
An Extension of Foster's Network Theorem

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For Paul Erdős on his 80th birthday

Consider an electrical network on n nodes with resistors r_{ij} between nodes i and j . Let R_{ij} denote the *effective resistance* between the nodes. Then Foster's Theorem [5] asserts that

$$\sum_{i \sim j} \frac{R_{ij}}{r_{ij}} = n - 1,$$

where $i \sim j$ denotes i and j are connected by a finite r_{ij} . In [10] this theorem is proved by making use of random walks. The classical connection between electrical networks and reversible random walks implies a corresponding statement for reversible Markov chains. In this paper we prove an elementary identity for ergodic Markov chains, and show that this yields Foster's theorem when the chain is time-reversible.

We also prove a generalization of a *resistive inverse* identity. This identity was known for resistive networks, but we prove a more general identity for ergodic Markov chains. We show that time-reversibility, once again, yields the known identity. Among other results, this identity also yields an alternative characterization of reversibility of Markov chains (see Remarks 1 and 2 below). This characterization, when interpreted in terms of electrical currents, implies the *reciprocity theorem* in single-source resistive networks, thus allowing us to establish the equivalence of *reversibility* in Markov chains and *reciprocity* in electrical networks.

1. Foster's Theorem

Let $P = (P_{ij})$ denote the n by n transition probability matrix of an ergodic Markov chain with stationary distribution π , and let us assume that $P_{ii} = 0$ for all i . Furthermore, let $H = (H_{ij})$ denote the expected first-passage matrix (also of size $n \times n$) of the above chain. Thus H_{ij} denotes the expected time it takes to reach state j from state i . We call the H_{ij} the *hitting times*. Here then is our result, which will easily imply Foster's Theorem.

Theorem 1. *With the notation above,*

$$\sum_{i,j} \pi_j P_{ji} H_{ij} = n - 1.$$

We give two elementary proofs of this identity. Having found the first, the author realized that the shorter second proof is hidden in a proof of [2, Theorem 1], unbeknown to the authors of [2].

Proof 1. Let N_k^{ij} denote the expected number of visits to k in a random walk from i to j . Then, [8, equation (34) (page 221)] implies that, for $k \neq j$,

$$\sum_i P_{ji} N_k^{ij} = \frac{\pi_k}{\pi_j}.$$

That is

$$\sum_i \pi_j P_{ji} N_k^{ij} = \pi_k.$$

Summing both sides over $k (\neq j)$,

$$\sum_{k \neq j} \sum_i \pi_j P_{ji} N_k^{ij} = 1 - \pi_j,$$

which implies

$$\sum_i \pi_j P_{ji} \sum_{k \neq j} N_k^{ij} = 1 - \pi_j,$$

so

$$\sum_i \pi_j P_{ji} H_{ij} = 1 - \pi_j.$$

Finally, summing over j , we obtain

$$\sum_{i,j} \pi_j P_{ji} H_{ij} = \sum_j (1 - \pi_j) = n - 1. \quad \square$$

Proof 2. Note simply that

$$\sum_{i,j} \pi_j P_{ji} H_{ij} = \sum_j \pi_j \left(\sum_i P_{ji} H_{ij} \right) = \sum_j \pi_j [H_{jj} - 1].$$

Since $H_{jj} = 1/\pi_j$, this implies that

$$\sum_{i,j} \pi_j P_{ji} H_{ij} = \sum_j \pi_j [1/\pi_j - 1] = n - 1. \quad \square$$

It is well known that a Markov chain is time-reversible if, and only if, it can be represented as a random walk on an undirected weighted graph. Moreover, if the weights are interpreted to be electrical conductors (inverses of *resistors*), there is a pleasant correspondence between the electrical properties of such resistor networks and reversible Markov chains ([4, 1, 10]).

More precisely, given an undirected graph with weight $c_{ij} = c_{ji}$ on the edge ij , define a random walk with transition probability matrix $P = (P_{ij})$, where $P_{ij} = c_{ij}/\sum_j c_{ij}$. If ij is not an edge then $P_{ij} = 0$; in particular, $P_{ii} = 0$ for all i . This walk has the stationary measure $\pi_i = \sum_j c_{ij}/C$, where $C = \sum_{i,j} c_{ij}$. Reversibility follows from the fact

that $\pi_i P_{ij} = \pi_j P_{ji} = c_{ij}/C$. Using the classical interpretation of c_{ij} as the conductance $1/r_{ij}$, Chandra *et al.* [1] showed that

$$H_{ij} + H_{ji} = CR_{ij}, \quad (1)$$

where R_{ij} is the *effective resistance* between i and j .

Given all this, it is easy to deduce Foster's Theorem from Theorem 1. Indeed, as P is reversible and (1) holds, we have

$$\begin{aligned} \sum_{i,j} \pi_j P_{ji} H_{ij} &= \sum_{i < j} [\pi_i P_{ij} H_{ji} + \pi_j P_{ji} H_{ij}] \\ &= \sum_{i < j} \frac{c_{ij}}{C} [H_{ji} + H_{ij}] \\ &= \sum_{i < j} \frac{c_{ij}}{C} [C \cdot R_{ij}] \\ &= \sum_{i < j} \frac{R_{ij}}{r_{ij}}. \end{aligned}$$

2. Reciprocity and reversibility

The following resistive inverse identity is well known in electrical network theory. Given conductances c_{ij} and an all-pairs effective resistance matrix $R = \{R_{ij}\}$, with $R_{ii} = 0$ for all i , define two $(n-1) \times (n-1)$ matrices $\bar{c} = (\bar{c}_{ij})$ and $\bar{R} = (\bar{R}_{ij})$ by the formulae

$$\bar{c}_{ii} = \sum_{\substack{j=1 \\ j \neq i}}^n c_{ij}, \quad \bar{c}_{ij} = -c_{ij}, \quad 1 \leq i, j \leq n-1,$$

$$\bar{R}_{ij} = [R_{in} + R_{nj} - R_{ij}]/2, \quad 1 \leq i, j \leq n-1.$$

Then \bar{c} is the inverse of \bar{R} :

$$\bar{c}\bar{R} = \bar{R}\bar{c} = I_{n-1},$$

where I_{n-1} is the identity matrix of order $n-1$. This identity can be generalized as follows.

Let $P = (P_{ij})$ be a probability transition matrix of an ergodic Markov chain on n states, with $P_{ii} = 0$ for all i . Define an $(n-1) \times (n-1)$ matrix $\bar{P} = (\bar{P}_{ij})$ by setting, for $1 \leq i, j \leq n-1$,

$$\bar{P}_{ii} = \pi_i \left(= \sum_{\substack{j=1 \\ j \neq i}}^n \pi_j P_{ij} \right) \quad \text{and} \quad \bar{P}_{ij} = -\pi_i P_{ij}.$$

Furthermore, for $1 \leq j, k \leq n-1$ and $j \neq k$, let $\bar{H}_{jj} = H_{jn} + H_{nj}$, and $\bar{H}_{jk} = H_{jn} + H_{nk} - H_{jk}$.

Theorem 2. *With the notation above,*

$$\bar{P}\bar{H} = \bar{H}\bar{P} = I_{n-1}.$$

Proof. The basic identity we use is the triangle inequality for hitting times. [9, Proposition 9-58] asserts that

$$H_{xz} + H_{zy} - H_{xy} = \frac{N_y^{xz}}{\pi_y}, \quad (2)$$

$$H_{xz} + H_{zx} = \frac{N_x^{xz}}{\pi_x}. \quad (3)$$

Recall that N_y^{xz} denotes the expected number of visits to y in a random walk from x to z . From (2) and (3) we have, for all j and k ,

$$\bar{H}_{jk} = \frac{N_k^{jn}}{\pi_k}.$$

Now consider the summation implicit in the statement of the theorem:

$$\begin{aligned} \sum_{j=1}^{n-1} \bar{P}_{ij} \bar{H}_{jk} &= \bar{P}_{ii} \bar{H}_{ik} + \sum_{\substack{j=1 \\ j \neq i}}^{n-1} \bar{P}_{ij} \bar{H}_{jk} \\ &= \frac{\pi_i}{\pi_k} \frac{N_k^{in}}{\pi_k} - \sum_{\substack{j=1 \\ j \neq i}}^{n-1} \pi_i P_{ij} \frac{N_k^{jn}}{\pi_k} \\ &= \frac{\pi_i}{\pi_k} \left[N_k^{in} - \sum_{\substack{j=1 \\ j \neq i}}^{n-1} P_{ij} N_k^{jn} \right]. \end{aligned}$$

By taking means conditional on the first outcome, we see that the last expression is equal to

$$\frac{\pi_i}{\pi_k} [\delta_{ik}] = \delta_{ik}.$$

An analogous argument shows that $\sum_{j=1}^{n-1} \bar{P}_{ji} \bar{H}_{kj} = \delta_{ik}$, completing the proof. \square

Remark 1. For reversible chains, we have $\bar{H}_{jk} = \bar{H}_{kj}$. This is because, for all i and j ,

$$H_{jn} + H_{nk} + H_{kj} = H_{jk} + H_{kn} + H_{nj}, \quad (4)$$

A proof of this can be found in [3], alternatively, (4) can be verified directly by using the formula for the hitting times in terms of either resistances (see [10]) or the *fundamental matrix* (see [8]). Thus the proof of Theorem 2 becomes simpler for the reversible case; in particular, we do not need to use equations (2) and (3). Note that the resistive inverse identity follows by using the analogs mentioned in the previous section: essentially, $\pi_i P_{ij} = c_{ij}/C$ and $H_{ij} + H_{ji} = CR_{ij}$, for all i, j .

Remark 2. Another interesting consequence of Theorem 2 is that the property in (4) is not only necessary but also sufficient to imply reversibility. Indeed, (4) implies that \bar{H} is symmetric, which, in turn, implies that \bar{P} is symmetric, i.e., $\pi_i P_{ij} = \pi_j P_{ji}$ for all i, j .

We now show that identity (4) has an interesting electrical interpretation. First, recall the following *reciprocity theorem* from electrical networks (see [7] for a proof).

Theorem 3. *The voltage V across any branch of a network, due to a single current source I anywhere else in the network, is equal to the voltage across the branch at which the source was originally located if the source is placed at the branch across which the voltage V was originally measured.*

Using the techniques from [4], it was shown in [10] that, for any network of unit resistors,

- (a) the induced voltage V_{zy} with a unit current flowing into x and out of y is equal to $N_z^{xy}/d(z)$, where $d(z)$ is the degree of z , and
- (b) the reciprocity theorem is equivalent to the fact that $N_z^{xy}/d(z) = N_x^{zy}/d(x)$ for all x, z and $y (\neq x, z)$.

Essentially the same proof can be used to show that the reciprocity theorem in *general* resistive networks (i.e., not necessarily with unit resistors) is equivalent to the statement that

$$N_z^{xy}/\pi(z) = N_x^{zy}/\pi(x)$$

holds for all x, z , and $y (\neq x, z)$. Using relation (2) above, it is easy to see that this is the same as identity (4). In view of Remarks 1 and 2 above, we have thus established the following assertion.

Corollary 4. *Reversibility in ergodic Markov chains is equivalent to reciprocity in electrical networks.*

Corollary 5. *Given P and π , the hitting times (H_{ij}) can be computed with a single matrix inversion, and conversely, given the hitting times, P and π can be computed with a single matrix inversion.*

Proof. In view of Theorem 2, we only need to show:

- (a) how to compute H from \bar{H} , and
 - (b) to compute P from \bar{P} .
- (a) For $1 \leq i, j \leq n-1$, we have

$$H_{in} = \sum_k N_k^{in} = \sum_k \pi_k \bar{H}_{ik},$$

$$H_{ni} = \bar{H}_{ii} - H_{in},$$

and

$$H_{ij} = H_{in} + H_{nj} - \bar{H}_{ij}.$$

Thus we can first compute H_{in} and H_{ni} , for all $i < n$, and then compute H_{ij} for $1 \leq i, j \leq n-1$.

- (b) We need to compute π_n and P_{ni} , since the rest of the information is available in \bar{P} . Since π is stochastic and $\pi P = \pi$, we have

$$\pi_n = 1 - \sum_{i < n} \pi_i = 1 - \sum_{i < n} \bar{P}_{ii}$$

and

$$\pi_n P_{ni} = \pi_i - \sum_{j \neq i, n} \pi_j P_{ji} = \sum_{j \neq n} \bar{P}_{ji}. \quad \square$$

Remark 3. We defined \bar{P} and \bar{H} by treating n as a special state of the chain. Clearly, we could have chosen any other state j and carried out a similar analysis.

Remark 4. In the reversible case, we can interpret \bar{H} as \bar{R} , and using part(a) of the proof of Corollary 5, we can write a formula for the hitting times in terms of effective resistances. This gives an alternative proof of the main result in [10]:

$$H_{ij} = \frac{1}{2} \sum_k c(k) \cdot [R_{ij} + R_{jk} - R_{ik}],$$

where $c(k)$ is the sum $\sum_w c_{kw}$ of the conductances at node k .

Remark 5. [8, Theorem 4.4.12] gives an alternative way of computing the chain, given all-pairs hitting times. However, the method outlined above seems simpler, since the solution can be written in essentially one equation – Theorem 2.

Finally we comment that these identities are useful in designing randomized *on-line algorithms* (essentially extending several results of [2]), and we refer to [11] for this work. The author thought of Theorem 2 while trying to extend the results of [2].

Note added in proof

Thanks to David Aldous, Theorem 1 has further been generalized as follows (see [11]). For any n -state ergodic Markov chain, we have $\sum_{i,j} \pi_i P_{ij} H_{ij} \leq n-1$, with equality only under reversibility of the chain.

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