Three-coloring triangle-free graphs on surfaces IV. Bounding face sizes of 4-critical graphs*

Zdeněk Dvořák[†] Daniel Král'[‡] Robin Thomas[§]

Abstract

Let G be a 4-critical graph with t triangles, embedded in a surface of genus g. Let c be the number of 4-cycles in G that do not bound a 2-cell face. We prove that

$$\sum_{f \text{ face of } G} (|f| - 4) \le \kappa(g + t + c - 1)$$

for a fixed constant κ, thus generalizing and strengthening several known results. As a corollary, we prove that every triangle-free graph G embedded in a surface of genus g contains a set of $O(g)$ vertices such that $G - X$ is 3-colorable.

1 Introduction

This paper is a part of a series aimed at studying the 3-colorability of graphs on a fixed surface that are either triangle-free, or have their triangles restricted in some way. Historically the first result in this direction is the following classical theorem of Grötzsch [7].

Theorem 1.1. *Every triangle-free planar graph is* 3*-colorable.*

Thomassen [14, 15, 17] found three reasonably simple proofs of this claim. Recently, two of us, in joint work with Kawarabayashi [1] were able to design a linear-time algorithm to 3-color triangle-free planar graphs, and as a by-product found perhaps a yet simpler proof of Theorem 1.1. Kostochka and Yancey [11] gave a completely different proof as a consequence of their results on critical

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†Computer Science Institute (CSI) of Charles University, Malostranské náměstí 25, 118 00 Prague, Czech Republic. E-mail: `rakdver@iuuk.mff.cuni.cz`. Supported by the Center of Excellence – Inst. for Theor. Comp. Sci., Prague, project P202/12/G061 of Czech Science Foundation.

‡Warwick Mathematics Institute, DIMAP and Department of Computer Science, University of Warwick, Coventry CV4 7AL, United Kingdom. E-mail: `D.Kral@warwick.ac.uk`.

§School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332. E-mail: `thomas@math.gatech.edu`. Partially supported by NSF Grants No. DMS-0739366 and DMS-1202640.

graphs. The statement of Theorem 1.1 cannot be directly extended to any surface other than the sphere. In fact, for every non-planar surface Σ there are infinitely many 4-critical graphs that can be embedded in Σ (a graph is 4-critical if it is not 3-colorable, but all its proper subgraphs are 3-colorable).

Gimbel and Thomassen [6] obtained the following elegant characterization of 4-critical triangle-free graphs embedded in the projective plane. A graph embedded in a surface is a *quadrangulation* if every face is bounded by a cycle of length four.

Theorem 1.2. *A triangle-free graph embedded in the projective plane is* 3*-colorable if and only if it has no subgraph isomorphic to a non-bipartite quadrangulation of the projective plane.*

However, on other surfaces the structure of triangle-free 4-critical graphs appears more complicated. Thus we aim not for a complete characterization, but instead for a quantitative bound that can be used in applications—roughly, that each 4-critical triangle-free graph embedded in a fixed surface has only a bounded number of faces of length greater than 4. One difficulty that needs to be dealt with is that the claim as stated is false: there exist 4-critical plane graphs with arbitrarily many faces of length 5 and with exactly 4 triangles, see [13] for a construction. By replacing an edge in each of the triangles by a suitably chosen triangle-free graph, one can obtain a 4-critical triangle-free graph embedded in a surface of bounded genus and with arbitrarily many 5-faces. However, the examples constructed in this way contain a large number of non-contractible 4-cycles. This turns out to be the case in general, as shown by the main result of this paper, the following.

Theorem 1.3. *There exists a constant κ with the following property. Let G be a graph embedded in a surface of Euler genus g. Let t be the number of triangles in G and let c be the number of* 4*-cycles in G that do not bound a* 2*-cell face. If G is* 4*-critical, then*

$$\sum_{f \text{ face of } G} (|f| - 4) \leq \kappa(g + t + c - 1).$$

The first step towards this result was obtained in the previous paper of this series [4], where we showed that Theorem 1.3 holds under the additional assumption that the girth of G is at least five (this special case strengthens a theorem of Thomassen [16]). The inclusion of triangles and non-facial 4-cycles in the statement is merely a technicality, as (≤ 4)-cycles can be eliminated by replacing their edges by suitable graphs of girth 5 at the expense of increasing the genus. Hence, the main contribution of this paper is dealing with 4-faces.

Theorem 1.3 is the cornerstone of this series. We will use it in future papers to deduce the following results. First, in [5] (whose preliminary version appeared as [2]) we will rely on the fact that the bound in Theorem 1.3 is linear in t to answer affirmatively a question of Havel [8] whether every planar graph with triangles sufficiently far apart is 3-colorable. Second, we will use Theorem 1.3

and another result about 3-coloring graphs with most faces bounded by 4-cycles to design a polynomial-time algorithm to test whether an input triangle-free graph embedded in a fixed surface is 3-colorable. That settles a problem of Gimbel and Thomassen [6] and completes one of two missing steps in a research program initiated by Thomassen [16]. The latter asks whether for fixed integers k and q the k-colorability of graphs of girth q on a fixed surface can be tested in polynomial time. (The other step concerns 4-coloring graphs on a fixed surface, and prospects for its resolution in the near future are not very bright at the moment.) With additional effort we will be able to implement our algorithm to run in linear time. Third, we will show that every triangle-free graph with an embedding of large edge-width in an orientable surface is 3-colorable. That generalizes a theorem of Hutchinson [9], who proved it under the stronger assumption that every face is even-sided.

More immediately, we apply Theorem 1.3 to prove another generalization of Grötzsch's theorem. Kawarabayashi and Thomassen [10] proved that there exists a function f such that every triangle-free graph G embedded in a surface of Euler genus g contains a set X of at most $f(g)$ vertices such that $G - X$ is 3-colorable. They prove the claim for a function $f(g) = O(g \log g)$, and believe that using a significantly more involved argument, they can improve the bound to linear (which is the best possible, as exemplified by a disjoint union of copies of the Grötzsch graph). Using the theory we develop, it is easy to prove this claim.

Theorem 1.4. *There exists a constant $\beta > 0$ with the following property. Every triangle-free graph G embedded in a surface of Euler genus g contains a set X of at most βg vertices such that $G - X$ is 3-colorable.*

The proof of Theorem 1.3 follows the method we developed in [3, 4]: we show that collapsing a 4-face (a standard reduction used when dealing with embedded triangle-free graphs) does not decrease a properly defined weight. By induction (with the base case given by the result of [4] for graphs of girth 5), this shows that the weight of every 4-critical embedded graph is bounded. As the contribution of each (≥ 5)-face to the weight is positive and linear in its size, the bound of Theorem 1.3 follows. Several technical difficulties are hidden in this brief exposition, however we worked out the solutions for them in [3, 4]. The main obstacle not encountered before is the need to avoid creating new non-facial 4-cycles in the reduction, which we deal with in Lemma 5.2.

Definitions and auxiliary results from [3, 4] that we are going to need in the proof are introduced in Sections 2 and 3. In Section 5, we prove a version of Theorem 1.3 generalized to allow some of the vertices to be precolored. Theorem 1.3 follows as a straightforward corollary as outlined in Section 4. Theorem 1.4 is proved in Section 6.

2 Definitions

All graphs in this paper are simple, with no loops or parallel edges.

A *surface* is a compact connected 2-manifold with (possibly null) boundary. Each component of the boundary is homeomorphic to a circle, and we call it a *cuff*. For non-negative integers a, b and c, let $\Sigma(a, b, c)$ denote the surface obtained from the sphere by adding a handles, b crosscaps and removing the interiors of c pairwise disjoint closed discs. A standard result in topology shows that every surface is homeomorphic to $\Sigma(a, b, c)$ for some choice of a, b and c. Note that $\Sigma(0, 0, 0)$ is a sphere, $\Sigma(0, 0, 1)$ is a closed disk, $\Sigma(0, 0, 2)$ is a cylinder, $\Sigma(1, 0, 0)$ is a torus, $\Sigma(0, 1, 0)$ is a projective plane and $\Sigma(0, 2, 0)$ is a Klein bottle. The *Euler genus* $g(\Sigma)$ of the surface $\Sigma = \Sigma(a, b, c)$ is defined as $2a + b$. For a cuff C of Σ, let $\hat{C}$ denote an open disk with boundary C such that $\hat{C}$ is disjoint from Σ, and let $\Sigma + \hat{C}$ be the surface obtained by gluing Σ and $\hat{C}$ together, that is, by closing C with a patch. Let $\hat{\Sigma} = \Sigma + \hat{C}_1 + \ldots + \hat{C}_c$, where $C_1, \ldots, C_c$ are the cuffs of Σ, be the surface without boundary obtained by patching all the cuffs.

Consider a graph G embedded in the surface Σ; when useful, we identify G with the topological space consisting of the points corresponding to the vertices of G and the simple curves corresponding to the edges of G. A *face* f of G is a maximal connected subset of $\hat{\Sigma} - G$ (let us remark that this definition is somewhat non-standard, and in particular the face is not necessarily a subset of the surface). The boundary of a face is equal to the union of closed walks of G, which we call the *boundary walks* of f (in case that G contains an isolated vertex, this vertex forms a boundary walk by itself).

An embedding of G in Σ is *normal* if each cuff C that intersects G either does so in exactly one vertex v or is equal to a cycle B in G. In the former case, we call the subgraph with vertex set $\{v\}$ and no edges a *vertex ring* (for briefness, we will sometimes refer to the vertex v as a vertex ring) and the face of G that contains C the *cuff face* of v. In the latter case, note that B is the boundary walk of the face $\hat{C}$ of G; we say that B is a *facial ring*. A *ring* is a vertex ring or a facial ring. A face of G is a *ring face* if it is equal to $\hat{C}$ for some ring C, and it is *internal* otherwise (in particular, the cuff face of a vertex ring is internal). We write $F(G)$ for the set of internal faces of G. A vertex v of G is a *ring vertex* if v is incident with a ring (i.e., it is drawn in the boundary of Σ), and *internal* otherwise. A cycle K in G *separates the surface* if $\hat{\Sigma} - K$ has two components, and K is *non-separating* otherwise. A subgraph H of G is *contractible* if there exists a closed disk $\Delta \subseteq \Sigma$ such that $H \subset \Delta$ (in particular, a cycle K is contractible if there exists a closed disk $\Delta \subseteq \Sigma$ with boundary equal to K). The subgraph H *surrounds a cuff* C if H is not contractible in Σ, but it is contractible in $\Sigma + \hat{C}$. We say that H *surrounds a ring* R if H surrounds the cuff incident with R. We say that H is *essential* if it is not contractible and it does not surround any cuff.

Let G be a graph embedded in a surface Σ, let the embedding be normal, and let $\mathcal{R}$ be the set of vertex rings and facial rings of this embedding. In those circumstances we say that G is a *graph in* Σ *with rings* $\mathcal{R}$. Furthermore, some vertex rings are designated as *weak vertex rings*.

The length $|R|$ of a facial ring is the number of edges of R. For a vertex ring R, we define $|R| = 0$ if R is weak and $|R| = 1$ otherwise. For an internal face f, by $|f|$ we mean the sum of the lengths of the boundary walks of f (in particular, if an edge appears twice in the boundary walks, it contributes 2 to $|f|$); if a boundary walk consists just of a vertex ring R, it contributes $|R|$ to $|f|$. For a set of rings $\mathcal{R}$, let us define $\ell(\mathcal{R}) = \sum_{R\in\mathcal{R}} |R|$.

Let G be a graph with rings $\mathcal{R}$. A *precoloring* of $\mathcal{R}$ is a proper 3-coloring of the graph $H = \bigcup\mathcal{R}$. Note that H is a (not necessarily induced) subgraph of G. A precoloring ϕ of $\mathcal{R}$ *extends to a* 3*-coloring of* G if there exists a proper 3-coloring ψ of G such that $\phi(v) \neq \psi(v)$ for every weak vertex ring v and $\phi(v) = \psi(v)$ for every other vertex v incident with one of the rings. The graph G is $\mathcal{R}$*-critical* if $G \neq H$ and for every proper subgraph G' of G that contains $\mathcal{R}$, there exists a precoloring of $\mathcal{R}$ that extends to a 3-coloring of G', but not to a 3-coloring of G. If $\mathcal{R}$ consists of a single ring R, then we abbreviate $\{R\}$-critical to R*-critical*. For a precoloring ϕ of the rings, the graph G is ϕ*-critical* if $G \neq H$ and ϕ does not extend to a 3-coloring of G, but it extends to a 3-coloring of every proper subgraph of G that contains $\mathcal{R}$.

Let us remark that if G is ϕ-critical for some ϕ, then it is $\mathcal{R}$-critical, but the converse is not true (for example, consider a graph consisting of a single facial ring with two chords). On the other hand, if ϕ is a precoloring of the rings of G that does not extend to a 3-coloring of G, then G contains a (not necessarily unique) ϕ-critical subgraph.

Let G be a graph in a surface Σ with rings $\mathcal{R}$. A face is *open* 2*-cell* if it is homeomorphic to an open disk. A face is *closed* 2*-cell* if it is open 2-cell and bounded by a cycle. A face f is *omnipresent* if it is not open 2-cell and each of its boundary walks is either a vertex ring or a cycle bounding a closed disk $\Delta \subseteq \hat{\Sigma} \setminus f$ containing exactly one ring.

3 Auxiliary results

First, let us state several simple properties of critical graphs (proofs can be found in [3]).

Lemma 3.1. *Let G be a graph in a surface Σ with rings $\mathcal{R}$. If G is $\mathcal{R}$-critical, then every internal vertex of G has degree at least three.*

Lemma 3.2. *Let G be a triangle-free graph in a surface Σ with rings $\mathcal{R}$. Suppose that each component of G is a planar graph containing exactly one of the rings. If G is $\mathcal{R}$-critical, then each component of G is* 2*-connected.*

Furthermore, the following claim was proved in [4].

Lemma 3.3. *Let G be an $\{R_1, R_2\}$-critical graph embedded in the cylinder, where each of R_1 and R_2 is a vertex ring. If every cycle of length at most* 4 *in G is non-contractible, then G consists of R_1, R_2 and an edge between them. In particular, neither R_1 nor R_2 is weak.*

Throughout the rest of the paper, let $s : \{2, 3, 4, \ldots\} \to \mathbb{R}$ be the function defined by $s(l) = 0$ for $l \leq 4$, $s(5) = 4/4113$, $s(6) = 72/4113$, $s(7) = 540/4113$,

$s(8) = 2184/4113$ and $s(l) = l - 8$ for $l \geq 9$. Based on this function, we assign weights to the faces. Let G be a graph embedded in Σ with rings $\mathcal{R}$. For an internal face f of G, we define $w(f) = s(|f|)$ if f is open 2-cell and $w(f) = |f|$ otherwise. We define $w(G, \mathcal{R})$ as the sum of $w(f)$ over all internal faces f of G.

The main result of [3] bounds the weight of graphs embedded in the disk with one ring.

Theorem 3.4. *Let G be a graph of girth at least 5 embedded in the disk with one ring R of length $l \geq 5$. If G is R-critical, then $w(G, \{R\}) \leq s(l-3) + s(5)$.*

This is extended to general surfaces in [4]. Let $\mathrm{gen}(g, t, t_0, t_1)$ be a function defined for non-negative integers g, t, t_0 and t_1 such that $t \geq t_0 + t_1$ as

$$\mathrm{gen}(g, t, t_0, t_1) = 120g + 48t - 4t_1 - 5t_0 - 120.$$

Let $\mathrm{surf}(g, t, t_0, t_1)$ be a function defined for non-negative integers g, t, t_0 and t_1 such that $t \geq t_0 + t_1$ as

- $\mathrm{surf}(g, t, t_0, t_1) = \mathrm{gen}(g, t, t_0, t_1) + 116 - 42t = 8 - 4t_1 - 5t_0$ if $g = 0$ and $t = t_0 + t_1 = 2$,
- $\mathrm{surf}(g, t, t_0, t_1) = \mathrm{gen}(g, t, t_0, t_1) + 114 - 42t = 6t - 4t_1 - 5t_0 - 6$ if $g = 0$, $t \leq 2$ and $t_0 + t_1 < 2$, and
- $\mathrm{surf}(g, t, t_0, t_1) = \mathrm{gen}(g, t, t_0, t_1)$ otherwise.

A graph G embedded in a surface Σ with rings $\mathcal{R}$ has *internal girth at least five* if every (≤ 4)-cycle in G is equal to one of the rings. Let $t_0(\mathcal{R})$ and $t_1(\mathcal{R})$ be the number of weak and non-weak vertex rings in $\mathcal{R}$, respectively.

Theorem 3.5 ([4, Theorem 6.2])**.** *There exists a constant η_0 with the following property. Let G be a graph embedded in a surface Σ with rings $\mathcal{R}$. If G is $\mathcal{R}$-critical and has internal girth at least five, then*

$$w(G, \mathcal{R}) \leq \eta_0 \mathrm{surf}(g(\Sigma), |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R})) + \ell(\mathcal{R}).$$

Let G be a graph embedded in a surface Σ with rings $\mathcal{R}$. Let J be a subgraph of G and let S be a subset of faces of J such that

J is equal to the union of the boundaries of the faces in S, each isolated vertex of J is a vertex ring and whenever C is a cuff intersecting a face $f \in S$, then C is incident with a vertex ring belonging to J. (1)

We define $G[S]$ to be the subgraph of G consisting of J and all the vertices and edges drawn inside the faces of S. Let $C_1, C_2, \ldots, C_k$ be the boundary walks of the faces in S (in case that a vertex ring $R \in \mathcal{R}$ forms a component of a boundary of a face in S, we consider R itself to be such a walk). We would like to view $G[S]$ as a graph with rings $C_1, \ldots, C_k$. However, the C_i's do not necessarily have to be disjoint, and they do not have to be cycles or isolated vertices. To overcome this difficulty we modify $G[S]$ by the following canonical construction. However, to better explain the intent of the construction, let us offer a couple of examples first. If J is a cycle and S consists of a closed 2-cell face of J, then we will regard $G[S]$ as a graph in a disk with one ring J. If J is a 2-sided non-separating cycle and S consists of the unique face of J, then we cut the surface open along J, thereby creating two copies C_1, C_2 of J in a surface Σ', and we wish to regard the new graph as a graph in Σ' with two rings C_1 and C_2.

The general construction is as follows. Let Z be the set of cuffs incident with the vertex rings that form a component of J by themselves, and let $\hat{Z} = \bigcup_{C\in Z} \hat{C}$. Suppose that $S = \{f_1, \ldots, f_m\}$. For $1 \le i \le m$, let Σ'_i be a surface with boundary B_i such that $\Sigma'_i \setminus B_i$ is homeomorphic to f_i. Let $\theta_i : \Sigma'_i \setminus B_i \to f_i$ be a homeomorphism that extends to a continuous mapping $\theta_i : \Sigma'_i \to \overline{f_i}$, where $\overline{f_i}$ denotes the closure of f_i. Let Σ_i be obtained from Σ'_i by deleting $\theta_i^{-1}(\hat{Z} \cap f_i)$, and let G_i be the inverse image of $G \cap \overline{f_i}$ under θ_i. Then G_i is a graph normally embedded in Σ_i. We say that the set of embedded graphs $\{G_i : 1 \le i \le m\}$ is a *G-expansion of S*. Note that there is a one-to-one correspondence between the boundary walks of the faces of S and the rings of the graphs in the G-expansion of S; however, each vertex of J may be split to several copies. For $1 \le i \le m$, we let $\mathcal{R}_i$ be the set of rings of G_i consisting of the facial rings formed by the cycles contained in the boundary of Σ'_i, and of the vertex rings formed by the vertices contained in $\theta_i(f_i \cap \bigcup Z)$, where each vertex ring is weak if and only if the corresponding vertex ring of $\mathcal{R}$ is weak. We say that the rings in $\mathcal{R}_i$ are the *natural rings* of G_i.

We use the following basic property of critical graphs proved in [4].

Lemma 3.6. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, and assume that G is $\mathcal{R}$-critical. Let J be a subgraph of G and S be a subset of faces of J satisfying (1). Let G' be an element of the G-expansion of S and $\mathcal{R}'$ its natural rings. If G' is not equal to the union of the rings in $\mathcal{R}'$, then G' is $\mathcal{R}'$-critical.*

A frequently used corollary of Lemma 3.6 concerns the case that J is a contractible cycle.

Lemma 3.7. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, and assume that G is $\mathcal{R}$-critical. Let C be a non-facial cycle in G bounding an open disk $\Delta \subseteq \hat{\Sigma}$ disjoint from the rings, and let G' be the graph consisting of the vertices and edges of G drawn in the closure of Δ. Then G' may be regarded as a graph embedded in the disk with one ring C, and as such it is C-critical.*

Furthermore, criticality is also preserved when cutting the surface, which was again shown in [4].

Lemma 3.8. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, and assume that G is $\mathcal{R}$-critical. Let c be a simple closed curve in Σ intersecting G in a set X of vertices. Let Σ_0 be one of the surfaces obtained from Σ by cutting along c. Let us split the vertices of G along c, let G' be the part of the resulting graph embedded in Σ_0, let X' be the set of vertices of G' corresponding to the vertices of X and let $\mathcal{R}' \subseteq \mathcal{R}$ be the the rings of G that are contained in Σ_0. Let Δ be an* *open disk or a disjoint union of two open disks disjoint from Σ_0 such that the boundary of Δ is equal to the cuff(s) of Σ_0 corresponding to c. Let $\Sigma' = \Sigma_0 \cup \Delta$. Let Y consist of all vertices of X' that are not incident with a cuff in Σ'. For each $y \in Y$, choose an open disk $\Delta_y \subset \Delta$ such that the closures of the disks are pairwise disjoint and the boundary of Δ_y intersects G' exactly in y. Let $\Sigma'' = \Sigma' \setminus \bigcup_{y \in Y} \Delta_y$ and $\mathcal{R}'' = \mathcal{R}' \cup Y$, where the elements of Y are considered to be non-weak vertex rings of the embedding of G' in Σ''. If G' is not equal to the union of the rings in $\mathcal{R}''$, then G' is $\mathcal{R}''$-critical.*

In particular, if G' is an component of a $\mathcal{R}$-critical graph, $\mathcal{R}'$ are the rings contained in G' and G' is not equal to the union of $\mathcal{R}'$, then G' is $\mathcal{R}'$-critical.

4 The main result

For technical reasons, we are going to prove the following generalization of Theorem 3.5 instead of Theorem 1.3.

Theorem 4.1. *There exists a constant η with the following property. Let G be a graph embedded in a surface Σ with rings $\mathcal{R}$. If G is $\mathcal{R}$-critical and triangle-free and contains no non-contractible 4-cycles, then*

$$w(G, \mathcal{R}) \leq \eta \mathrm{surf}(g(\Sigma), |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R})) + \ell(\mathcal{R}).$$

The following section is devoted to the proof of Theorem 4.1. Here, we show how it implies Theorem 1.3.

Proof of Theorem 1.3. Let η be the constant of Theorem 4.1; we can assume that $\eta \geq 1/80$. Let $\kappa = 1600\eta/s(5)$. Let $J \subseteq G$ be the union of all triangles and all 4-cycles that do not bound a 2-cell face. Note that $|E(J)| \leq 4c+3t$. Let S be the set of all faces of J. Let $\{G_1, \ldots, G_k\}$ be the G-expansion of (J, S), where G_i is embedded in a surface Σ_i with rings $\mathcal{R}_i$ for $1 \leq i \leq k$. Note that G_i is either equal to its rings or $\mathcal{R}_i$-critical and all its rings are facial. Let the graph G'_i embedded in Σ_i with rings $\mathcal{R}'_i$ be obtained from G_i by, for each ring $R \in \mathcal{R}_i$ of length at most 4, subdividing an edge of R by $5 - |R|$ new vertices. Observe that G'_i is either equal to its rings or $\mathcal{R}'_i$-critical, triangle-free and contains no non-contractible 4-cycles, and that $\ell(\mathcal{R}'_i) \leq 5\ell(\mathcal{R}_i)/3$. By Theorem 4.1, we have

$$\begin{aligned}
z_i &:= \sum_{f \text{ internal face of } G'_i} (|f| - 4) \\
&\leq \frac{1}{s(5)} \sum_{f \text{ internal face of } G'_i} w(f) = \frac{1}{s(5)} w(G'_i, \mathcal{R}'_i) \\
&\leq \frac{120\eta(g(\Sigma_i) + |\mathcal{R}'_i| - 1) + \ell(\mathcal{R}'_i)}{s(5)} \\
&\leq \frac{120\eta(g(\Sigma_i) + \ell(\mathcal{R}'_i) - 1)}{s(5)},
\end{aligned}$$

because $\eta \geq 1/80$ and $\ell(\mathcal{R}'_i) \geq 3|\mathcal{R}'_i|$. Note that $\sum_{i=1}^k g(\Sigma_i) \leq g$ and $\sum_{i=1}^k \ell(\mathcal{R}'_i) \leq \frac{5}{3}\sum_{i=1}^k \ell(\mathcal{R}_i) = \frac{10|E(J)|}{3}$. We conclude that

$$\begin{aligned}\sum_{f \text{ face of } G} (|f|-4) &\leq \sum_{i=1}^k z_i \\ &\leq \frac{120\eta(g+10|E(J)|/3-1)}{s(5)} \\ &\leq \frac{120\eta(g+40c/3+10t-1)}{s(5)} \\ &\leq \kappa(g+c+t-1).\end{aligned}$$

□

5 The proof

The basic cases of Theorem 4.1 follow from Theorem 1.2, as well as from the following result of [6].

Theorem 5.1. *Let G be a triangle-free graph embedded in a disk with one ring R of length at most 6. If G is R-critical, then $|R| = 6$ and all internal faces of G have length 4.*

Next, we are going to show a key lemma that enables us to reduce 4-faces. Let G be a graph embedded in a surface Σ with rings $\mathcal{R}$. A non-contractible 4-cycle $C = w_1w_2w_3w_4$ can be *flipped to a 4-face* if C surrounds a ring, $w_1w_2w_3$ is part of a boundary walk of a face f_1, $w_1w_4w_3$ is part of a boundary walk of a face f_2, C separates f_1 from f_2, and $w_1w_2w_3$ and $w_1w_4w_3$ are the only paths of length at most 2 in G between w_1 and w_3. See the left part of Figure 1 for an illustration.

A 4-face $f = w_1w_2w_3w_4$ is *ring-bound* if (up to a relabelling of the vertices of f)

- w_1 is a vertex ring and there exists a cycle C of length at most 6 surrounding w_1 such that $w_2w_3w_4 \subset C$; or,
- f is incident with vertices of two distinct rings; or,
- w_1 and w_3 are ring vertices.

Let Π be a surface with boundary and c a simple curve intersecting the boundary of Π exactly in its ends. The topological space obtained from Π by cutting along c (i.e., removing c and adding two new pieces of boundary corresponding to c) is a union of at most two surfaces. If $\Pi_1, \ldots, \Pi_k$ are obtained from Π by repeating this construction, we say that they are *fragments* of Π. Consider a graph H embedded in Π with rings $\mathcal{Q}$, and let f be an internal face of H. For each facial walk t of f, we perform the following: if t consists only of

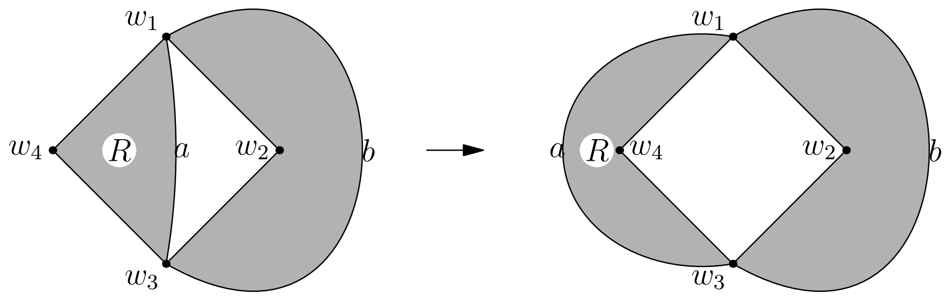

Figure 1: Non-contractible 4-cycle $w_1w_2w_3w_4$ which can be flipped to a 4-face, and the corresponding transformation.

a vertex ring incident with the cuff C, then we remove $\hat{C}$ from f. Otherwise, we add a simple closed curve tracing t (if an edge appears twice in t, then it will correspond to two disjoint parts of the curve). We define Π_f to be the resulting surface. Note that the cuffs of Π_f correspond to the facial walks of f.

Let G and G' be graphs embedded in a surface Σ with rings $\mathcal{R}$. Suppose that there exists a collection $\{(J_f, S_f) : f \in F(G')\}$ of subgraphs J_f of G and sets S_f of faces of J_f satisfying (1) such that

- for every $f \in F(G')$, the boundary of S_f is not equal to the union of $\mathcal{R}$,
- for every $f \in F(G')$, the surfaces embedding the components of the G-expansion of S_f are fragments of Σ_f, and
- for every face $h \in F(G)$ which is not 2-cell of length four, there exists a unique $f \in F(G')$ such that h is a face of the G-expansion of S_f.

We say that this collection forms a *cover of G by faces of G'*. We say that an element (J_f, S_f) is *non-trivial* if the G-expansion of (J_f, S_f) does not consist of only one cycle. We define the *elasticity* $\mathrm{el}(f)$ of a face $f \in F(G')$ to be $\left(\sum_{h\in S_f} |h|\right) - |f|$.

Lemma 5.2. *Let G be a graph embedded in a surface Σ with rings $\mathcal{R}$. Suppose that G is $\mathcal{R}$-critical and triangle-free, contains no non-contractible 4-cycles, and every connected essential subgraph of G has at least 13 edges. If G has an internal 4-face that is not ring-bound, then there exists a triangle-free $\mathcal{R}$-critical graph G' embedded in Σ with rings $\mathcal{R}$ such that $|E(G')| < |E(G)|$, and a collection $\{(J_f, S_f) : f \in F(G')\}$ forming a cover of G by faces of G', satisfying the following conditions.*

- *G' has at most one non-contractible 4-cycle, and if it has one, it can be flipped to a 4-face.*
- *The cover satisfies $\sum_{f\in F(G')} el(f) \le 4$, and if f is omnipresent or closed 2-cell, then $el(f) \in \{0, 2\}$.*

- *For every $f \in F(G')$, if f is closed 2-cell and el$(f) = 2$, then (J_f, S_f) is non-trivial.*

Proof. Let $h_0 = v_1v_2v_3v_4$ be an internal 4-face of G that is not ring-bound. By symmetry, we can assume that v_3 and v_4 are internal vertices. As G is triangle-free, v_1 and v_3 are non-adjacent and v_2 and v_4 are non-adjacent. For $i \in \{1, 2\}$, we say that v_i is *problematic* if either v_i is a vertex ring, or G contains a path P_i of length at most four joining v_i with v_{i+2} such that P_i together with $v_iv_{5-i}v_{i+2}$ forms a non-contractible cycle C_i. By switching the labels of vertices if necessary, we can assume that if v_1 is problematic, then v_2 is problematic as well.

Suppose that v_1 is a vertex ring. Then v_1 is problematic, and thus v_2 is problematic. As h_0 is not ring-bound, v_2 is an internal vertex, and thus G contains a non-contractible cycle C_2 as above; and furthermore, C_2 does not surround v_1. However, then C together with the edge v_1v_2 forms a connected essential subgraph of G with at most 7 edges, which is a contradiction. We conclude that v_1 is not a vertex ring; and by symmetry, if both v_1 and v_2 are problematic, then v_2 is not a vertex ring.

Let G_0 be the graph obtained from G by identifying v_1 with v_3 to a new vertex z and suppressing the arising parallel edges. Note that every 3-coloring of G_0 corresponds to a 3-coloring of G in which v_1 and v_3 have the same color as z. Let ψ be a precoloring of $\mathcal{R}$ that does not extend to G, and observe that ψ also does not extend to G_0. Therefore, G_0 has a ψ-critical subgraph G'.

Consider a face $f \in F(G')$. Let J_f be the subgraph of G obtained by applying the following construction to each boundary walk C of f. We repeatedly replace every subwalk of C of the form x, z, y as follows. If both x and y are adjacent to v_1 in G, then we replace x, z, y by x, v_1, y. Otherwise, if both x and y are adjacent to v_3 in G, then we replace x, z, y by x, v_3, y. Thus we may assume that x is adjacent to v_i and y is adjacent to v_j in G, where $\{i, j\} = \{1, 3\}$. In that case we replace x, z, y by x, v_i, v_4, v_j, y. Thus we eventually convert C to a walk in G.

It should be noted that even though f is a face of G', it may correspond to two faces of J_f when z is not adjacent to v_2 in G' and the boundary of f contains both v_2 and z. Let S_f denote the set of the faces of J_f corresponding to f. The construction implies that the surfaces embedding the components of the G-expansion of S_f are fragments of Σ_f (note that here we use the fact that neither v_1 nor v_3 is a vertex ring, as otherwise if say v_1 were a vertex ring and the boundary of f contained two consecutive edges xz and zy that both correspond to edges of G incident with v_3, then v_1 would become an isolated vertex in J_f as a result of the replacement of x, z, y by x, v_3, y, thus increasing the number of boundary components). Furthermore, every internal face h of G except possibly for h_0 is contained in the G-expansion of S_f for a unique face $f \in F(G')$. Therefore, the collection $\{(J_f, S_f) : f \in F(G')\}$ forms a cover of G by faces of G'.

By the construction of J_f, a face $f \in F(G')$ has non-zero elasticity if and only if if f contains two consecutive edges e_1 and e_2 incident with z such that

e_1 is incident with v_1 in G and e_2 is incident with v_3 in G. There are at most two such pairs of edges in the cyclic order around z, and thus the sum of the elasticities of faces of G' is at most 4. Furthermore, if both of these pairs of edges are incident with one face f, then z is incident twice with f and thus f is neither closed 2-cell nor omnipresent. It follows that if f is closed 2-cell or omnipresent, then its elasticity is at most two.

Suppose that f is a closed 2-cell face of G' and the elasticity of f is not 0. Then J_f contains the path $v_1v_4v_3$. If some member of the G-expansion of (J_f, S_f) contains h_0, then (J_f, S_f) is non-trivial, as J_f contains at least two edges not incident with h_0. Suppose that no member of the G-expansion of (J_f, S_f) contains h_0, and thus some member contains the other face h incident with v_1v_4. Note that since v_4 is an internal vertex, by Lemma 3.1, it is incident with an edge $e \in E(G)$ distinct from v_1v_4 and v_3v_4. If v_4 belongs to the the boundary walk of f, then $|S_f| = 2$. Otherwise, the edge e belongs to a member of the G-expansion of (J_f, S_f). We conclude that in all the cases, (J_f, S_f) is non-trivial.

If G' is triangle-free and contains no non-contractible 4-cycles, then the claim of the lemma holds. Therefore, suppose that G' contains either a triangle or a non-contractible 4-cycle. Then G contains a path P_1 of length 3 or 4 joining v_1 with v_3 such that P_1 together with $v_1v_4v_3$ forms a cycle C_1 such that if $|C_1| = 6$, then C_1 is non-contractible. Suppose that $|C_1| = 5$ and C_1 is contractible, and let $\Delta \subset \Sigma$ be a closed disk bounded by C_1. Recall that v_4 is incident with an edge e distinct from v_1v_4 and v_3v_4, and thus either e or h_0 is contained in the interior of Δ. Consequently, Lemma 3.7 implies that the subgraph of G drawn in Δ is C_1-critical. However, as $|C_1| = 5$, this contradicts Theorem 5.1. We conclude that C_1 is non-contractible, and thus v_1 is problematic. It follows that v_2 is problematic as well, and thus there exists a path P_2 joining v_2 with v_4 which combines with $v_2v_3v_4$ to a non-contractible ($\leq$6)-cycle C_2.

Note that the subgraph H of G formed by h_0, P_1 and P_2 has at most 12 edges, and thus it is not essential. It follows that there exists a cuff C (incident with a ring R) and an open disk $\Lambda \subset \Sigma + \hat{C}$ with H contained in Λ. Consider the drawing of H in Λ, and for $1 \leq i \leq 4$, let h_i be its face incident with v_iv_{i+1} (where $v_5 = v_1$) distinct from h_0. The faces $h_1, \ldots, h_4$ are pairwise distinct, as each two of them are separated by C_1 or C_2. As G is triangle-free, each of the faces has length at least four. We have

$$16 \leq |h_1| + |h_2| + |h_3| + |h_4| \leq |h_0| + 2(|P_1| + |P_2|) \leq 20. \tag{2}$$

Furthermore, the boundaries of at least two of the faces of H surround R. Since G does not contain non-contractible 4-cycles, at least two faces of H have length at least 5. We conclude that H has no face distinct from $h_0, \ldots, h_4$. By symmetry, we can assume that R is contained in h_3, and thus $|h_3| \geq 5$. Since both C_1 and C_2 are non-contractible, they surround R, and thus the boundary of h_1 surrounds R as well, and $|h_1| \geq 5$. Note that $|h_1|+|h_3| \leq |P_1|+|P_2|+2 \leq 10$, and thus $|h_1| = |h_3| = 5$ and $|P_1| = |P_2| = 4$. As the subdisk of Λ bounded by C_2 consists of the closure of the faces h_2 and h_3 and $|C_2| = 6$, it follows that $|h_2|$ and $|h_3|$ have the same parity, i.e., $|h_2|$ is odd. Similarly, $|h_4|$ is odd. We conclude that $|h_2| = |h_4| = 5$, and by (2), it follows that P_1 and P_2 are edge-disjoint. That is, P_1 and P_2 intersect in a single vertex w_1. Observe that w_1 is the middle vertex of P_1 and P_2.

The cycles bounding h_2 and h_4 are contractible, and thus by Lemma 3.7 and Theorem 5.1, h_2 and h_4 are faces of G. Let $Q_1 = v_1w_2w_1$ be the subpath of P_1 between v_1 and w_1. If G contained another path Q_1' of length at most two between v_1 and w_1, then the cycle consisting of Q_1 and Q_1' would either be a triangle, a non-contractible 4-cycle, or a 4-face showing that w_2 is an internal vertex of degree two, which is a contradiction. Therefore, Q_1 is the only path of length at most two between v_1 and w_1 in G. Similarly, the subpath $v_3w_4w_1$ of P_2 is the only path of length at most two between v_3 and w_1 in G. Therefore, G' is triangle-free and it contains exactly one non-contractible 4-cycle $w_1w_2w_3w_4$, where $w_3 = z$. Because of the faces h_2 and h_4 of G, the paths $w_1w_2w_3$ and $w_1w_4w_3$ are parts of boundaries of faces of G' separated by $w_1w_2w_3w_4$. Furthermore, $w_1w_2w_3w_4$ surrounds the ring R. We conclude that $w_1w_2w_3w_4$ can be flipped to a 4-face. □

As the next step, let us prove a stronger variant of Theorem 4.1 in the case Σ is the disk. Let $\mathcal{G}_{r,k}$ denote the set of R-critical graphs of girth at least r embedded in the disk with one facial ring R of length k. For a graph $G \in \mathcal{G}_{4,k}$ let $S(G)$ denote the multiset of lengths of the internal (≥ 5)-faces of G. Let $\mathcal{S}_{r,k} = \{S(G) : G \in \mathcal{G}_{r,k}\}$. Note that by Theorem 3.4, the set $\mathcal{S}_{5,k}$ is empty for $5 \leq k \leq 7$ and finite for every $k \geq 8$, the maximum of each element of $\mathcal{S}_{5,k}$ is at most $k-3$, and if the maximum of $S \in \mathcal{S}_{5,k}$ is equal to $k-3$, then $S = \{5, k-3\}$. Furthermore, $\mathcal{S}_{4,4} = \mathcal{S}_{4,5} = \emptyset$ and $\mathcal{S}_{4,6} = \{\emptyset\}$ by Theorem 5.1.

Let S_1 and S_2 be multisets of integers. We say that S_2 is a *one-step refinement* of S_1 if there exists $k \in S_1$ and a set $Z \in \mathcal{S}_{4,k} \cup \mathcal{S}_{4,k+2}$ such that $S_2 = (S_1 \setminus \{k\}) \cup Z$. We say that S_2 is an *refinement* of S_1 if it can be obtained from S_2 by a (possibly empty) sequence of one-step refinements.

Lemma 5.3. *For every $k \geq 7$, each element of $\mathcal{S}_{4,k}$ other than $\{k-2\}$ is a refinement of an element of $\mathcal{S}_{4,k-2} \cup \mathcal{S}_{5,k}$. In particular, if $S \in \mathcal{S}_{4,k}$, then the maximum of S is at most $k-2$, and if the maximum is equal to $k-2$, then $S = \{k-2\}$.*

Proof. Suppose for a contradiction that there exists a graph $G \in \mathcal{G}_{4,k}$ for some $k \geq 7$ such that $S(G) \neq \{k-2\}$ and $S(G)$ is not a refinement of an element of $\mathcal{S}_{4,k-2} \cup \mathcal{S}_{5,k}$. Let G be chosen so that k is minimum, and subject to that G has the smallest number of edges. As $S(G) \notin \mathcal{S}_{5,k}$, G contains a 4-face $h_0 = v_1v_2v_3v_4$. As G is embedded in the disk, G has no essential subgraph.

If h_0 is ring-bound, then say v_1 and v_3 belong to the ring R of G, and thus G contains a path P of length 1 or 2 intersecting R exactly in the endpoints of P. Let C_1 and C_2 be the cycles of $R + P$ distinct from R, and let G_1 and G_2 be the subgraphs of G drawn in the closed disks bounded by C_1 and C_2, respectively. We have $|C_1| + |C_2| = k + 2|P|$. Suppose that say $|C_1| = 4$;

hence, C_1 bounds a face by Lemma 3.7 and Theorem 5.1, and $S(G) = S(G_2)$. If C_2 bounds a face, then $|P| = 1$, as otherwise the middle vertex of P would contradict Lemma 3.1. In that case, $S(G) = \{k-2\}$, which is a contradiction. Therefore, $G_2 \neq C_2$, and thus G_2 is C_2-critical by Lemma 3.7. If $|P| = 2$, then $|C_2| = k$, and G_2 contradicts the minimality of G. If $|P| = 1$, then $|C_2| = k-2$ and $S(G) \in \mathcal{S}_{4,k-2}$, which is again a contradiction.

Therefore, we can assume that $|C_1| \geq 5$ and by symmetry, $|C_2| \geq 5$. If P is a chord, then $Z = \{|C_1|, |C_2|\}$ belongs to $\mathcal{S}_{5,k}$, and since $S(G_i)$ is either $\{|C_i|\}$ or belongs to $\mathcal{S}_{4,|C_i|}$ for $i \in \{1,2\}$, it follows that $S(G) = S(G_1) \cup S(G_2)$ is a refinement of the element Z of $\mathcal{S}_{5,k}$. This contradicts the assumption that G is a counterexample. Finally, consider the case that $|P| = 2$. As the middle vertex of P has degree at least three, by symmetry we can assume that C_2 does not bound a face, and thus G_2 is C_2-critical. In particular $|C_2| \geq 6$. If $|C_2| = 6$, then $S(G_2) = \emptyset$, $S(G) = S(G_1)$ and either $S(G_1) = \{k-2\}$ or $S(G_1)$ belongs to $\mathcal{S}_{4,k-2}$, which contradicts the assumption that G is a counterexample. If $|C_2| \geq 7$, then note that $\{|C_1|, |C_2| - 2\}$ belongs to $\mathcal{S}_{5,k}$ and that $S(G_2)$ is a refinement of $\{|C_2| - 2\}$; consequently, $S(G)$ is a refinement of an element of $\mathcal{S}_{5,k}$, which is a contradiction.

It follows that h_0 is not ring-bound, and thus Lemma 5.2 applies; let G' be the corresponding R-critical graph and let $\{(J_f, S_f) : f \in F(G')\}$ be the cover of G by faces of G'. By Lemma 3.2, each face of G' is closed 2-cell, and thus each face of G' has elasticity at most 2. For each face $f \in F(G')$, the G-expansion of (J_f, S_f) consists of several graphs $H_1, \ldots, H_t$ embedded in disks with rings R_1, $\ldots$, R_t such that $|R_1| + \ldots + |R_t| = |f| + \mathrm{el}(f)$, where $|R_i| > 4$ for $1 \leq i \leq m$ and $|R_i| = 4$ for $m+1 \leq i \leq t$. Observe that if $t \geq 2$, then $\{|R_1|, \ldots, |R_m|\}$ is an element of $\mathcal{S}_{4,|f|}$ (corresponding to a ring of length $|f|$ with $2t - 3 + \mathrm{el}(f)/2$ chords), and thus $S(H_1) \cup \ldots \cup S(H_t)$ is a refinement of $\{|f|\}$. This is clearly true as well if $t = 1$ and $\mathrm{el}(f) = 0$. If $t = 1$ and $\mathrm{el}(f) = 2$, then Lemma 5.2 asserts that (J_f, S_f) is non-trivial. It follows that H_1 is R_1-critical and $S(H_1)$ belongs to $\mathcal{S}_{4,|f|+2}$, and thus $S(H_1)$ is a refinement of $\{|f|\}$. We conclude that $S(G)$ is a refinement of $S(G')$.

As $S(G)$ is not a refinement of an element of $\mathcal{S}_{4,k-2} \cup \mathcal{S}_{5,k}$, $S(G')$ is not a refinement of an element of $\mathcal{S}_{4,k-2} \cup \mathcal{S}_{5,k}$. By the minimality of G, it follows that $S(G') = \{k-2\}$. Let f be the $(k-2)$-face of G' and let $H_1, \ldots, H_t$ and m be as in the previous paragraph, and observe that $S(G) = S(H_1) \cup \ldots \cup S(H_m)$. If $t \geq 2$, then this implies that $S(G)$ is a refinement of an element of $\mathcal{S}_{4,k-2}$, which is a contradiction. If $t = 1$, then H_1 is a proper subgraph of G containing all internal (≥ 5)-faces of G. If $\mathrm{el}(f) = 2$, then (J_f, S_f) is non-trivial, and thus H_1 is R_1-critical, $|R_1| = k$ and H_1 is a counterexample with fewer edges than G, which is a contradiction. Therefore, $\mathrm{el}(f) = 0$. As $S(G) \neq \{k-2\}$, we have $H_1 \neq R_1$. However, then H_1 is R_1-critical and $S(G) = S(H_1)$ belongs to $\mathcal{S}_{4,k-2}$, which is a contradiction. □

For a multiset Z, let $s(Z) = \sum_{z \in Z} s(z)$.

Corollary 5.4. *Let G be a triangle-free graph embedded in the disk with one ring R of length $l \geq 4$. If G is R-critical, then $w(G, \{R\}) \leq s(l-2)$.*

Proof. We proceed by induction on l. The claim holds for $l \leq 6$ by Theorem 5.1. Suppose that $l \geq 7$. By Lemma 5.3, we have that $S(G)$ is either $\{l-2\}$ or a refinement of $Z \in \mathcal{S}_{4,l-2} \cup \mathcal{S}_{5,l}$. In the former case, we have $w(G, \{R\}) = s(l-2)$, and thus we can assume the latter.

If $Z \in \mathcal{S}_{4,l-2}$, then $s(Z) \leq s(l-4) < s(l-2)$ by induction. If $Z \in \mathcal{S}_{5,l}$, then $s(Z) \leq s(l-3)+s(5) \leq s(l-2)$ by Theorem 3.4. Since $S(G)$ is a refinement of Z, there exist multisets $Z = Z_0, Z_1, \ldots, Z_m = S(G)$ such that Z_{i+1} is a one-step refinement of Z_i for $0 \leq i \leq m-1$. We claim that $s(Z_0) \geq s(Z_1) \geq \ldots \geq s(Z_m)$. Indeed, all elements of $Z_0, \ldots, Z_m$ are smaller or equal to $l-3$, and $s(Z_{i+1}) = s(Z_i) - s(a) + s(A)$ for some $a \in Z_i$ and $A \in \mathcal{S}_{4,a} \cup \mathcal{S}_{4,a+2}$; therefore, by induction hypothesis we have $s(A) \leq s(a)$. We conclude that $S(G) \leq s(Z) \leq s(l-2)$. $\square$

We will need the following properties of the function surf, which were proved in [4].

Lemma 5.5. *If g, g', t, t_0, t_1, t'_0, t'_1 are non-negative integers, then the following holds:*

(a) *Assume that if $g = 0$ and $t \leq 2$, then $t_0 + t_1 < t$. If $t \geq 2$, $t'_0 \leq t_0$, $t'_1 \leq t_1$ and $t'_0 + t'_1 \geq t_0 + t_1 - 2$, then* $\mathrm{surf}(g, t-1, t'_0, t'_1) \leq \mathrm{surf}(g, t, t_0, t_1) - 1$.

(b) *If $g' < g$ and either $g' > 0$ or $t \geq 2$, then* $\mathrm{surf}(g', t, t_0, t_1) \leq \mathrm{surf}(g, t, t_0, t_1) - 120(g - g') + 32$.

(c) *Let g'', t', t'', t''_0 and t''_1 be nonnegative integers satisfying $g = g' + g''$, $t = t' + t''$, $t_0 = t'_0 + t''_0$, $t_1 = t'_1 + t''_1$, either $g'' > 0$ or $t'' \geq 1$, and either $g' > 0$ or $t' \geq 2$. Then,* $\mathrm{surf}(g', t', t'_0, t'_1) + \mathrm{surf}(g'', t'', t''_0, t''_1) \leq \mathrm{surf}(g, t, t_0, t_1) - \delta$, *where $\delta = 16$ if $g'' = 0$ and $t'' = 1$ and $\delta = 56$ otherwise.*

(d) *If $g \geq 2$, then* $\mathrm{surf}(g-2, t, t_0, t_1) \leq \mathrm{surf}(g, t, t_0, t_1) - 124$

Consider a graph H embedded in a surface Π with rings $\mathcal{Q}$, and let f be an internal face of H. Let a_0 and a_1 be the number of weak and non-weak rings, respectively, that form one of the facial walks of f by themselves. Let a be the number of facial walks of f. We define $\mathrm{surf}(f) = \mathrm{surf}(g(\Pi_f), a, a_0, a_1)$.

Let G_1 be a graph embedded in Σ_1 with rings $\mathcal{R}_1$ and G_2 a graph embedded in Σ_2 with rings $\mathcal{R}_2$. Let $m(G_i)$ denote the number of edges of G_i that are not contained in the boundary of Σ_i. Let us write $(G_1, \Sigma_1, \mathcal{R}_1) \prec (G_2, \Sigma_2, \mathcal{R}_2)$ to denote that the quadruple $(g(\Sigma_1), |\mathcal{R}_1|, m(G_1), |E(G_1)|)$ is lexicographically smaller than $(g(\Sigma_2), |\mathcal{R}_2|, m(G_2), |E(G_2)|)$.

Proof of Theorem 4.1. Let η_0 be the constant from Theorem 3.5 and let $\eta = 7\eta_0 + 36$. We proceed by induction and assume that the claim holds for all graphs G' embedded in surfaces Σ' with rings $\mathcal{R}'$ such that $(G', \Sigma', \mathcal{R}') \prec (G, \Sigma, \mathcal{R})$. Let $g = g(\Sigma)$, $t_0 = t_0(\mathcal{R})$ and $t_1 = t_1(\mathcal{R})$. By Theorem 1.1, Corollary 5.4 and Theorem 1.2, the claim holds if $g + |\mathcal{R}| \leq 1$, hence assume that $g + |\mathcal{R}| \geq 2$. Similarly, if $g = 0$ and $|\mathcal{R}| = 2$, then we can assume that $t_0 + t_1 \leq 1$ by Lemma 3.3. First, we aim to prove that we can assume that all internal faces in the embedding of G are closed 2-cell. For later use, we will consider a more general setting.

Let H be a triangle-free graph embedded in Π with rings $\mathcal{Q}$ without non-contractible 4-cycles, such that at least one internal face of H is not open 2-cell and no face of H is omnipresent. If H is $\mathcal{Q}$-critical and $(H,\Pi,\mathcal{Q}) \preceq (G,\Sigma,\mathcal{R})$, then

$$w(H,\mathcal{Q}) \le \ell(\mathcal{Q}) + \eta\Big(\mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 7 - \sum_{h\in F(H)} \mathrm{surf}(h)\Big). \tag{3}$$

Proof. We prove the claim by induction. Consider for a moment a triangle-free graph H' embedded in a surface Π' with rings $\mathcal{Q}'$ without non-contractible 4-cycles, where $(H',\Pi',\mathcal{Q}') \prec (H,\Pi,\mathcal{Q})$, such that either $H' = \mathcal{Q}'$ or H' is $\mathcal{Q}'$-critical. We claim that

$$w(H',\mathcal{Q}') \le \ell(\mathcal{Q}') + \eta\Big(\mathrm{surf}(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}')) - \sum_{h\in F(H')} \mathrm{surf}(h)\Big). \tag{4}$$

The claim obviously holds if $H' = \mathcal{Q}'$, since $w(H',\mathcal{Q}') \le \ell(\mathcal{Q}')$ in that case; hence, it suffices to consider the case that H' is $\mathcal{Q}'$-critical. If at least one internal face of H' is not open 2-cell and no face of H' is omnipresent, then this follows by an inductive application of (3) (we could even strengthen the inequality by 7η). If all internal faces of H' are open 2-cell, then note that $\mathrm{surf}(h) = 0$ for every $h \in H'$, and since $(H',\Pi',\mathcal{Q}') \prec (G,\Sigma,\mathcal{R})$, we can apply Theorem 4.1 inductively to obtain (4). Finally, suppose that H' has an omnipresent face f, let $\mathcal{Q}' = \{Q_1,\ldots,Q_t\}$ and for $1 \le i \le t$, let f_i be the boundary walk of f such that f_i and Q_i are contained in a closed disk $\Delta_i \subset \Pi' + \hat{Q}_i$. Since all components of H' are planar and contain only one ring, Lemma 3.2 implies that all internal faces of H' distinct from f are closed 2-cell. Furthermore, each vertex ring forms component of the boundary of f by itself, hence $\mathrm{surf}(f) = \mathrm{surf}(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}'))$. If Q_i is a facial ring, then by applying Corollary 5.4 to the subgraph H'_i of H' embedded in Δ_i, we conclude that the weight of H'_i is at most $s(|Q_i|)$ and that $|f_i| \le |Q_i|$. Note that $s(|Q_i|) - s(|f_i|) \le |Q_i| - |f_i|$. Therefore, we again obtain (4):

$$\begin{aligned}
w(H',\mathcal{Q}') &\le |f| + \sum_{i=1}^{t} s(|Q_i|) - s(|f_i|) \le |f| + \sum_{i=1}^{t} |Q_i| - |f_i| \\
&= \ell(\mathcal{Q}') \\
&= \ell(\mathcal{Q}') + \eta\Big(\mathrm{surf}(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}')) - \sum_{h\in F(H')} \mathrm{surf}(h)\Big).
\end{aligned}$$

Let us now return to the graph H. Since H is $\mathcal{Q}$-critical, Theorem 1.1 implies that no component of H is a planar graph without rings. Let f be a face of H

which is not open 2-cell. Since H has such a face and f is not omnipresent, we have $g(\Pi) > 0$ or $|\mathcal{Q}| > 2$. Let c be a simple closed curve in f infinitesimally close to a facial walk W of f. Cut Π along c and cap the resulting holes by disks (c is always a 2-sided curve). Let Π_1 be the surface obtained this way that contains W, and if c is separating, then let Π_2 be the other surface. Since f is not omnipresent, we can choose W so that either $g(\Pi_1) > 0$ or Π_1 contains at least two rings of $\mathcal{Q}$. Let us discuss several cases:

- *c is separating and H is contained in* Π_1. Let H_1 denote the embedding of H in Π_1. In this case f has only one facial walk, and since f is not open 2-cell, Π_2 is not the sphere. It follows that $g(\Pi_1) = g(\Pi) - g(\Pi_2) < g(\Pi)$, and thus $(H_1, \Pi_1, \mathcal{Q}) \prec (H, \Pi, \mathcal{Q})$. Note that the weights of the faces of the embedding of H in Π and of H_1 in Π_1 are the same, with the exception of f whose weight in Π is $|f|$ and weight in Π_1 is $s(|f|) \geq |f| - 8$. By (4), we have

$$
\begin{aligned}
w(H, \mathcal{Q}) &\leq w(H_1, \mathcal{Q}) + 8 \\
&\leq \ell(\mathcal{Q}) + 8 \\
&\quad + \eta\Big(\mathrm{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) + \mathrm{surf}(f) - \sum_{h \in F(H)} \mathrm{surf}(h)\Big).
\end{aligned}
$$

  Note that $\mathrm{surf}(f) = 120g(\Pi_2) - 72$. By Lemma 5.5(b), we conclude that

$$
w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + 8 + \eta\Big(\mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 40 - \sum_{h \in F(H)} \mathrm{surf}(h)\Big).
$$

- *c is separating and Π_2 contains a nonempty part H_2 of H.* Let H_1 be the part of H contained in Π_1. Let $\mathcal{Q}_i$ be the subset of $\mathcal{Q}$ belonging to Π_i and f_i the face of H_i corresponding to f, for $i \in \{1, 2\}$. Note that f_1 is an open disk, hence $\mathrm{surf}(f_1) = 0$. Using (4), we get

$$
\begin{aligned}
w(H, \mathcal{Q}) \quad \leq \quad & w(f) - w(f_1) - w(f_2) + \ell(\mathcal{Q}_1) + \ell(\mathcal{Q}_2) + \\
& + \eta \sum_{i=1}^{2} \mathrm{surf}(g(\Pi_i), |\mathcal{Q}_i|, t_0(\mathcal{Q}_i), t_1(\mathcal{Q}_i)) + \\
& + \eta\Big(\mathrm{surf}(f) - \mathrm{surf}(f_2) - \sum_{h \in F(H)} \mathrm{surf}(h)\Big).
\end{aligned}
$$

  Note that $w(f) - w(f_1) - w(f_2) \leq 16$ and $\ell(\mathcal{Q}_1) + \ell(\mathcal{Q}_2) = \ell(\mathcal{Q})$. Also, $\mathrm{surf}(f) - \mathrm{surf}(f_2) \leq 48$, and when $g(\Pi_f) = 0$ and f has only two facial walks, then $\mathrm{surf}(f) - \mathrm{surf}(f_2) \leq 6$.

  By Lemma 5.5(c), we have

$$
\sum_{i=1}^{2} \mathrm{surf}(g(\Pi_i), |\mathcal{Q}_i|, t_0(\mathcal{Q}_i), t_1(\mathcal{Q}_i)) \leq \mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - \delta,
$$

where $\delta = 16$ if $g(\Pi_2) = 0$ and $|\mathcal{Q}_2| = 1$ and $\delta = 56$ otherwise. Note that if $g(\Pi_2) = 0$ and $|\mathcal{Q}_2| = 1$, then $g(\Pi_f) = 0$ and f has only two facial walks. We conclude that $\mathrm{surf}(f) - \mathrm{surf}(f_2) - \delta \leq -8$. Therefore, $w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + 16 + \eta\Big(\mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 8 - \sum_{h \in F(H)} \mathrm{surf}(h)\Big)$.

- *c is not separating.* Let f_1 be the face of H (in the embedding in Π_1) bounded by W and f_2 the other face corresponding to f. Again, note that $\mathrm{surf}(f_1) = 0$. By (4) applied to H embedded in Π_1, we obtain the following for the weight of H in Π:

$$\begin{aligned} w(H, \mathcal{Q}) \quad \leq \quad & w(f) - w(f_1) - w(f_2) + \ell(\mathcal{Q}) + \\ & + \eta\mathrm{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) + \\ & + \eta\Big(\mathrm{surf}(f) - \mathrm{surf}(f_2) - \sum_{h \in F(H)} \mathrm{surf}(h)\Big). \end{aligned}$$

  Since c is two-sided, $g(\Pi_1) = g(\Pi) - 2$, and

$$\mathrm{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) = \mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 124$$

  by Lemma 5.5(d). Since $\mathrm{surf}(f) - \mathrm{surf}(f_2) \leq 48$ and $w(f) - w(f_1) - w(f_2) \leq 16$, we have $w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + 16 + \eta\Big(\mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 76 - \sum_{h \in F(H)} \mathrm{surf}(h)\Big)$.

The results of all the subcases imply (3). $\square$

Next, we consider the case that one of the faces is omnipresent.

Let H be a triangle-free graph embedded in Σ with rings $\mathcal{R}$ and without non-contractible 4-cycles, and let f be an omnipresent face of H. If H is $\mathcal{R}$-critical, then

$$w(H, \mathcal{R}) \leq \ell(\mathcal{R}) - 2 = \ell(\mathcal{R}) - 2 + \eta\Big(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - \sum_{h \in F(H)} \mathrm{surf}(h)\Big). \tag{5}$$

Proof. Since H is $\mathcal{R}$-critical and f is an omnipresent face, each component of H is planar and contains exactly one ring. In particular, all internal faces of H distinct from f are closed 2-cell. For $R \in \mathcal{R}$, let H_R be the component of H containing R. Exactly one boundary walk W of f belongs to H_R. Cutting along W and capping the hole by a disk, we obtain an embedding of H_R in a disk with one ring R. Let f_R be the face of this embedding bounded by W. Note that either $H_R = R$ or H_R is R-critical. If R is a vertex ring, then we have $H_R = R$; hence, every vertex ring in $\mathcal{R}$ forms a facial walk of f, and $\mathrm{surf}(f) = \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1)$. Consequently, $\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) = \sum_{h \in F(H)} \mathrm{surf}(h)$, and it suffices to prove the first inequality of the claim.

Suppose that $H_R \neq R$ for a ring $R \in \mathcal{R}$. By Corollary 5.4, we have $w(H_R, \{R\}) \le s(|R|-2)$. Since f_R is a face of H_R, it follows that $|f_R| \le |R|-2$. Furthermore, $w(H_R, \{R\}) - w(f_R) \le s(|R|-2) - s(|f_R|) \le |R| - |f_R| - 2$. On the other hand, if $H_R = R$, then $w(H_R, \{R\}) - w(f_R) = 0 = |R| - |f_R|$. As H is $\mathcal{R}$-critical, there exists $R \in \mathcal{R}$ such that $H_R \neq R$, and thus

$$\begin{aligned} w(H, \mathcal{R}) &= w(f) + \sum_{R \in \mathcal{R}} (w(H_R, \{R\}) - w(f_R)) \\ &\le |f| + \sum_{R \in \mathcal{R}} (|R| - |f_R|) - 2 \\ &= \ell(\mathcal{R}) - 2. \end{aligned}$$

□

Finally, consider the case that the embedding is open 2-cell but not closed 2-cell.

Let H be an $\mathcal{R}$-critical graph embedded in Σ with rings $\mathcal{R}$ so that all internal faces of H are open 2-cell. If H is $\mathcal{R}$-critical, has internal girth at least five, $|E(H)| \le |E(G)|$ and an internal face f of H is not closed 2-cell, then $w(H, \mathcal{R}) \le \ell(\mathcal{R}) + \eta\Big(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 1/2\Big)$.

(6)

Proof. Since f is not closed 2-cell, there exists a vertex v appearing at least twice in the facial walk of f. There exists a simple closed curve c going through the interior of f and joining two of the appearances of v. Cut the surface along c and patch the resulting hole(s) by disk(s). Let v_1 and v_2 be the two vertices to that v is split. For each of v_1 and v_2, if it is not incident with a cuff, drill a new hole next to it in the incident patch.

If c is separating, then let H_1 and H_2 be the resulting graphs embedded in the two surfaces Σ_1 and Σ_2 obtained by this construction; if c is not separating, then let H_1 be the resulting graph embedded in a surface Σ_1. We choose the labels so that $v_1 \in V(H_1)$. If c is two-sided, then let f_1 and f_2 be the faces to that f is split by c, where f_1 is a face of H_1. If c is one-sided, then let f_1 be the face in Σ_1 corresponding to f. Note that $|f_1| + |f_2| = |f|$ in the former case, and thus $w(f) - w(f_1) - w(f_2) \le 16$. In the latter case, we have $w(f) = w(f_1)$.

If c is separating, then for $i \in \{1, 2\}$, let $\mathcal{R}_i$ consist of the rings of $\mathcal{R} \setminus \{v\}$ contained in Σ_i, and if none of these rings contains v, then also of the vertex ring v_i. Here, v_i is weak if v is an internal vertex, Σ_{3-i} is a cylinder and the ring of H_{3-i} distinct from v_{3-i} is a vertex ring. If c is not separating, then let $\mathcal{R}_1$ consist of the rings of $\mathcal{R} \setminus \{v\}$ as well as those of v_1 and v_2 that are not incident with any of the rings in this set. In this case, we treat v_1 and v_2 as non-weak vertex rings.

Suppose first that c is not separating. By Lemma 3.8, H_1 is $\mathcal{R}_1$-critical. Note that H_1 has at most two more rings (of length 1) than H and $g(\Sigma_1) \in$

$\{g-1, g-2\}$ (depending on whether c is one-sided or not), and that H_1 has at least two rings. If H_1 has only one more ring than H, then

$$\begin{aligned}
\mathrm{surf}(g(\Sigma_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) &\leq \mathrm{surf}(g-1, |\mathcal{R}|+1, t_0, t_1+1)\\
&\leq \mathrm{gen}(g-1, |\mathcal{R}|+1, t_0, t_1+1)+32\\
&= \mathrm{gen}(g, |\mathcal{R}|, t_0, t_1)-44\\
&= \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1)-44.
\end{aligned}$$

Let us now consider the case that H_1 has two more rings than H (i.e., that v is an internal vertex). If $g(\Sigma_1) = 0$ and $|\mathcal{R}_1| = 2$, then note that both rings of H_1 are vertex rings. Lemma 3.3 implies that H_1 has only one edge; but the corresponding edge in H would form a loop, which is a contradiction. Consequently, we have $g(\Sigma_1) > 1$ or $|\mathcal{R}_1| \geq 3$, and

$$\begin{aligned}
\mathrm{surf}(g(\Sigma_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) &\leq \mathrm{surf}(g-1, |\mathcal{R}|+2, t_0, t_1+2)\\
&= \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1)-32.
\end{aligned}$$

By induction, we can apply Theorem 4.1 inductively for H_1, concluding that $w(H,\mathcal{R}) \leq \ell(\mathcal{R}) + 18 + \eta\Big(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 32\Big)$, and the claim follows.

Next, we consider the case that c is separating. Let us remark that H_i is $\mathcal{R}_i$-critical for $i \in \{1, 2\}$. This follows from Lemma 3.8 unless v_i is a weak vertex ring. If v_i is a weak vertex ring, then Σ_{3-i} is a cylinder and the ring R_{3-i} of H_{3-i} distinct from v_{3-i} is a vertex ring. By Lemma 3.3, H_{3-i} is a single edge and R_{3-i} is not weak. Note that every precoloring ψ_i of $\mathcal{R}_i$ corresponds to a precoloring ψ of $\mathcal{R}$ defined by $\psi(R_{3-i}) = \psi_i(v_i)$ and $\psi \restriction R = \psi_i \restriction R$ for $R \in \mathcal{R} \setminus \{R_{3-i}\}$, and thus H_i is $\mathcal{R}_i$-critical. Thus, we can apply Theorem 4.1 inductively for H_1 and H_2, and we have

$$\begin{aligned}
w(H,\mathcal{R}) &= w(H_1,\mathcal{R}_1) + w(H_2,\mathcal{R}_2) + w(f) - w(f_1) - w(f_2)\\
&\leq \ell(\mathcal{R}) + 18 + \eta\sum_{i=1}^{2} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i))
\end{aligned}$$

Therefore, since $\eta \geq 36$, it suffices to prove that

$$\sum_{i=1}^{2} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \leq \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 1. \tag{7}$$

Note that if $g(\Sigma_i) = 0$, then $|\mathcal{R}_i| \geq 2$ for $i \in \{1, 2\}$. If $|\mathcal{R}_1| + |\mathcal{R}_2| = |\mathcal{R}| + 1$, we have

$$\begin{aligned}
\sum_{i=1}^{2} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) &\leq \sum_{i=1}^{2}(\mathrm{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + 32)\\
&= \mathrm{gen}(g, |\mathcal{R}|, t_0, t_1) - 12\\
&= \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 12.
\end{aligned}$$

Therefore, we can assume that $|\mathcal{R}_1| + |\mathcal{R}_2| = |\mathcal{R}| + 2$, i.e., v is an internal vertex. Suppose that for both $i \in \{1, 2\}$, we have $g(\Sigma_i) > 0$ or $|\mathcal{R}_i| > 2$. Then,

$$\begin{aligned}\sum_{i=1}^{2} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) &= \sum_{i=1}^{2} \mathrm{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \\ &= \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 32.\end{aligned}$$

and the claim follows.

Hence, we can assume that say $g(\Sigma_1) = 0$ and $|\mathcal{R}_1| = 2$. Then, $\mathcal{R}_1 = \{v_1, R_1\}$ for some ring R_1, $g(\Sigma_2) = g$ and $|\mathcal{R}_2| = |\mathcal{R}|$. Since H_1 is $\mathcal{R}_1$-critical, Lemma 3.3 implies that R_1 is not a weak vertex ring. If R_1 is a vertex ring, then v_2 is a weak vertex ring of $\mathcal{R}_2$ which replaces the non-weak vertex ring R_1. Therefore, $\mathrm{surf}(g(\mathcal{R}_2), |\mathcal{R}_2|, t_0(\mathcal{R}_2), t_1(\mathcal{R}_2)) = \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 1$. Furthermore, $\mathrm{surf}(g(\mathcal{R}_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) = \mathrm{surf}(0, 2, 0, 2) = 0$, and the claim follows.

Finally, consider the case that R_1 is a facial ring. By symmetry, we can assume that if $g(\Sigma_2) = 0$ and $|\mathcal{R}_2| = 2$, then $\mathcal{R}_2$ contains a facial ring. Since $\mathcal{R}_2$ is obtained from $\mathcal{R}$ by replacing a facial ring R_1 by a non-weak vertex ring v_2, we have $\mathrm{surf}(g(\mathcal{R}_2), |\mathcal{R}_2|, t_0(\mathcal{R}_2), t_1(\mathcal{R}_2)) = \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 4$. Furthermore, $\mathrm{surf}(g(\mathcal{R}_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) = \mathrm{surf}(0, 2, 0, 1) = 2$. Consequently,

$$\sum_{i=1}^{2} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \le \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 2.$$

Therefore, inequality (7) holds. This proves (6). $\square$

By (3), (5) and (6), we have either $w(G, \mathcal{R}) < \eta\mathrm{surf}(g(\Sigma), |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R})) + \ell(\mathcal{R})$ or every internal face of G is closed 2-cell. From now on, assume that the latter holds.

Suppose that there exists a path $P \subset G$ of length at most 12 with ends in distinct rings $R_1, R_2 \in \mathcal{R}$. By choosing the shortest such path, we can assume that P intersects no other rings. If R_1 or R_2 is a vertex ring, first replace it by a facial ring of length three by adding new vertices and edges in the incident cuff. Let $J = P \cup \bigcup_{R \in \mathcal{R}} R$ and let $S = \{f\}$, where f is the face of J incident with edges of P. Let $\{(G')\}$ be the G-expansion of S, let Σ' be the surface in that G' is embedded and let $\mathcal{R}'$ be the natural rings of G'. Note that $g(\Sigma') = g$, $|\mathcal{R}'| = |\mathcal{R}| - 1$, $\ell(\mathcal{R}') \le \ell(\mathcal{R}) + 30$ and $t_0(\mathcal{R}') + t_1(\mathcal{R}') \ge t_0 + t_1 - 2$. Since $(G', \Sigma', \mathcal{R}') \prec (G, \Sigma, \mathcal{R})$, by induction and by Lemma 5.5(a) we have $w(G, \mathcal{R}) = w(G', \mathcal{R}') \le \eta\mathrm{surf}(g, |\mathcal{R}| - 1, t_0(\mathcal{R}'), t_1(\mathcal{R}')) + \ell(\mathcal{R}) + 30 \le \eta\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + \ell(\mathcal{R})$. Therefore, we can assume that no such path exists in G.

Let $R \in \mathcal{R}$ be a facial ring of G. Let P be either a path joining two distinct vertices u and v of R, or a cycle containing a vertex $u = v$ of R, or a cycle together with a path joining one of its vertices to a vertex $u = v$ of R. Suppose that $|E(P)| \le 12$, P intersects R only in u and v and that the boundaries of all (at most two) internal faces of $R \cup P$ are non-contractible. Since the distance between any two rings in G is at least 13, all vertices of $V(P) \setminus \{u, v\}$

are internal. Let J be the subgraph of G consisting of P and of the union of the rings, and let S be the set of internal faces of J. Clearly, S and J satisfy (1). Let $\{G_1, \ldots, G_k\}$ be the G-expansion of S, and for $1 \le i \le k$, let Σ_i be the surface in that G_i is embedded and let $\mathcal{R}_i$ be the natural rings of G_i. Note that $\sum_{i=1}^k t_0(\mathcal{R}_i) = t_0$ and $\sum_{i=1}^k t_1(\mathcal{R}_i) = t_1$. Let $r = \left(\sum_{i=1}^k |\mathcal{R}_i|\right) - |\mathcal{R}|$ and observe that either $r = 0$ and $k = 1$, or $r = 1$ and $1 \le k \le 2$ (depending on whether the curves in $\hat{\Sigma}$ corresponding to cycles in $R \cup P$ distinct from R are one-sided, two-sided and non-separating or two-sided and separating). Furthermore, $\sum_{i=1}^k g(\Sigma_i) = g + 2k - r - 3$.

We claim that $(G_i, \Sigma_i, \mathcal{R}_i) \prec (G, \Sigma, \mathcal{R})$ for $1 \le i \le k$. This is clearly the case, unless $g(\Sigma_i) = g$. Then, we have $k = 2$, $r = 1$ and $g(\Sigma_{3-i}) = 0$. Since the boundaries of internal faces of $R \cup P$ are non-contractible, Σ_{3-i} is not a disk, hence $|\mathcal{R}_{3-i}| \ge 2$ and $|\mathcal{R}_{3-i}| < |\mathcal{R}|$, again implying $(G_i, \Sigma_i, \mathcal{R}_i) \prec (G, \Sigma, \mathcal{R})$.

By induction, we have $w(G_i, \mathcal{R}_i) \le \ell(\mathcal{R}_i) + \eta \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i))$, for $1 \le i \le k$. Since every internal face of G is an internal face of G_i for some $i \in \{1, \ldots, k\}$ and $\sum_{i=1}^k \ell(\mathcal{R}_i) \le \ell(\mathcal{R}) + 24$, we conclude that $w(G, \mathcal{R}) \le \ell(\mathcal{R}) + 24 + \eta \sum_{i=1}^k \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i))$. Note that for $1 \le i \le k$, we have that Σ_i is not a disk and $\mathcal{R}_i$ contains at least one facial ring, and thus $\mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \le \mathrm{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + 30$. Therefore,

$$
\begin{aligned}
&\sum_{i=1}^k \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \\
&\quad = \sum_{i=1}^k (\mathrm{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + 30) \\
&\quad \le \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + 120(2k - r - 3) + 48r - 120(k-1) + 60 \\
&\quad = \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + 120k - 72r - 180 \\
&\quad \le \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 12.
\end{aligned}
$$

The inequality of Theorem 4.1 follows. Therefore, we can assume that G contains no such subgraph.

Suppose that G contains a connected essential subgraph H with at most 12 edges. We can assume that H is minimal. Given the already excluded cases, all vertices of H are internal and H satisfies one of the following (see [12]):

(a) H is a cycle, or

(b) H is the union of two cycles C_1 and C_2 intersecting in exactly one vertex, and C_1 and C_2 surround distinct rings, or

(c) H is the union of two vertex-disjoint cycles C_1 and C_2 and of a path between them, and C_1 and C_2 surround distinct rings, or

(d) H is the theta-graph and each of the three cycles in G surrounds a different ring (in this case, Σ is the sphere and $|\mathcal{R}| = 3$).

Let J be the subgraph of G consisting of H and of the union of the rings, and let S be the set of internal faces of J. Clearly, S and J satisfy (1). Let $\{G_1, \ldots, G_k\}$ be the G-expansion of S, and for $1 \le i \le k$, let Σ_i be the surface in that G_i is embedded and let $\mathcal{R}_i$ be the natural rings of G_i. Let $r = \left(\sum_{i=1}^{k} |\mathcal{R}_i|\right) - |\mathcal{R}|$. Note that $\sum_{i=1}^{k} t_0(\mathcal{R}_i) + \sum_{i=1}^{k} t_1(\mathcal{R}_i) = t_0 + t_1$ and $\sum_{i=1}^{k} \ell(\mathcal{R}_i) \le \ell(\mathcal{R}) + 24$.

Let us first consider the case (a). Observe that we have either $r = 1$ and $k = 1$, or $r = 2$ and $1 \le k \le 2$, and that $\sum_{i=1}^{k} g(\Sigma_i) = g - r + 2k - 2$. If $g(\Sigma_1) = g$, then $k = 2$ and $g(\Sigma_2) = 0$; furthermore, Σ_2 has at least three cuffs, since H does not surround a ring. Thus, if $g(\Sigma_1) = g$, then $r = 2$ and consequently $|\mathcal{R}_1| = |\mathcal{R}| + r - |\mathcal{R}_2| = |\mathcal{R}| + 2 - |\mathcal{R}_2| < |\mathcal{R}|$. The same argument can be applied to Σ_2 if $k = 2$, hence $(G_i, \Sigma_i, \mathcal{R}_i) \prec (G, \Sigma, \mathcal{R})$ for $1 \le i \le k$.

By induction, we conclude that

$$w(G, \mathcal{R}) \le \ell(\mathcal{R}) + 24 + \eta \sum_{i=1}^{k} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)).$$

For $1 \le i \le k$, let $\delta_i = 72$ if $g(\Sigma_i) = 0$ and $|\mathcal{R}_i| = 1$, let $\delta_i = 30$ if $g(\Sigma_i) = 0$ and $|\mathcal{R}_i| = 2$, and let $\delta_i = 0$ otherwise, and note that since $\mathcal{R}_i$ contains a facial ring, we have $\mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), r_1(\mathcal{R}_i)) = \mathrm{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), r_1(\mathcal{R}_i)) + \delta_i$.

If $k = 2$, then recall that since H does not surround a ring, we have either $g(\Sigma_i) > 0$ or $|\mathcal{R}_i| \ge 3$ for $i \in \{1, 2\}$; hence, $\delta_1 + \delta_2 \le 0$. If $k = 1$, then recall that we can assume that G is not embedded in the projective plane with no rings by Theorem 1.2; hence, then either $g(\Sigma_1) > 0$, or $|\mathcal{R}_1| \ge 2$. Consequently, we have $\delta_1 \le 30$.

Combining the inequalities, we obtain $\sum_{i=1}^{k} \delta_i \le 60 - 30k$, and

$$\begin{aligned}
&\sum_{i=1}^{k} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \\
&\quad= \sum_{i=1}^{k} \mathrm{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + \delta_i \\
&\quad\le \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + 120(2k - r - 2) + 48r - 120(k - 1) + \sum_{i=1}^{k} \delta_i \\
&\quad= \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + 90k - 72r - 60 \\
&\quad\le \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 24.
\end{aligned}$$

If H satisfies (b), (c) or (d), then $k = 3$, $r = 3$ and say Σ_1 and Σ_2 are cylinders and Σ_3 is obtained from Σ by replacing two cuffs by one. Therefore,

$$\sum_{i=1}^{k} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i))$$

$$
\begin{aligned}
&= \mathrm{surf}(g, |\mathcal{R}| - 1, t_0, t_1) + 2\mathrm{surf}(0, 2, 0, 0) \\
&\le \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + 12 - 18 \\
&\le \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 6.
\end{aligned}
$$

In all the cases, this implies the inequality of Theorem 4.1. Therefore, assume that

$$\textit{every connected essential subgraph of } G \textit{ has at least } 13 \textit{ edges.} \tag{8}$$

Consider now a path P of length one or two intersecting $\mathcal{R}$ exactly in its endpoints u and v. Both ends of P belong to the same ring R; let P, P_1 and P_2 be the paths in $R \cup P$ joining u and v. Recall that we can assume that $P \cup P_2$ is a contractible cycle. Let J be the subgraph of G consisting of P and of the union of the rings, and let S be the set of internal faces of J. Clearly, S and J satisfy (1). Let $\{G_1, G_2\}$ be the G-expansion of S, and for $1 \le i \le 2$, let Σ_i be the surface in that G_i is embedded and let $\mathcal{R}_i$ be the natural rings of G_i. Note that say Σ_2 is a disk and its ring has length $|P| + |P_2|$, and Σ_1 is homeomorphic to Σ. Therefore, by induction and Corollary 5.4, we have $w(G, \mathcal{R}) = w(G_1, \mathcal{R}_1) + w(G_2, \mathcal{R}_2) \le \ell(\mathcal{R}_1) + \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + s(\ell(\mathcal{R}_2))$. Furthermore, $\ell(\mathcal{R}_1) + s(\ell(\mathcal{R}_2)) \le \ell(\mathcal{R}_1) + \ell(\mathcal{R}_2) - 4 = \ell(R) + 2|P| - 4 \le \ell(R)$, and the claim of the theorem follows. Therefore, we can assume that

$$G \textit{ contains no path of length one or two intersecting } \mathcal{R} \textit{ exactly in its endpoints.} \tag{9}$$

Note that every 4-cycle in G is contractible, and thus it bounds a face by Lemma 3.7 and Theorem 5.1. Let us consider the case that every 4-face in G is ring-bound. By (8) and (9), it follows that every 4-face is incident with a vertex ring. For each vertex ring $R \in \mathcal{R}$ incident with a 4-face $Rw_2w_3w_4$, let J_R be the edge Rw_2 together with the (≤ 6)-cycle surrounding R and containing the path $w_2w_3w_4$. For any other ring $R \in \mathcal{R}$, let $J_R = \emptyset$. Let $J = \bigcup_{R \in \mathcal{R}} R \cup J_R$ and let S be the set of all internal faces of J. Let $\{G_1, \ldots, G_k\}$ be the G-expansion of S, and for $1 \le i \le k$, let Σ_i be the surface in that G_i is embedded and let $\mathcal{R}_i$ be the natural rings of G_i, labelled so that Σ_1 is homeomorphic to Σ and $\Sigma_2, \ldots, \Sigma_k$ are disks bounded by rings of length at most 8, corresponding to the vertex rings of G incident with 4-faces. Note that G_1 has internal girth at least five, and thus $w(G_1, \mathcal{R}_1) \le \ell(R_1) + \eta_0 \mathrm{surf}(g, t, 0, 0)$ by Theorem 3.5. Furthermore, $w(G_i, \mathcal{R}_i) \le s(6)$ for $2 \le i \le k$ by Corollary 5.4. Therefore,

$$
\begin{aligned}
w(G, \mathcal{R}) &= \sum_{i=1}^{k} w(G_i, \mathcal{R}_i) \\
&\le \ell(\mathcal{R}_1) + \eta_0 \mathrm{surf}(g, t, 0, 0) + ts(6) \\
&\le \ell(\mathcal{R}) + \eta_0 \mathrm{surf}(g, t, 0, 0) + t(8 + s(6)) \\
&\le \ell(\mathcal{R}) + \eta \mathrm{surf}(g, t, t_0, t_1).
\end{aligned}
$$

Finally, suppose that G contains a 4-face which is not ring-bound. Let G' be the $\mathcal{R}$-critical graph embedded in Σ such that $|E(G')| < |E(G)|$, and let

$\{(J_f, S_f) : f \in F(G')\}$ be the cover of G by faces of G', obtained by Lemma 5.2. If G' does not contain any non-contractible 4-cycle, then let $G_0 = G'$ and $\mathcal{R}_0 = \mathcal{R}$. Otherwise, consider the unique non-contractible 4-cycle $C = w_1w_2w_3w_4$ in G', which can be flipped to a 4-face, and let $R \in \mathcal{R}$ be the ring surrounded by C. Let Δ be the closed disk bounded by C in $\Sigma + \hat{C}_R$, where C_R is the cuff corresponding to R. By symmetry, we can assume that a face f_1 whose boundary contains the path $w_1w_2w_3$ is a subset of Δ, and the face f_2 whose boundary contains the path $w_1w_4w_3$ is not a subset of Δ. See Figure 1, which also illustrates the transformation described in the rest of this paragraph. By (8), if w_4 is not incident with R, then w_4 is an internal vertex, and thus it has a neighbor in G' distinct from w_1 and w_3 drawn inside Δ. By Theorem 5.1, we conclude that the subgraph H of G' drawn in Δ is connected, and thus f_1 is open 2-cell. Let a denote the walk such that the boundary of f_1 is the concatenation of a and $w_1w_2w_3$. Note that w_1 and w_3 form a 2-cut in G'. Let G_0 be the embedding of G' in Σ obtained by mirroring $H - w_2$ (i.e., the cyclic orders of neighbors of vertices in $H - \{w_1, w_2, w_3\}$ are reversed, and the orders of the neighbors of w_1 and w_3 in $H - w_2$ are reversed). Let $\mathcal{R}_0$ be the rings of G_0 obtained from $\mathcal{R}$ by mirroring R. Note that G_0 contains no non-contractible 4-cycle, as C is the only 4-cycle in G' containing w_1 and w_3 and C becomes a 4-face in G_0. Let f be the face of G_0 whose boundary contains the walk a. Note that $\Sigma_f = \Sigma_{f_2}$. We have $w(G', \mathcal{R}) = w(G_0, \mathcal{R}) - w(f) + w(f_1) + w(f_2)$. If f_2 is not open 2-cell, then $w(f) - w(f_1) - w(f_2) = |f| - s(|f_1|) - |f_2| = |f_1| - 4 - s(|f_1|) \geq 0$. If f_2 is open 2-cell, then $w(f) - w(f_1) - w(f_2) = s(|f|) - s(|f_1|) - s(|f_2|) = s(|f_1| + |f_2| - 4) - s(|f_1|) - s(|f_2|) \geq 0$. Therefore,

$$w(G', \mathcal{R}) \leq w(G_0, \mathcal{R}_0). \tag{10}$$

Note that (10) holds trivially when $G_0 = G'$.

For $f \in F(G')$, let $\{G^f_1, \ldots, G^f_{k_f}\}$ be the G-expansion of S_f and for $1 \leq i \leq k_f$, let Σ^f_i be the surface in that G^f_i is embedded and let $\mathcal{R}^f_i$ denote the natural rings of G^f_i. We have

$$w(G, \mathcal{R}) = \sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G^f_i, \mathcal{R}^f_i). \tag{11}$$

Consider a face $f \in F(G')$. We have $g(\Sigma_f) \leq g$. If $g(\Sigma_f) = g$, then every component of G' is planar, and since G' is $\mathcal{R}$-critical and triangle-free, Theorem 1.1 implies that each component of G' contains at least one ring of $\mathcal{R}$; consequently, f has at most $|\mathcal{R}|$ facial walks and Σ_f has at most $|\mathcal{R}|$ cuffs. Since the surfaces embedding the components of the G-expansion of S_f are fragments of Σ_f, we have $(G^f_i, \Sigma^f_i, \mathcal{R}^f_i) \prec (G, \Sigma, \mathcal{R})$ for $1 \leq i \leq k_f$: otherwise, we would have $m(G^f_i) = m(G)$, hence by the definition of G-expansion, the boundary of S_f would have to be equal to the union of rings in $\mathcal{R}$, contrary to the definition of a cover of G by faces of G'.

Therefore, we can apply Theorem 4.1 inductively for G^f_i and we get $w(G^f_i, \mathcal{R}^f_i) \leq \ell(\mathcal{R}^f_i) + \eta \mathrm{surf}(g(\Sigma^f_i), |\mathcal{R}^f_i|, t_0(\mathcal{R}^f_i), t_1(\mathcal{R}^f_i))$. Observe that since $\{\Sigma^f_1, \ldots, \Sigma^f_{k_f}\}$ are fragments of Σ_f, we have

$$\sum_{i=1}^{k_f} \mathrm{surf}(g(\Sigma_i^f), |\mathcal{R}_i^f|, t_0(\mathcal{R}_i^f), t_1(\mathcal{R}_i^f)) \leq \mathrm{surf}(f),$$

and we obtain

$$\sum_{i=1}^{k_f} w(G_i^f, \mathcal{R}_i^f) \leq |f| + \mathrm{el}(f) + \eta \mathrm{surf}(f). \tag{12}$$

In case that f is closed 2-cell, all fragments of f are disks and we can use Corollary 5.4 instead of Theorem 4.1, getting a stronger inequality $w(G_i^f, \mathcal{R}_i^f) \leq s(\ell(\mathcal{R}_i^f) - \delta_i)$ for $1 \leq i \leq k_f$, where $\delta_i = 0$ if G_i^f is equal to its ring and $\delta_i = 2$ otherwise. Furthermore, if $\mathrm{el}(f) > 0$, then Corollary 5.4 implies that (J_f, S_f) is non-trivial and $\mathrm{el}(f) = 2$, and thus summing these inequalities, we can strengthen (12) to

$$\sum_{i=1}^{k_f} w(G_i^f, \mathcal{R}_i^f) \leq w(f). \tag{13}$$

As the sum of the elasticities of the faces of G' is at most 4 by Lemma 5.2, inequalities (11) and (12) give

$$\begin{aligned} w(G, \mathcal{R}) &\leq \sum_{f \in F(G')} (w(f) + \mathrm{el}(f) + \eta \mathrm{surf}(f)) \\ &\leq w(G', \mathcal{R}) + 4 + \eta \sum_{f \in F(G')} \mathrm{surf}(f). \end{aligned}$$

If G' has a face that is neither open 2-cell nor omnipresent, then (3) applied to G_0 and (10) imply that

$$w(G', \mathcal{R}) \leq \ell(\mathcal{R}) + \eta\Big(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 7 - \sum_{f \in F(G')} \mathrm{surf}(f)\Big),$$

and consequently G satisfies the outcome of Theorem 4.1. Therefore, we can assume that all internal faces of G' are either open 2-cell or omnipresent. Similarly, using (6) we can assume that if no face of G' is omnipresent, then all of them are closed 2-cell.

Suppose first that G' has no omnipresent face. Using (11) and (13), applying Theorem 4.1 inductively for G_0 and using (10), we have

$$\begin{aligned} w(G, \mathcal{R}) &= \sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \mathcal{R}_i^f) \\ &\leq \sum_{f \in F(G')} w(f) \\ &= w(G', \mathcal{R}) \leq w(G_0, \mathcal{R}_0) \\ &\leq \ell(\mathcal{R}) + \eta \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1). \end{aligned}$$

It remains to consider the case that G' has an omnipresent face h. Then, every component of G is a plane graph with one ring, and by Lemma 3.2, we conclude that every internal face of G different from h is closed 2-cell. By Lemma 5.2, we have $\mathrm{el}(h) \le 2$. By (11), (12), (13) and (10), we have

$$\begin{aligned} w(G,\mathcal{R}) &\le w(h) + \mathrm{el}(h) + \sum_{f \in F(G'), f \neq h} w(f) \\ &= w(G',\mathcal{R}) + \mathrm{el}(h) \le w(G_0,\mathcal{R}_0) + \mathrm{el}(h). \end{aligned}$$

However, by (5), we have $w(G_0,\mathcal{R}_0) \le \ell(\mathcal{R}) + \eta\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 2$, and the claim of the theorem follows. □

For use in a future paper of this series, let us formulate a corollary of Theorem 4.1. Let G be a graph embedded in a surface Σ with only facial rings. For a real number η and an internal face f of G, let $w_\eta(f) = w(f) + \eta\mathrm{surf}(f)$. We define $w_\eta(G)$ as the sum of $w_\eta(f)$ over the internal faces f of G.

Corollary 5.6. *There exists a constant $\eta > 0$ such that the following holds. Let G be a triangle-free graph embedded in a surface Σ without non-contractible 4-cycles and with only facial rings $\mathcal{R}$. Let B be the union of the rings $\mathcal{R}$. If G is $\mathcal{R}$-critical, then $w_\eta(G) \le w_\eta(B)$.*

Proof. Choose η as the constant of Theorem 4.1. Suppose first that G is embedded in the disk with one ring of length $l \ge 4$. Since G is $\mathcal{R}$-critical, it is connected and thus all faces of G are open 2-cell, and $w_\eta(f) = w(f)$ for every internal face f of G. Note furthermore that $w_\eta(B) = s(l)$. Hence, $w_\eta(G) = w(G,\mathcal{R}) \le s(l-2) \le w_\eta(B)$ by Corollary 5.4.

Hence, we can assume that Σ is not the disk, and thus

$$w_\eta(B) = \eta\mathrm{surf}(g(\Sigma), |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R})) + \ell(\mathcal{R}).$$

If all internal faces of the embedding of G in Σ are open 2-cell, then $w_\eta(G) = w(G,\mathcal{R})$ and the claim follows from Theorem 4.1. Otherwise, the claim follows from (3) or (5). □

6 Forcing 3-colorability by removing $O(g)$ vertices

Finally, we show that in a triangle-free graph of genus g, it suffices to remove $O(g)$ vertices to make it 3-colorable.

Proof of Theorem 1.4. Let κ be the constant of Theorem 1.3. We let $\beta = \max(5\kappa, 4)$. We prove Theorem 1.4 by induction on the number of vertices. For $g = 0$, the claim holds by Grötzsch's theorem, hence assume that $g > 0$. Let G be a triangle-free graph embedded in a surface Σ of Euler genus g, such that Theorem 1.4 holds for all graphs with less than $|V(G)|$ vertices. Without loss of generality, we can assume that all the faces in the embedding of G are open 2-cell.

Suppose that G contains a non-facial 4-cycle K. If K is non-contractible, then $G - V(K)$ can be embedded in a surface of Euler genus at most $g - 1$. By the induction hypothesis, there exists a set X' of size at most $\beta(g-1)$ such that $G - V(K) - X'$ is 3-colorable, and we can set $X = V(K) \cup X'$.

Hence, assume that K is contractible. Let G' be obtained from G by removing vertices and edges contained in the open disk bounded by K. By the induction hypothesis, there exists a set X of size at most βg such that $G' - X$ is 3-colorable. We claim that $G - X$ is also 3-colorable. Indeed, consider any 3-coloring φ of $G' - X$ and let H be the subgraph of G drawn in the closed disk bounded by K. Let ψ be a 3-coloring of K that matches φ on $V(K) \setminus X$. By Theorem 5.1, ψ extends to a 3-coloring φ' of H. Hence, φ together with the restriction of φ' to $V(H) \setminus X$ gives a 3-coloring of $G - X$.

Therefore, we can assume that every 4-cycle in G bounds a face. If G is 4-critical, then let $X = \{v\}$ for an arbitrary vertex $v \in V(G)$. By the definition of a 4-critical graph, $G - X$ is 3-colorable. Hence, assume that G is not a 4-critical graph.

Let $G_0 \subseteq G$ be a maximal subgraph of G such that every component of G_0 is 4-critical. Let X consist of all vertices of G_0 that are incident with a face of length greater than 4. Firstly, we bound the size of X. Let $H_1, \ldots, H_k$ be the components of G_0. For $i = 1, \ldots, k$, let us obtain an embedding of H_i in a surface Σ_i such that every face is open 2-cell from the embedding of H_i in Σ as follows: as long as there exists a face f that is not open 2-cell, cut along a closed non-contractible curve contained in f and cap the resulting holes by disks. Since H_i is 4-critical, triangle-free and every 4-cycle in H_i bounds a face, Theorem 1.3 implies that $|X \cap V(H_i)| \le 5\kappa g(\Sigma_i)$. Observe that since $H_1, \ldots, H_k$ are vertex-disjoint, we have $g \ge g(\Sigma_1) + \ldots + g(\Sigma_k)$, and thus $|X| \le 5\kappa g$ as required.

Consider any $i \in \{1, \ldots, k\}$. If $V(H_i) \cap X = \emptyset$, then all faces of H_i have length 4. Since all 4-cycles in G bound faces, it follows that $G = H_i$. However, this contradicts the assumption that G is not 4-critical. Therefore, X intersects every component of G_0.

We claim that $G - X$ is 3-colorable. Indeed, if that is not the case, then $G - X$ contains a 4-critical subgraph G'. By the maximality of G_0, it follows that there exists $i \in \{1, \ldots, k\}$ such that G' intersects H_i. Since $V(H_i) \cap X \neq \emptyset$, we have $G' \neq H_i$, and since G' and H_i are 4-critical, it follows that $G' \not\subseteq H_i$. Since G' is 4-critical, it is connected, and thus there exists $uv \in E(G)$ such that $u \in V(G') \setminus V(H_i)$ and $v \in V(G') \cap V(H_i)$. Let f be the face of H_i containing u, and note that v is incident with f. Since $v \in V(G')$, we have $v \notin X$, and thus f is a 4-face. However, then the 4-cycle bounding the face f separates u from all vertices of X in G, which contradicts the assumption that all 4-cycles in G bound faces. $\square$

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