true for all $1 \leq m \leq k$.

Induction Step. LHS of $P(k+1)$ is a_{k+1}

$= 2a_k - a_{k-1}$ (from Eqn 1, as $k+1 > 1$)

$= 2k - (k-1)$ (from induction hypothesis)

$= k+1$ which is the RHS of $P(k+1)$.

So, $P(k+1)$ is true, and so, by mathematical induction $P(n)$ is true for all integers $n \geq 0$.

b. If $a_0 = a_1 = 5$ then we have to prove $P(n)$ for all $n \geq 0$, where $P(n)$ is: $a_n = 5$. We will prove it by strong induction on n.

Base Case. We have two base cases here, $P(0)$ and $P(1)$. $P(0)$ is $a_0 = 5$ and $P(1)$ is $a_1 = 5$ which are true as they are given.

Induction Hypothesis. Let $P(m)$ be true for all $1 \leq m \leq k$.

Induction Step. LHS of $P(k+1)$ is a_{k+1}

$= 2a_k - a_{k-1}$ (from Eqn 1, as $k+1 > 1$)

$= 2 \cdot 5 - 5$ (from induction hypothesis)

$= 5$ which is the RHS of $P(k+1)$.

So, $P(k+1)$ is true, and so, by mathematical induction $P(n)$ is true for all integers $n \geq 0$.