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Random Dyadic Tilings of the Unit Square

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The Model

A **dyadic interval** is an interval from $\frac{a}{2^s}$ to $\frac{a+1}{2^s}$, where s is a nonnegative integer and a is an integer with $0 \le a < 2^s$.

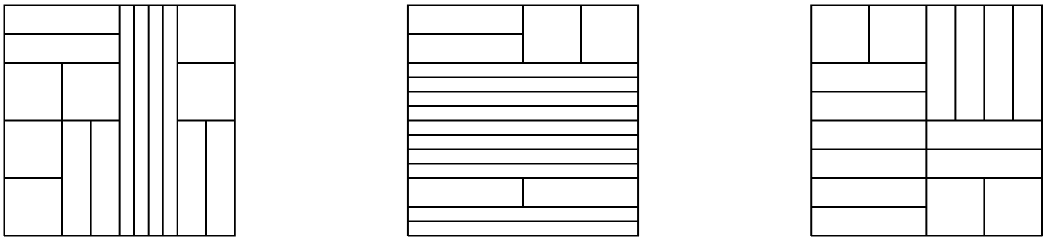
A **dyadic rectangle** is a region with dimensions

$$R = \left[\frac{a}{2^s}, \frac{a+1}{2^s}\right] \times \left[\frac{b}{2^t}, \frac{b+1}{2^t}\right]$$

where s, t and a, b are integers with $0 \le a < 2^s$ and $0 \le b < 2^t$.

An **n-tiling of the unit square** is a set of 2^n dyadic rectangles, each of area 2^{-n} (whose union is the unit square).

Examples:



a. b. c.

A tiling has a **vertical fault line** if the line $x = \frac{1}{2}$ cuts through none of its rectangles. Similarly, **horizontal fault line**.

Theorem: Every tiling has either a vertical fault line or a horizontal fault line. (It may have both.)

A Recurrence for Dyadic Tilings:

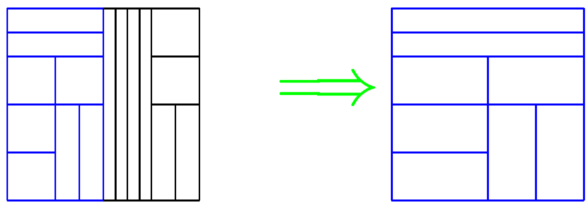
Let T_k be the set of $k-tilings$ and let A_k be the number.

$$A_0 = 1,$$
$$A_1 = 2,$$
$$A_2 = 7,$$
$$A_3 = 82,$$
$$A_4 = 11047,$$
$$A_5 = 198860242,$$
$$A_6 = 64197955389505447, \ldots$$

Theorem: [CLSW, LSV] For $n \geq 2$,

$$A_n = 2A_{n-1}^2 - A_{n-2}^4.$$

This follows from the observation that the **left half** of a tiling in T_n with a vertical cut can be **dilated** to a tiling of T_{n-1}:

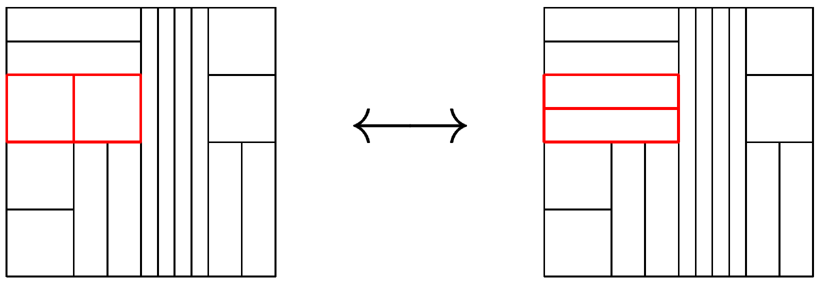


The asymptotic behavior of A_n is

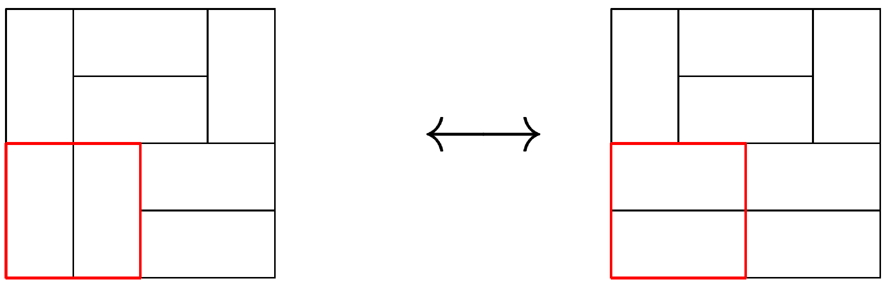
$$A_n \sim \phi^{-1} \rho^{2^n}$$

where $\rho = 1.84454757\cdots$
and $\phi = (1+\sqrt{5})/2 = 1.6180\cdots$ is the golden ratio.

Local Moves



Dyadic Tilings and Rotations



Domino Tilings and Rotations

Questions:

1. How can we sample from T_n?
2. What does a random sample in T_n look like?
3. What does T_∞ look like?

Outline:

Combinatorial structures:

The height function

Tree representations

I. Recursive sampling algorithms

II. Dynamic sampling algorithms

III. Properties of random tilings

The Lattice of Tilings

Define the **height** $h(t)$ of a dyadic $2^{-k} \times 2^{-l}$ rectangle t with area 2^{-n} to be $k = n - l$.

Let the **total height** $H(T)$ of a tiling T to be the sum of the heights of all rectangles in it.

- $0 \leq H(T) \leq n2^n, \qquad T \in \mathcal{T}_n.$
- $H(T) = 2^n \int_{[0,1]^2} h(T)(p)\, dp.$

Partial order

Let $T_1 \preceq T_2$ if $h(T_1(p)) \leq h(T_2(p))$ for all $p \in [0,1]^2$.

Theorem: The partial order on $\mathcal{T}_n$ defines a distributive lattice.

The **join** $T_1 \vee T_2$ is $\{\max(T_1(p), T_2(p)) : p \in [0,1]^2\}$,

where $\max(T_1(p), T_2(p))$ is the tile with larger height.

The **meet** $T_1 \wedge T_2$ is $\{\min(T_1(p), T_2(p)) : p \in [0,1]^2\}$.

- There are unique highest and lowest elements in $\mathcal{T}_n$: the highest tiling is the all vertical tiling and the lowest is the all horizontal tiling
- The meet and join always yield valid tilings.
- The lattice is distributive.

More on the Height Function

Let $\widetilde{T}_k$ denote the special tiling with

$$2^{-k} \times 2^{k-n}$$

rectangles, $k = 0, \ldots, n$; thus $\widetilde{T}_k$ has height function constant k.

So $\widetilde{T}_0$ is the lowest tiling and $\widetilde{T}_n$ is the highest.

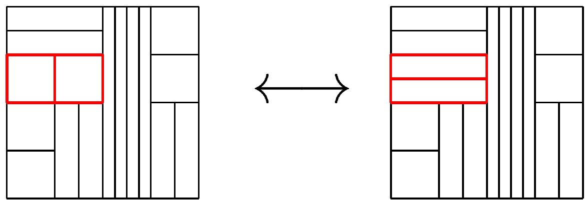
Theorem: An n-tiling T has a horizontal cut iff $T \preceq \widetilde{T}_{n-1}$.

(It has a vertical cut iff $T \succeq \widetilde{T}_1$.)

Proof: T has a horizontal cut iff it contains no $2^{-n} \times 1$ rectangle, i.e. if and only if $h(T)(p) \leq n - 1$ for every $p \in [0, 1]^2$. $\square$

Theorem: Suppose that T_1, T_2 are n-tilings with $n \geq 2$. If $T_1 \preceq T_2$, T_1 has a horizontal cut and T_2 has a vertical cut, then there exists a tiling T_3 with both vertical and horizontal cuts such that $T_1 \preceq T_3 \preceq T_2$.

Let G_n be the (oriented) graph which connect tilings that differ by an **elementary rotation** changing one edge.



Theorem: Let $T_1, T_2 \in \mathcal{T}_n$. Then $T_1 \preceq T_2$ iff there exists an oriented path from T_1 to T_2 in the directed graph G_n. Every such path has length $\frac{1}{2}H(T_2) - \frac{1}{2}H(T_1)$.

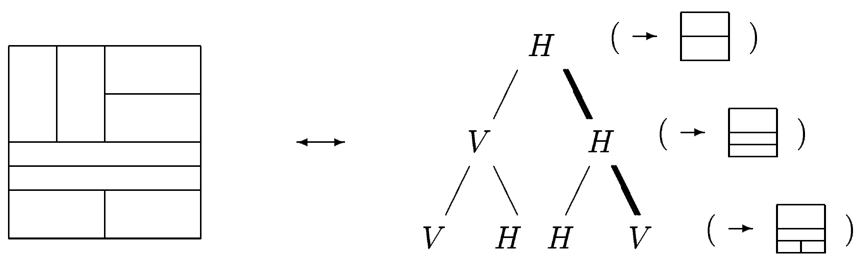
Tree Representations: HV-Trees

A complete binary tree of height n whose $2^n - 1$ nodes are labeled H or V defines an n-tiling by the following procedure:

Algorithm (HV-Tree $\rightarrow$ Tiling):

1. If the tree is empty ($n = 0$) then Exit.
2. If the root is labeled H, make a horizontal cut.

   If the root is labeled V, make a vertical cut.
3. Continue recursively with the two halves separately, using the left and right subtrees.



Conversely, every n-tiling is produced in this way by some labeled complete binary tree.

The tree is in general not unique!!!

Definition: A complete binary tree whose nodes are labeled H or V is an ***HV*-tree** if there is no node labeled H which has two children labeled V.

(I.e., we take the vertical cut if possible!!)

Theorem: There is a bijection between $\mathcal{T}_n^{HV}$ (the set of HV-trees of height n) and $\mathcal{T}_n$.

I. Recursive Algorithms for Sampling

Probabilities at the root:

$$p_n = \mathbf{P}(\text{a random tiling in } \mathcal{T}_n \text{ has a vertical cut})$$
$$= \frac{A_{n-1}^2}{A_n}, \qquad n \geq 0.$$

(The prob. of a horizontal cut is the same.)

We have $p_0 = 0$, $p_1 = 1/2$, $p_2 = 4/7$, $\ldots$.

From $A_n = 2A_{n-1}^2 - A_{n-2}^4$, we find:

$$p_n = \frac{1}{2 - p_{n-1}^2}, \qquad n \geq 1.$$

Note: It follows easily that p_n increases to the smallest positive root of $x = \frac{1}{2-x^2}$, i.e.

$$p_n \to \phi^{-1} = \phi - 1 = (\sqrt{5} - 1)/2, \qquad \text{as } n \to \infty.$$

A Recursive Construction

The ***type*** of a node in a HV-tree is one of the four symbols V, H_{HH}, H_{HV}, H_{VH}, chosen according to the following rules:

- If the node is labeled V, its type is V.
- If the node is labeled H and it is not a leaf, its type is H_{xy}, where x and y are the labels of its children.
- If the node is labeled H and it is a leaf, its type is H_{HH}.

The number of trees of type V is A_{n-1}^2 (i.e., no constraints on subtrees T_1 and T_2).

The total number of trees of the other types is $A_n - A_{n-1}^2$.

Therefore, the distribution $\tau^{(n)}$ for labels at the root are:

$$\begin{aligned}
V &: \quad A_{n-1}^2 = p_n A_n \\
H_{HH} &: \quad (A_{n-1} - A_{n-2}^2)^2 = p_n (1 - p_{n-1})^2 A_n \\
H_{HV} &: \quad A_{n-2}^2 (A_{n-1} - A_{n-2}^2) = p_n p_{n-1} (1 - p_{n-1}) A_n \\
H_{VH} &: \quad A_{n-2}^2 (A_{n-1} - A_{n-2}^2) = p_n p_{n-1} (1 - p_{n-1}) A_n
\end{aligned}$$

Recursive Generation of Random Tilings

Probabilities at all other nodes:

Let $\tau^{(n)}$ denote a random type $\tau \in \{V, H_{HH}, H_{HV}, H_{VH}\}$ with the distribution given by

$$\mathbf{P}(\tau^{(n)} = V) = p_n,$$

$$\mathbf{P}(\tau^{(n)} = H_{HH}) = p_n(1 - p_{n-1})^2,$$

$$\mathbf{P}(\tau^{(n)} = H_{HV}) = \mathbf{P}(\tau^{(n)} = H_{VH}) = p_n p_{n-1}(1 - p_{n-1}).$$

We also need **conditional probabilities**:

Let $\tau_H^{(n)}$ denote $\tau^{(n)}$ conditioned on $\tau^{(n)} \neq V$:

$$\mathbf{P}(\tau_H^{(n)} = H_{HH}) = (1 - p_{n-1})/(1 + p_{n-1})$$

$$\mathbf{P}(\tau_H^{(n)} = H_{HV}) = \mathbf{P}(\tau_H^{(n)} = H_{VH}) = p_{n-1}/(1 + p_{n-1}).$$

Recursively Generating Tilings

Recursive Algorithm:

1. Select randomly a type for the root with the distribution $\tau^{(n)}$.
2. Recursively assign types to all other nodes such that if a node of height k, $1 \leq k < n$, is assigned a type τ, then its left and right child get types τ_1 and τ_2 selected as follows:

   $\tau = V$: Choose τ_1 and τ_2, independently, both with the distribution of $\tau^{(n-k)}$.

   $\tau = H_{HH}$: Choose τ_1 and τ_2, independently, both with the distribution of $\tau_H^{(n-k)}$.

   $\tau = H_{HV}$: Choose τ_1 with the distribution of $\tau_H^{(n-k)}$ and let $\tau_2 = V$.

   $\tau = H_{VH}$: Let $\tau_1 = V$ and choose τ_2 with the distribution of $\tau_H^{(n-k)}$.

3. All vertices with type V are labeled V; the others are labeled H.

Generating Asymptotic Tilings

$$\begin{aligned}
\mathbf{P}(\tau^{(\infty)} = V) &= \phi^{-1} = \phi - 1, \\
\mathbf{P}(\tau^{(\infty)} = H_{HH}) &= \phi^{-5} = 5\phi - 8, \\
\mathbf{P}(\tau^{(\infty)} = H_{HV}) &= \phi^{-4} = 5 - 3\phi, \\
\mathbf{P}(\tau^{(\infty)} = H_{VH}) &= \phi^{-4} = 5 - 3\phi,
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}(\tau_H^{(\infty)} = V) &= 0, \\
\mathbf{P}(\tau_H^{(\infty)} = H_{HH}) &= \phi^{-3} = 2\phi - 3, \\
\mathbf{P}(\tau_H^{(\infty)} = H_{HV}) &= \phi^{-2} = 2 - \phi, \\
\mathbf{P}(\tau_H^{(\infty)} = H_{VH}) &= \phi^{-2} = 2 - \phi.
\end{aligned}$$

Recursive Asymptotic Algorithm:

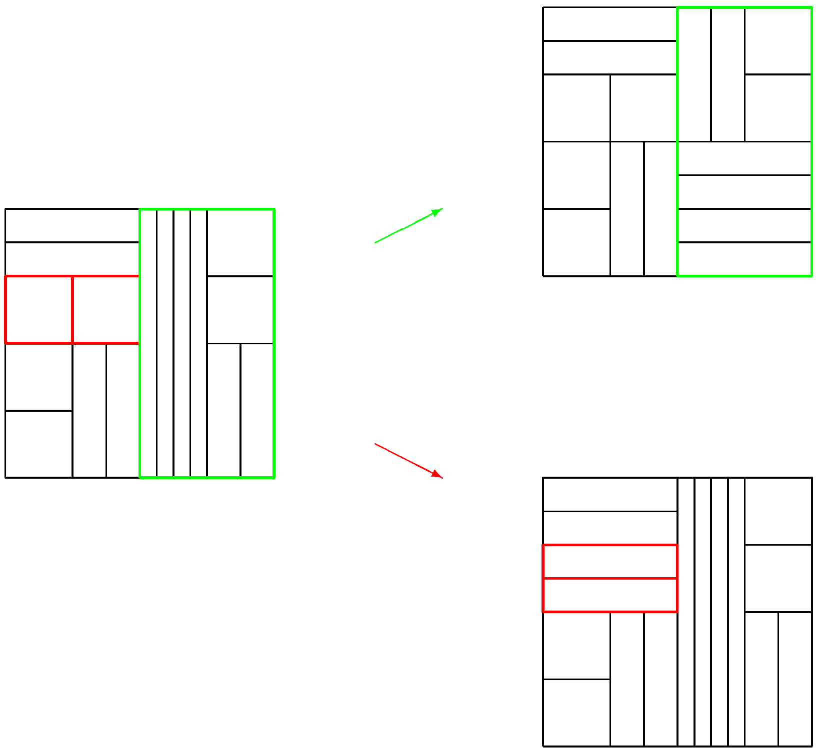
This is the same as the Recursive Algorithm, but using the distributions $\tau^{(\infty)}$ and $\tau_H^{(\infty)}$.

II. Dynamic Sampling Algorithms

Markov chain 1 (Rotations):

Repeat:

- Choose a dyadic rectangle within the square of any size
- Choose a direction $d \in \{0^o, 90^o, 180^o, 270^o\}$.
- "Rotate" the subtiling within this rectangle by d, if possible.



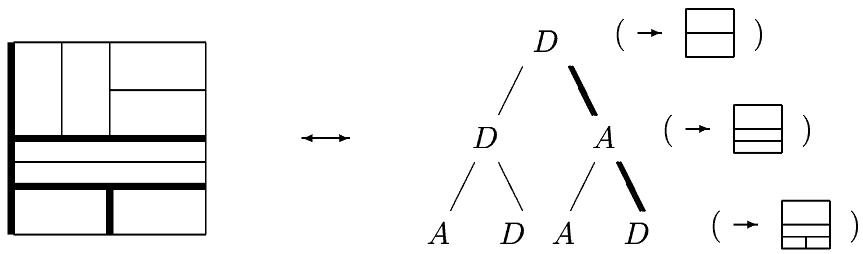
AD-Trees

Consider complete binary trees with the labels A **(agree)** or D **(disagree)** using *relative* orientation of a cut relative to its parent.

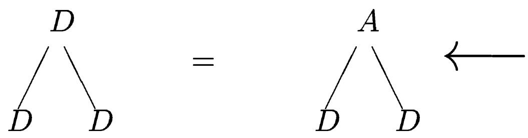
Algorithm:

1. Initialize by defining the parent cut to be the square's left edge.
2. If the tree is empty $(n = 0)$ then Exit.
3. If the root is labeled A, make a cut parallel to the parent cut.

   If the root is labeled D, make a cut orthogonal to the parent.
4. Continue recursively (from Step 2) with the two halves (setting the parent cut equal to the cut just made).



But note that:



A "badly labeled subtree"

Definition: A complete binary tree whose nodes are labeled A or D is an ***AD*-tree** if there is no node labeled A which has two children labeled D.

Theorem: There is a bijection between $\mathcal{T}_n^{AD}$ (the set of AD-trees) and $\mathcal{T}_n$.

Markov chain 2 (on AD-trees):

This MC is motivated by the AD-**tree** representation of tilings.

Repeat:

- Choose a node v in the AD-**tree**.
- Choose a label $b \in \{A, D\}$.
- Relabel v with b with prob. $1/2$ – unless it creates a *badly labeled subtree!*

Analysis of MC 2 on AD-Trees

Let $\Phi(x, y)$ be the Hamming distance between trees x and y. (I.e., the number of vertices which are assigned different labels.)

Lemma: Let $x, y \in \mathcal{T}_n^{AD}$ be any two configurations. Then there is a sequence of states $z_0, z_1, \ldots, z_d$ such that $z_0 = x$, $z_d = y$, $d = \Phi(x, y)$ and for all $0 \leq i < d$, $\Phi(z_i, z_{i+1}) = 1$.

Corollary: The Markov chain $\widetilde{\mathcal{M}}_n$ is ergodic and converges to the uniform distribution on $\mathcal{T}_n^{AD}$.

Q: How quickly??

Bounding the Mixing Rate (for MC 2)

The **variation distance** is:

$$\Delta_x(t) = \frac{1}{2}\sum_{y\in\Omega} |P^t(x,y) - \pi(y)|.$$

The **mixing time** of a Markov chain is:

$$\tau(\epsilon) = \max_x \min\{t : \ \Delta_x(t') \leq \epsilon \text{ for all } t' \geq t\}.$$

If $\tau(\epsilon)$ is polylogarithmic in the size of Ω, for fixed ϵ, then we say that the Markov chain is **rapidly mixing**.

* Recall that Ω is doubly exponential in n, so $\tau(\epsilon)$ will be exponential in n.

...it takes O(2^n) time just to write down a configuration!!

Path Coupling:

- Let Φ be a metric on $\Omega \times \Omega$ taking values in $\{0, \ldots, B\}$.
- Let $U \subseteq \Omega \times \Omega$ such that:
  1. for all x_t, y_t there exists **a path** $x_t = z_0, z_1, \ldots, z_r = y_t$ between x_t and y_t such that $(z_i, z_{i+1}) \in U$
  2. $\sum_{i=0}^{r-1} \Phi(z_i, z_{i+1}) = \Phi(x_t, y_t)$.
- $\mathbf{E}(\Delta\Phi(x_t, y_t)) \leq 0$ for all $(x_t, y_t) \in U$,
- $\mathbf{P}[\Phi(x_{t+1}, y_{t+1}) \neq \Phi(x_t, y_t)] \geq \alpha \ (> 0)$ whenever $x_t \neq y_t$.

Theorem: [Bubley, Dyer, Greenhill] The mixing time satisfies:

$$\tau(\epsilon) \leq \left\lceil \frac{\mathrm{e}B^2}{\alpha} \right\rceil \lceil \ln \epsilon^{-1} \rceil.$$

Our coupling for Alg 2:

To couple: Choose the **same vertex** and the **same label** b.

We have:

Path: Recall the Hamming distance Φ has the path property.

Diameter: $B \leq n2^n$.

Variance: $\alpha \geq 2^{-n}$.

So it suffices to show:

Expected change:

$$\mathbf{E}(\Delta\Phi(x_t, y_t)) \leq 0 \text{ for all } (x_t, y_t) \in U.$$

Showing that $\mathbf{E}(\Delta\Phi(x_t, y_t)) \leq 0$

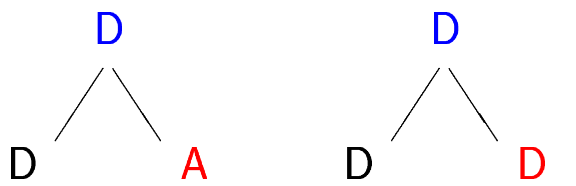
Let x_t, y_t have distance $\Phi(x_t, y_t) = 1$ and differ at vertex w.

Good Case:

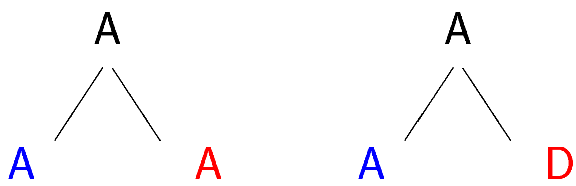
1. $v = w$: then $x_{t+1} = y_{t+1}$ for either choice of b.

(Potentially) Bad Cases:

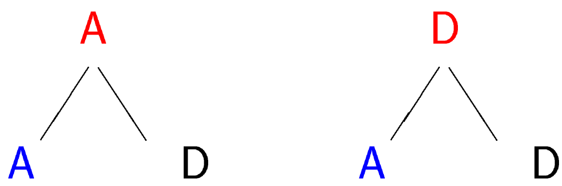
1. $v = p(w)$ (parent):



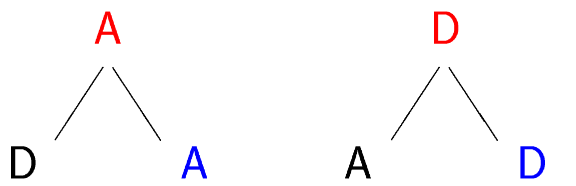
2. $v = s(w)$ (sibling):



3. $v = l(w)$ (left child):



4. $v = r(w)$ (right child):



However: bad cases 1 and 2 cannot simultaneously occur! Nor can cases 3 and 4!

So there are:

- at most 2 bad cases, with relative weight $1/2$,
- and 1 good case, with relative weight 1.

Theorem: The mixing time of the Markov chain satisfies

$$\tau(\epsilon) \leq 2^{3n}\mathrm{e}\lceil \ln \epsilon^{-1} \rceil.$$

The Natural MC on Tilings (MC 1)

The number b_n of subrectangles with area at least $2 \cdot 2^{-n}$ is:

$$b_n = \sum_{i=1}^{n-1} (k+1)2^k = (n-1)2^n.$$

The transition probabilities $P_n(\cdot)$ of our MC 1 are:
$P_n(T_1, T_2) =$

$$\begin{cases} 1/4b_n & \text{if } T_1, T_2 \text{ differ by rotating a} \\ & \text{subtiling by } \pm 90^\circ \text{ or } 180^\circ; \\ 1 - \sum_{T' \neq T_1} P_n(T_1, T') & \text{if } T_1 = T_2 \\ 0 & \text{o.w..} \end{cases}$$

Comparison of Markov Chains

We know $\widetilde{P}$ (on AD-trees) is rapidly mixing.
We want to know about P (rotations on dyadic tilings).

Theorem: [Diaconis, Saloff-Coste] Let (P, π, Ω) and $(\widetilde{P}, \pi, \Omega)$ be two reversible Markov chains such that $\widetilde{P}(x, y) \neq 0$ implies $P(x, y) \neq 0$ for all $x, y \in \Omega$. Let $\pi_* = \min_{x \in \Omega} \pi(x)$. Then, for $0 < \epsilon < 1/2$,

$$\tau(\epsilon) \leq \frac{4 \ln(1/(\epsilon \pi_*))}{\ln(1/2\epsilon)} \, A \, \widetilde{\tau}(\epsilon),$$

where

$$A = \max_{x \neq y, \widetilde{P}(x,y) > 0} \frac{\widetilde{P}(x, y)}{P(x, y)}.$$

Back to Tilings and AD-Trees

Let $x \neq y \in \Omega$ be tilings s.t. $P(x,y) > 0$.

We find

$$\frac{\widetilde{P}(x,y)}{P(x,y)} = \frac{(2|V_n|)^{-1}}{(4b_n)^{-1}} = \frac{4(n-1)\,2^n}{2(2^n-1)} \leq 2n.$$

Also,

$$\pi_*^{-1} \leq 2^{2^n}.$$

Hence:

$$\tau(\epsilon) \leq c(\epsilon) n\, 2^n\, \widetilde{\tau}(\epsilon),$$

for some constant $c(\epsilon)$,

* Thus MC 1 is also rapidly mixing.

III. What do Random Dyadic Tilings Look Like?

Total Height:

The **normalized height function** is

$$\tilde{H}(T) = 2^{-n}H(T) - n/2, \qquad T \in \mathcal{T}_n.$$

This gives us that $-n/2 \le \tilde{H}(T) \le n/2$.

By symmetry, $\mathbf{E}\,\tilde{H}_n = 0$.

Theorem: There exists a sym. r.v. $\tilde{H}_\infty$ s.t.

- As $n \to \infty$, $\tilde{H}_n \xrightarrow{d} \tilde{H}_\infty$.
- For any real t,
$$\mathbf{E}\exp(t\tilde{H}_n) \le \exp(\tfrac{1}{4}\phi^4 t^2), \qquad 1 \le n \le \infty.$$
- For any $a \ge 0$,
$$\mathbf{P}(\tilde{H}_n \ge a) \le \exp(-\phi^{-4}a^2), \qquad 1 \le n \le \infty.$$
- $\mathbf{Var}\,\tilde{H}_\infty = \mathbf{E}\,\tilde{H}_\infty^2 = (6\phi - 2)/11$
$$= (3\sqrt{5} + 1)/11 = 0.7007458\cdots.$$

For the unnormalized height (in $\{0,\ldots,n\}$), if heights were independent, the variance would be at most $n^2 2^n$.

Here we find:

$$\begin{aligned}\mathbf{Var}\, H_n &= 2^{2n}\,\mathbf{Var}\,\tilde{H}_n \\ &\sim 2^{2n}\,\mathbf{Var}\,\tilde{H}_\infty = 2^{2n}(3\sqrt{5}+1)/11\end{aligned}$$

Hence there is very high correlation.

Long thin rectangles force lots of other long thin rectangles!

"Struts"

A subrectangle of the unit square is a **strut** if it spans the unit square vertically (i.e., its height is n).

Let $S_n(T)$ be the number of struts in a random tiling T of $\mathcal{T}_n$.

What is the distribution of $S_n(T)$??

- T has a horizontal cut iff there are no struts, i.e. if $S_n(T) = 0$. Hence,

$$\mathbf{P}(S_n = 0) = p_n \to \phi - 1.$$

- Tiles are struts iff in the HV-tree all nodes on the path are labeled V (and this produces two struts).

  Thus S_n equals twice the number of such paths in a random HV-tree.

Theorem: $S_n/(\sqrt{5} - 1)^n \xrightarrow{d} Z$ as $n \to \infty$, for some random variable Z such that:

- $\mathbf{P}(Z = 0) = \lim_{n\to\infty} \mathbf{P}(S_n = 0) = \phi - 1$.
- $\mathbf{E}\, Z = \beta$ and $\mathbf{Var}\, Z = 2\phi\beta^2$,
  where $\beta = \prod_{n=1}^{\infty}(p_n\phi) = 0.702845\cdots$.