Abstract

We will prove the van den Berg-Kesten Inequality in the special case of increasing (or decreasing) events.

Let us recall the FKG condition and inequality:
A positive probability measure μ on a space Ω satisfies the FKG condition if

$$\forall a, b \in \Omega,\ \ \mu(a \cap b)\mu(a \cup b) \geq \mu(a)\mu(b).$$

Theorem 1 (FKG Inequality) *Suppose $A, B \subseteq \Omega$ are increasing (or decreasing) sets. Then if μ satisfies the FKG Condition then*

$$\mu(A \cap B) \geq \mu(A)\mu(B).$$

If A is increasing and B is decreasing,

$$\mu(A \cap B) \leq \mu(A)\mu(B)$$

The Random Cluster Model

Consider a finite set X, and define $\Omega = \{0,1\}^X$. The model depends on two parameters, p and q, $0 \leq p \leq 1$, $q \geq 0$. Let the weight, $wt(\omega)$ of each configuration, namely

$$wt(\omega) = p^{|S(\omega)|}(1-p)^{(|X|-|S(\omega)|)}q^{|C(\omega)|}$$

where $S(\omega)$ is the set of open bonds in ω, and $C(\omega)$ is the set of connected components in ω.

Then, letting $Z = \sum_\omega wt(\omega)$, we may define a probability measure $\mathbb{P}_{p,q}$ on Ω by

$$\mathbb{P}_{p,q}(\omega) = \frac{1}{Z} wt(\omega)$$

When $q = 1$ we have the percolation product measure $\mathbb{P}_p$. When $q > 1$ (especially if q is large), the model favours configurations with many components; and when $q < 1$, the model favours few components.(We will revisit this model when we discuss the Ising Model.)

Exercise: Show that when $q \geq 1$, $\mathbb{P}_{p,q}$ satisfies the FKG condition.

It is not difficult to show that this is equivalent to the statement

$$\forall a, b \in \Omega,\ \ |C(a \cup b)| + |C(a \cap b)| \geq |C(a)| + |C(b)|$$

Open Question: Prove that when $q \leq 1$, we have negative correlation; that is, for bonds e_1 and e_2,

$$\mathbb{P}_{p,q}(e_1, e_2 \text{ present in } \omega) \leq \mathbb{P}_{p,q}(e_1 \text{ present in } \omega)\mathbb{P}_{p,q}(e_2 \text{ present in } \omega)$$

A special case to be considered is when $q \to 0$, (i.e we'll only see connected configurations), and $p \to 0$, (i.e we'll only see spanning trees). This case of negative correlation on the state space of spanning trees of a graph was shown by Feder and Mihail.

The BK Inequality

The BK Inequality will give a result of a similar flavour to the FKG Inequality, but in the reverse direction.

The setting is our familiar space of configurations, ($\Omega = \{0,1\}^{[m]}$), bond sites on a lattice, with the percolation product measure, $\mathbb{P}_p$. The BK inequality involves not the intersection of two sets, $A \cap B$, but the set operation, disjoint occurrence, $A \circ B$. Consider the following specific example, after which a more precise definition will follow.

For u, v, points in our underlying lattice, let $A(u,v)$ denote the set of configurations $\omega \in \Omega$ with a path in the lattice from u to v, that is, configurations containing bonds connecting u to v. Then $A(u,v) \cap A(x,y)$ denotes the set of configurations $\omega \in \Omega$ with paths in the lattice from u to v, and from x to y, that is, configurations containing bonds connecting both u to v, and x to y. And, we define $A(u,v) \circ A(x,y)$ as the set of configurations with edge disjoint paths in the lattice, from u to v and from x to y.

More generally, we have the following definitions.

Definition 1 *Given a configuration* $\omega \in \Omega$ *and a set of points* $I \subset [m]$, *define the* cylinder $[\omega]_I$ *by*

$$[\omega]_I = \{\tilde{\omega} : \tilde{\omega}_i = \omega_i, \ \forall i \in I\}$$

A set $A \subset \Omega$ *is a* cylinder set *if* $\exists \omega \in \Omega$ *and* $I \subseteq [n]$ *such that* $A = [\omega]_I$.

Definition 2 *Given events* $A, B \subseteq \Omega$

$$A \circ B = \{\omega : \exists I \subset [m], [\omega]_I \subseteq A, [\omega]_{I^c} \subseteq B\}.$$

BK Inequality Let $m \in \mathbb{N}$, let $\Omega = \{0,1\}^{[m]}$, and let the probability measure on Ω be the percolation product measure $\mathbb{P}_p$. Then, for all $A, B \subset \Omega$,

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B)$$

Example. Let $n = 4$, and let A and B be subsets of $\{0,1\}^{[4]}$. The following examples all satisfy the BK Inequality. Indeed $A \circ B \subseteq A \cap B$.

(a) $A = \{1100, 1101, 1110, 1111\} = \{11**\}, B = \{0011, 0111, 1011, 1111\} = \{**11\}$. Then $A \circ B = \{1111\} = A \cap B$, and $\mathbb{P}_p(A \circ B) = 1/16 \leq (1/4)^2 = \mathbb{P}_p(A)\mathbb{P}_p(B)$.

(b) $A = \{11**\}, B = \{**11\} \cup \{**01\}$. Then $A \circ B = \{1111, 1101\} = A \cap B$ and $\mathbb{P}_p(A \circ B) = 1/8 \leq (1/4)(1/2) = \mathbb{P}_p(A)\mathbb{P}_p(B)$.

(c) $A = \{*10*\} \cup \{*00*\} \cup \{111*\}, B = \{**11\} \cup \{**01\}$. Then $A \circ B = \{1111, 1101, 0101\}$ and $\mathbb{P}_p(A \circ B) = 3/16 \leq 3/8 = (3/4)(1/2) = \mathbb{P}_p(A)\mathbb{P}_p(B)$, but $A \cap B = \{1111, 0001, 0101, 1001, 1101\}$.

A simplification due to van den Berg and Fiebig reduces the general BK inequality to the case where all configurations are equally likely. We will prove it in the uniform case, referring to this theorem to infer the inequality for all percolation product measures.

Recall that a general percolation measure on $\Omega = \{0,1\}^{[n]}$ assigns to each bond b weight $\mu_b(1) = p_b$ of being open and $\mu_b(0) = 1 - p_b$ of being closed, where $\mathbb{P}_p = \prod_{b \in [n]} \mu_b$ and $p = (p_1, ..., p_n)$ is any vector of bond probabilities. The uniform case assigns $p_i = 1/2$, for all i.

Theorem 2 (van den Berg, Fiebig) *It suffices to show that the BK Inequality holds in the uniform measure, with $p = 1/2$.*

Proof 2. We will prove this in the case that all probabilities p_i are of the form $\frac{a_i}{2^k}$, for some integer k and integers a_i.

Assume that the BK inequality holds for the uniform distribution over configurations. Let $\Omega = \{0,1\}^{[n]}$ with weights $\mathbb{P}_p$ (as above) and let $A, B \subset \Omega$ be events. We will construct events $\hat{A}$ and $\hat{B}$ events in a larger state space $\hat{\Omega} = \{0,1\}^{[kn]}$ as follows: $u \in \hat{A}$ if $\exists w \in A$ such that for all $0 \leq j < n$, if $w_j = 1$ then $u_{jk+1}, ..., u_{j(k+1)} < a_i$ (when taken as a k bit binary number) and if $w_j = 0$ then $u_{jk+1}, ..., u_{j(k+1)} \geq a_i$. $\hat{B}$ is defined analogously. Then we have that

$$\mathbb{P}_{1/2}(\hat{A}) = \mathbb{P}_p(A)$$

and

$$\mathbb{P}_{1/2}(\hat{B}) = \mathbb{P}_p(B).$$

Now, suppose that for $u \in \hat{\Omega}$ there exists $w \in A \circ B$ such that, for all $0 \leq j < n$, if $w_j = 1$ then $u_{jk+1}, ..., u_{j(k+1)} < a_i$ and if $w_j = 0$ then $u_{jk+1}, ..., u_{j(k+1)} \geq a_i$. Then we can conclude that $u \in \hat{A} \circ \hat{B}$. Hence we have

$$\mathbb{P}_p(A \circ B) \;\leq\; \mathbb{P}_{1/2}(\hat{A} \circ \hat{B}) \;\leq\; \mathbb{P}_{1/2}(\hat{A})\mathbb{P}_{1/2}(\hat{B}) \;=\; \mathbb{P}_p(A)\mathbb{P}_p(B),$$

where the second inequality follows from our assumtion that the BK inequality holds for the uniform distribution.

□

This simplification of the BK Inequality is useful because it lets us consider the inequality from a combinatorial perspective. When $p = 1/2$, the BK Inequality for increasing events gives

$$\frac{|A \circ B|}{2^m} \leq \frac{|A|}{2^m}\frac{|B|}{2^m}$$

that is

$$2^m |A \circ B| \leq |A||B|. \tag{1}$$

Van den Berg and Kesten proved that the BK Inequality holds whenever A and B are increasing events, and conjectured that it holds in general. Below we present their proof which relies on a clever duplication of the state space.

Theorem 3 *The BK Inequality holds when A and B are increasing sets.*

Proof 3. Define a product space $\Omega^2 = \Omega \times \Omega$ (where $\Omega = \{0,1\}^{[m]}$ and $[m] = \{1, 2, \ldots, m\}$) and a product measure on the space, $\mathbb{P}^2 = \mathbb{P}_p \times \mathbb{P}_p$. Let $\langle x; y\rangle = \langle x; y\rangle_0 \in \Omega^2$, and denote $\langle x; y\rangle_i$ by

$$\begin{aligned}
\langle x; y\rangle_0 &= (x_1, x_2, \ldots, x_m;\ y_1, y_2, \ldots, y_m)\\
\langle x; y\rangle_1 &= (y_1, x_2, \ldots, x_m;\ x_1, y_2, \ldots, y_m)\\
\langle x; y\rangle_k &= (y_1, \ldots, y_k, x_{k+1}, \ldots, x_m;\ x_1, \ldots, x_k, y_{k+1}, \ldots, y_m)
\end{aligned}$$

Using this notation, define membership in $A', B'_k \subseteq \Omega^2$ by

$\langle x; y\rangle \in A'$ if $x \in A$

for all k, $\langle x; y\rangle \in B'_k$ if $[\langle x; y\rangle_k]_{[m]} \in B$ (i.e. if $(y_1, \ldots, y_k, x_{k+1}, \ldots, x_m) \in B$)

for all k, $\langle x; y\rangle \in A' \circ B'_k$ if $\exists I_1, I_2 \subseteq [m]$, such that $I_1 \cap I_2 \cap \{k+1, \ldots, m\} = \emptyset$, $[\langle x; y\rangle]_{I_1} \in A'$, (which implies $[x]_{I_1} \in A$) and $[\langle x; y\rangle]_{I_2} \in B'_k$, (which implies $[y_1, \ldots, y_k, x_{k+1}, \ldots, x_m]_{I_2} \in B$.

Here are two important observations:

(i) Note that $\langle x; y\rangle \in A' \circ B'_m$ if and only if $x \in A$ and $y \in B$. But they do not using any bits in common. Thus, $\mathbb{P}^2(A' \circ B'_m) = \mathbb{P}_p(A)\mathbb{P}_p(B)$.

(ii) We also have $\langle x, y\rangle \in (A' \circ B'_0) \Leftrightarrow x \in (A \circ B)$. In this case we have $I_1 \cap I_2 = \emptyset$, and $I_1 \cup I_2 = [m]$, so that $\mathbb{P}^2(A' \circ B'_0) = \mathbb{P}_p(A \circ B)$.

If we can show that for $1 \le k \le m$

$$\mathbb{P}^2(A' \circ B'_{k-1}) \le \mathbb{P}^2(A' \circ B'_k) \text{ for } 1 \le k \le m \tag{2}$$

it would imply our result because we would have

$$\begin{aligned}
\mathbb{P}_p(A \circ B) &= \mathbb{P}^2(A' \circ B')\\
&= \mathbb{P}^2(A' \circ B'_0) \le \mathbb{P}^2(A' \circ B'_1) \le \ldots \le \mathbb{P}^2(A' \circ B'_m) = \mathbb{P}_p(A)\mathbb{P}_p(B)
\end{aligned}$$

The proof of the validity of Equation 2 begins here. Let e_k be the $(0,1)$ vector of length m with zeroes in every coordinate *except* the kth coordinate, which is 1. The symbol $\oplus$ denotes bitwise symmetric difference. Let $\langle x; y\rangle \in (A' \circ B'_{k-1})$.
We'll exhibit a map $\Phi : \Omega^2 \longrightarrow \Omega^2$ which will be defined piecewise on a partition of $(A' \circ B'_{k-1})$. As we shall see, it will be injective, measure-preserving, and map elements from $(A' \circ B'_{k-1})$ into $(A \circ B_k)$.

- **Case (1):** $\langle x \oplus e_k; y\rangle \in (A' \circ B'_k)$.

  Then x_k, the kth bit of x, is irrelevant to membership in $(A' \circ B'_k)$. We therefore obtain $\langle x; y\rangle \in (A' \circ B'_k)$. Let Φ be defined as the identity map on this set.

In all other cases the value of the kth bit of x is relevant to membership in $(A' \circ B'_k)$. Then it must be that $x_k = 1$, (since A and B are increasing events, so if $x_k = 0$ implies membership in $(A' \circ B'_k)$, so does $x_k = 1$, contradicting the fact that the kth bit is relevant).

- **Case (2):** $\langle x \oplus e_k; y \rangle \notin (A' \circ B'_k)$ and $x_k = y_k = 1$.

  Then $\langle x; y \rangle \in (A' \circ B'_k)$ because nothing changed when we exchanged x_k and y_k. Again we let Φ be defined as the identity map on this set.

- **Case (3):** $\langle x \oplus e_k; y \rangle \notin (A' \circ B'_k)$ and $x_k = 1$ but $y_k = 0$.

  There are two subcases.

  (a) There exist sets $I_1, I_2 \subseteq [m]$ such that $[\langle x; y \rangle]_{I_1} \subseteq A'$, $[\langle x; y \rangle]_{I_2} \subseteq B'_k, I_1 \cap I_2 \cap \{k, \ldots, m\} = \emptyset$ and $[x]_{I_1} \in A'$: Then $\langle x; y \rangle \in (A' \circ B'_k)$, because the kth bit is irrelevant to membership in B'_{k-1}.

  (b) For all sets $I_1, I_2 \subseteq [m]$ such that $[\langle x; y \rangle]_{I_1} \in A', [\langle x; y \rangle]_{I_2} \in B'_k$, $I_1 \cap I_2 \cap \{k, \ldots, m\} = \emptyset$ we have $k \in I_1$: This implies $k \in I_2$, so that $\langle x; y \rangle \notin (A' \circ B'_k)$ (recall that membership in $(A' \circ B'_k)$ depends on the value of x_k). However, consider $\langle \hat{x}; \hat{y} \rangle = \langle x \oplus e_k; y \oplus e_k \rangle$ (that is, exchange the kth bits of x and y, since $x_k = 1$ and $y_k = 0$). Since $\langle x; y \rangle \in (A' \circ B'_{k-1})$, $\langle \hat{x}; \hat{y} \rangle \in (A' \circ B'_k)$ (replacing the kth bit of x with the kth bit of y). In this case we define Φ by $\Phi(\langle x; y \rangle = \langle \hat{x}; \hat{y} \rangle$.

To summarize these cases, $\Phi : A' \circ B'_{k-1} \to A' \circ B'_k$ by:

$$\Phi(\langle x; y \rangle) \mapsto \begin{array}{ll} \langle x \oplus e_k; y \oplus e_k \rangle & \text{if } (\langle x; y \rangle) \text{ falls in case (3b)} \\ \langle x; y \rangle & \text{otherwise} \end{array}$$

Notice that the map is injective, since if $\langle x; y \rangle$ falls into case (3b) then $\langle x; y \rangle \notin (A' \circ B'_k)$ so we cannot have two configurations mapping to the same point. Furthermore, Φ is measure-preserving (as it is either the identity map or it exchanges the kth bits of x_k and y_k which does not change the measure). Hence, $\mathbb{P}^2(\Phi(\langle x; y \rangle)) = \mathbb{P}^2(\langle x; y \rangle)$. This implies that since Φ maps elements from $(A' \circ B'_{k-1})$ into $(A \circ B_k)$, $\mathbb{P}^2(A' \circ B'_{k-1}) \leq \mathbb{P}^2(A \circ B_k)$.

This completes the proof of Equation 2, and therefore the proof of Theorem 3.

□