THE EXTREMAL FUNCTIONS FOR TRIANGLE-FREE GRAPHS WITH EXCLUDED MINORS[1]

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Abstract

We prove two results:

1. A graph G on at least seven vertices with a vertex v such that $G - v$ is planar and t triangles satisfies $|E(G)| \leq 3|V(G)| - 9 + t/3$.
2. For $p = 2, 3, \ldots, 9$, a triangle-free graph G on at least $2p - 5$ vertices with no K_p-minor satisfies $|E(G)| \leq (p-2)|V(G)| - (p-2)^2$.

1 Introduction

All *graphs* in this paper are finite and simple. *Cycles* have no "repeated" vertices. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. An *H-minor* is a minor isomorphic to H. Mader [6] proved the following beautiful theorem.

Theorem 1.1. *For $p = 2, 3, \ldots, 7$, a graph with no K_p-minor and $V \geq p - 1$ vertices has at most $(p-2)V - \binom{p-1}{2}$ edges.*

For large p however, a graph on V vertices with no K_p-minor can have up to $\Omega(p\sqrt{\log p}V)$ edges as shown by several people (Kostochka [4, 5], and Fernandez de la Vega [2] based on Bollobás, Catlin and Erdös [1]),. Already for $p = 8, 9$, there are K_p-minor-free graphs on V vertices with strictly more than $(p-2)V - \binom{p-1}{2}$ edges, but the exceptions are known. Given a graph G and a positive integer k, we define *(G, k)-cockades* recursively as follows. A graph isomorphic to G is a (G, k)-cockade. Moreover, any graph isomorphic to one obtained by identifying complete subgraphs of size k of two (G, k)-cockades is also a (G, k)-cockade, and every (G, k)-cockade is obtained this way. The following is a theorem of Jørgensen [3].

Theorem 1.2. *A graph on $V \geq 7$ vertices with no K_8-minor has at most $6V - 21$ edges, unless it is a $(K_{2,2,2,2,2}, 5)$-cockade.*

The next theorem is due to Song and the first author [10].

Theorem 1.3. *A graph on $V \geq 8$ vertices with no K_9-minor has at most $7V - 28$ edges, unless it is a $(K_{1,2,2,2,2,2}, 6)$-cockade or isomorphic to $K_{2,2,2,3,3}$.*

[1] 12 January 2018, revised 14 July 2018.
[2] Partially supported by NSF under Grant No. DMS-1700157.

The first author and Zhu [11] conjecture the following generalization.

Conjecture 1.4. *A graph on $V \geq 9$ vertices with no K_{10}-minor has at most $8V-36$ edges, unless it is isomorphic to one of the following graphs:*

(1) *a $(K_{1,1,2,2,2,2,2,2}, 7)$-cockade,*

(2) $K_{1,2,2,2,3,3}$*,*

(3) $K_{2,2,2,2,2,3}$*,*

(4) $K_{2,2,2,2,2,3}$ *with an edge deleted,*

(5) $K_{2,3,3,3,3}$*,*

(6) $K_{2,3,3,3,3}$ *with an edge deleted,*

(7) $K_{2,2,3,3,4}$*, and*

(8) *the graph obtained from the disjoint union of $K_{2,2,2,2}$ and C_5 by adding all edges joining them.*

McCarty and the first author studied the extremal functions for *linklessly embeddable graphs*: graphs embeddable in 3-space such that no two disjoint cycles form a non-trivial link. Robertson, Seymour, and the first author [9] showed that a graph is linklessly embeddable if and only if it has no minor isomorphic to a graph in the *Petersen family*, which consists of the seven graphs (including the Petersen graph) that can be obtained from K_6 by ΔY- or $Y\Delta$-transformations. Thus, Mader's theorem implies that a linklessly embeddable graph on V vertices has at most $4V-10$ edges. McCarty and the first author [8] proved the following.

Theorem 1.5. *A bipartite linklessly embeddable graph on $V \geq 5$ vertices has at most $3V-10$ edges, unless it is isomorphic to $K_{3,V-3}$.*

In the same paper McCarty and the first author made the following three conjectures.

Conjecture 1.6. *A triangle-free linklessly embeddatble graph on $V \geq 5$ vertices has at most $3V-10$ edges, unless it is isomorphic to $K_{3,V-3}$.*

As a possible approach to Conjecture 1.6 McCarty and the first author proposed the following.

Conjecture 1.7. *A linklessly embeddable graph on $V \geq 7$ vertices with t triangles has at most $3V-9+t/3$ edges.*

The third conjecture of McCarty and the first author is as follows.

Conjecture 1.8. *For $p = 2, 3, \ldots, 8$, a bipartite graph on $V \geq 2p-5$ vertices with no K_p-minor has at most $(p-2)V-(p-2)^2$ edges.*

1.1 Our results

We first give a partial result to Conjectures 1.6 and 1.7. An *apex graph* is a graph G with a vertex a such that $G-a$ is planar. All apex graphs are linklessly embeddable. We show that Conjectures 1.6 and 1.7 hold for apex graphs:

Theorem 1.9. *A triangle-free apex graph on $V \geq 5$ vertices has at most $3V-10$ edges, unless it is isomorphic to $K_{3,V-3}$. Moreover, an apex graph on $V \geq 7$ vertices with t triangles has at most $3V-9+t/3$ edges.*

Let us remark that the assumption that $V \geq 7$ is necessary: let G be the graph obtained from K_6 by deleting a perfect matching. Then G has six vertices, 12 edges and eight triangles; thus $|E(G)| = 12 \not\leq 35/3 = 3V-9+t/3$.

Our second result proves a generalization of Conjecture 1.8 to triangle-free graphs for values of p up to 9:

Theorem 1.10. *For $p = 2,3,\ldots,9$, a triangle-free graph with no K_p-minor on $V \geq 2p-5$ vertices has at most $(p-2)V-(p-2)^2$ edges.*

We prove Theorem 1.9 in Section 2 and Theorem 1.10 in Section 3.

2 Proof of Theorem 1.9

For an integer V, by V^+ we denote $\max\{V,0\}$, and we define $\psi(V) := (7-V)^+ + (5-V)^+$. We need the following lemma.

Lemma 2.1. *Let $V_1, V_2 \geq 2$ be integers, and let $V = V_1+V_2-1$. Then*

$$\max\{\psi(V_1),1\} + \max\{\psi(V_2),1\} \leq \psi(V) + 10$$

with equality if and only if $V \leq 5$.

Proof. Assume first that both V_1, V_2 are at most five. If $V \geq 6$, then $V_1+V_2 \geq 7$ and we have

$$\max\{\psi(V_1),1\} + \max\{\psi(V_2),1\} = \psi(V_1)+\psi(V_2) = 7-V-1+17-(V_1+V_2) \leq \psi(V)+9.$$

If $V \leq 5$, then

$$\max\{\psi(V_1),1\} + \max\{\psi(V_2),1\} = \psi(V_1)+\psi(V_2) = 7-V-1+5-V-1+12 = \psi(V)+10.$$

We may therefore assume that say $V_2 \geq 6$. Then

$$\begin{aligned}\max\{\psi(V_1),1\} + \max\{\psi(V_2),1\} &= \max\{(5-V_1)^+ + (7-V_1)^+,1\} + \max\{(7-V_2)^+,1\}\\ &\leq 3+5+1 = 9,\end{aligned}$$

as desired. □

Let G be an apex graph on V vertices and E edges with a vertex a such that $G-a$ is planar. Let $G^\circ := G-a$ be embedded in the plane, and let $V^\circ := |V(G^\circ)|$ and $E^\circ := |E(G^\circ)|$. Note that $V = V^\circ+1$ and $E = E^\circ + d(a)$.

2.1 Triangle-free case

First suppose that G is triangle-free and that $V \geq 5$. Then $N(a)$ is an independent set. As G° is triangle-free, planar and has at least three vertices, it follows from Euler's formula that $E^\circ \leq 2V^\circ - 4$, so

$$E = E^\circ + d(a) \leq 2V^\circ - 4 + d(a) = 2V - 6 + d(a)$$

If $d(a) \leq V - 4$, then we are done. As $d(a) \leq V - 1$, we just need to check 3 cases:

1. $d(a) = V - 1$. As $N(a)$ is independent, G° is the empty graph on $V - 1$ vertices, so $E = V - 1 \leq 3V - 10$, since $V \geq 5$.

2. $d(a) = V - 2$. Then a is adjacent to all but one vertex u in G°. Since $d(u) \leq V - 2$ and $N(a)$ is independent, it follows that $E \leq 2V - 4 \leq 3V - 10$, unless $V = 5$, in which case $E \leq 3V - 10$, except when G is isomorphic to $K_{2,3}$, as desired.

3. $d(a) = V - 3$. Then a is adjacent to all but two vertices u, v in G°. Since G° is triangle-free and $N(a)$ is independent, if u is adjacent to v, then $E^\circ \leq V^\circ - 1$, in which case $E = E^\circ + V - 3 \leq 2V - 5 \leq 3V - 10$, since $V \geq 5$; and if u is not adjacent to v, then $E^\circ \leq 2(V^\circ - 2)$, in which case $E \leq E^\circ + V - 3 \leq 3V - 9$, with equality if and only if G° is isomorphic to $K_{2,V-3}$ and G is isomorphic to $K_{3,V-3}$.

Therefore $E \leq 3V - 10$, unless G is isomorphic to $K_{3,V-3}$, as desired.

2.2 General case

Now suppose that G has t triangles. Let t° denote the number of triangular faces of G° and let t_a denote the number of triangles of G incident with a. Let $t' = t^\circ + t_a$. Since $t' \leq t$, it would suffice to show that

$$E \leq 3V - 9 + t'/3 \tag{1}$$

However, this inequality does not always hold. Consider a graph G obtained from $K_{3,V-3}$ with bipartition $(\{a, b, c\}, \{v_1, \ldots, v_{V-3}\})$ by adding the edge bc and any subset of the edges $\{v_1v_2, v_2v_3, \ldots, v_{V-4}v_{V-3}\}$. This gives an apex graph, where $G - a$ is planar, with $E = 3V - 9 + t'/3 + 1/3$, violating the inequality (1). Let us call any graph isomorphic to such a graph *exceptional*. What we will show is that every graph G on at least seven vertices satisfies (1), unless G is exceptional. Note that this proves Theorem 1.9, since an exceptional graph has at least two triangles which are not counted in t', and hence satisfies the inequality in Theorem 1.9.

In fact, we prove a stronger statement, and for the sake of the inductive argument we allow graphs on fewer than seven vertices. Let $\mathcal{F}$ denote the set of faces of G°. Define

$$\phi(G, a) := \frac{t_a}{3} - \sum_{f \in \mathcal{F}} \frac{|f| - 4}{3} = \frac{t_a}{3} + \frac{t^\circ}{3} - \sum_{\substack{f \in \mathcal{F} \\ |f| \geq 5}} \frac{|f| - 4}{3} \leq \frac{t'}{3}.$$

We prove the following:

Theorem 2.2. *Let G, a, V, E be as before. and let $V \geq 2$. Then*

(1) *if G is exceptional, then $E = 3V - 9 + \phi(G,a) + 1/3$.*

Otherwise

(2) $E \le 3V - 9 + \phi(G,a) + \psi(V)/3$,

(3) *if $G - a$ has at least one non-neighbour of a, then $E \le 3V - 9 + \phi(G,a) + (7-V)^+/3$, and*

(4) *if $G - a$ has at least two non-neighbours of a, then $E \le 3V - 9 + \phi(G,a)$.*

Proof. We proceed by induction on $V + E$. If $V = 2$ and $E = 0$, then

$$E = 0 = 6 - 9 + 4/3 + (7-2)/3 = 3V - 9 + \phi(G,a) + (7-V)^+/3.$$

If $V = 2$ and $E = 1$, then

$$E = 1 = 6 - 9 + 4/3 + (7-2)/3 + (5-2)/3 = 3V - 9 + \phi(G,a) + \psi(V)/3.$$

We may therefore assume that $V \ge 3$ and that the theorem holds for all graphs G' with $|V(G')| + |E(G')| < V + E$. We suppose for a contradiction that the theorem does not hold for G. It follows that G is not exceptional, because exceptional graphs satisfy the theorem. Let $G^\circ := G - a$, V° and E° be as before.

Claim 2.2.1. *The graph G° has no cut-edges.*

Proof. Suppose $e = xy$ is a cut-edge of G° incident with a face f_e. Let C_1 be the connected component of $G^\circ - e$ containing x, and let $C_2 = G^\circ - V(C_1)$. Define $G_i := G[V(C_i) \cup \{a\}]$, $V_i := |V(G_i)|$ and $E_i := |E(G_i)|$ for $i = 1, 2$. Let $\mathcal{F}_i$ denote the set of faces of C_i, let f_i denote the face of C_i that contains f_e, and let $t_{a,i}$ denote the number of triangles incident with a in G_i for $i = 1, 2$. Then $V = V_1 + V_2 - 1$, $|f_e| = |f_1| + |f_2| + 2$, $\mathcal{F} = ((\mathcal{F}_1 \cup \mathcal{F}_2) \setminus \{f_1, f_2\}) \cup \{f_e\}$, and $t_{a,1} + t_{a,2} \le t_a - \epsilon$, where $\epsilon = 1$ if a is adjacent to every vertex of $G - a$ and $\epsilon = 0$ otherwise, and so

$$\begin{aligned}
\phi(G_1,a) + \phi(G_2,a) &= \frac{t_{a,1} + t_{a,2}}{3} - \sum_{f \in \mathcal{F}_1} \frac{|f| - 4}{3} - \sum_{f \in \mathcal{F}_2} \frac{|f| - 4}{3} \\
&\le \frac{t_a}{3} - \frac{\epsilon}{3} - \left(\sum_{f \in \mathcal{F}} \frac{|f| - 4}{3} \right) + \frac{|f_e| - 4}{3} - \frac{|f_1| + |f_2| - 8}{3} \\
&= \phi(G,a) + 2 - \epsilon/3.
\end{aligned}$$

By Lemma 2.1 $\max\{\psi(V_1), 1\} + \max\{\psi(V_1), 1\} \le \psi(V) + 10$, with equality if and only if $V \le 5$. Note that $E = E_1 + E_2 + 1$. By the induction hypothesis each G_i satisfies

$$E_i \le 3V_i - 9 + \phi(G_i, a) + \max\{\psi(V_i), 1\}/3,$$

where for $V_i \le 4$ equality holds only if a is adjacent to every vertex of $G_i - a$; thus

$$\begin{aligned}
E &= E_1 + E_2 + 1 \\
&\le 3(V_1 + V_2) - 18 + \phi(G_1,a) + \phi(G_2,a) + (\max\{\psi(V_1), 1\} + \max\{\psi(V_2), 1\})/3 + 1 \\
&\le 3V - 15 + \phi(G,a) + 3 + (\psi(V) + 10 - \epsilon)/3.
\end{aligned}$$

It follows that $E \leq 3V - 9 + \phi(G, a) + \psi(V)/3$, a contradiction, because if equality holds in the two inequalities above, then $V \leq 5$, which implies that $V_1, V_2 \leq 4$, and hence a is adjacent to every vertex of $G - a$, and consequently $\epsilon = 1$. This proves the claim in the case when either $V \geq 7$ or a is adjacent to every vertex of $G - a$.

We may therefore assume that $V \leq 6$ and that a is not adjacent to every vertex of $G - a$. Assume next that a is adjacent to all but one vertex of $G - a$. By the symmetry we may assume that a is adjacent to every vertex of $G_1 - a$ and all but one vertex of $G_2 - a$. Then

$$\psi(V_1) + (7 - V_2)^+ = 7 - V - 1 + 12 - V_1 \leq (7 - V)^+ + 9,$$

and hence

$$\begin{aligned} E &= E_1 + E_2 + 1 \\ &\leq 3(V_1 + V_2) - 18 + \phi(G_1, a) + \phi(G_2, a) + (\psi(V_1) + (7 - V_2)^+)/3 + 1 \\ &\leq 3V - 15 + \phi(G, a) + 3 + ((7 - V)^+ + 9)/3 \\ &\leq 3V - 9 + \phi(G, a) + (7 - V)^+/3, \end{aligned}$$

a contradiction.

We may therefore assume that a is not adjacent to at least two vertices of $G - a$. Assume next that a is not adjacent to at least two vertices vertices of $G_2 - a$. Then

$$\begin{aligned} E &= E_1 + E_2 + 1 \\ &\leq 3(V_1 + V_2) - 18 + \phi(G_1, a) + \phi(G_2, a) + \psi(V_1)/3 + 1 \\ &\leq 3V - 15 + \phi(G, a) + 3 + 8/3 \\ &\leq 3V - 9 + \phi(G, a), \end{aligned}$$

a contradiction.

We may therefore assume that a is not adjacent to exactly one vertex of $G_i - a$ for $i = 1, 2$. We have

$$\begin{aligned} E &= E_1 + E_2 + 1 \\ &\leq 3(V_1 + V_2) - 18 + \phi(G_1, a) + \phi(G_2, a) + ((7 - V_1)^+ + (7 - V_2)^+)/3 + 1 \\ &\leq 3V - 15 + \phi(G, a) + 3 + 10/3 \\ &= 3V - 9 + \phi(G, a) + 1/3, \end{aligned}$$

with equality if and only if $V_1 = V_2 = 2$, in which case G is exceptional, in either case a contradiction. Thus the claim holds. $\square$

Claim 2.2.2. *We have that $t_a = 0$; that is, $N(a)$ is independent. In particular, $\phi(G, a) = \sum_{f \in \mathcal{F}} \frac{4 - |f|}{3}$.*

Proof. Suppose there exist adjacent vertices $x, y \in N(a)$ As xy is not a cut-edge of G° by Claim 2.2.1, it is incident with two distinct faces f_1, f_2. Let $G' := G - xy$, let $E' := |E(G')|$ and let f' be the new face obtained in $G^\circ - xy$. Let $\mathcal{F}'$ denote the set of faces of $G^\circ - xy$ and let t'_a denote the number of triangles of G' incident with a. Then $t'_a = t_a - 1$, $|f_1| + |f_2| = |f'| + 2$, and $\mathcal{F} = (\mathcal{F}' \setminus \{f'\}) \cup \{f_1, f_2\}$, so

$$\begin{aligned}
\phi(G', a) &= \frac{t'_a}{3} - \sum_{f \in \mathcal{F}'} \frac{|f| - 4}{3} \\
&= \frac{t_a}{3} - \frac{1}{3} - \left(\sum_{f \in \mathcal{F}} \frac{|f| - 4}{3} \right) + \frac{|f_1| + |f_2| - 8}{3} - \frac{|f'| - 4}{3} \\
&= \phi(G, a) - 1
\end{aligned}$$

Since G does not satisfy the theorem, it is not exceptional, and so neither is G'. Let $x := \psi(V)$ if a is adjacent to every vertex of $G - a$, let $x := (7 - V)^+$ if a is adjacent to all but one vertex of $G - a$ and let $x := 0$ otherwise. By the induction hypothesis

$$\begin{aligned}
E &= E' + 1 \\
&\leq 3V - 9 + \phi(G', a) + 1 + x \\
&= 3V - 9 + \phi(G, a) + x,
\end{aligned}$$

a contradiction. $\square$

Claim 2.2.3. *The graph G° has no isolated vertices.*

Proof. Suppose for a contradiction that v is an isolated vertex of G°. Let $G' = G - v$, let $V' := |V(G')|$ and let $E' = |E(G')|$. Then $\phi(G', a) = \phi(G, a)$. Let $x' := \psi(V')$ and $x := \psi(V)$ if a is adjacent to every vertex of $G - a$, let $x' := (7 - V')^+$ and $x := (7 - V)^+$ if a is adjacent to all but one vertex of $G - a$ and let $x = x' := 0$ otherwise. If v is adjacent to a, then by the induction hypothesis

$$\begin{aligned}
E &= E' + 1 \\
&\leq 3V' - 9 + \phi(G', a) + 1 + \max\{x', 1\}/3 \\
&\leq 3V - 3 - 9 + \phi(G, a) + 1 + x/3 + 2/3 \\
&\leq 3V - 9 + \phi(G, a) + x/3,
\end{aligned}$$

and if v is not adjacent to a, then

$$\begin{aligned}
E = E' &\leq 3V' - 9 + \phi(G', a) + \max\{\psi(V'), 1\}/3 \\
&\leq 3V - 3 - 9 + \phi(G, a) + 8/3 \\
&\leq 3V - 9 + \phi(G, a),
\end{aligned}$$

a contradiction in either case. $\square$

Claim 2.2.4. *If $v \in N(a)$, then v has at least three neighbours in G°; that is, $d(v) \geq 4$.*

Proof. Since G° has no cut-edges by Claim 2.2.1 and no isolated vertices by Claim 2.2.3, v has at least two neighbours in G°. Suppose it has exactly two neighbours, and let f_1, f_2 be the two faces of G° incident to v. Let $G' = G - v$, let $V' := |V(G')|$, let $E' = |E(G')|$ and let f' denote the new face in $G^\circ - v$. Then $|f_1| + |f_2| = |f'| + 4$, and

$$\begin{aligned}\phi(G', a) &= -\sum_{f \in \mathcal{F}'} \frac{|f| - 4}{3} \\ &= -\left(\sum_{f \in \mathcal{F}} \frac{|f| - 4}{3}\right) + \frac{|f_1| + |f_2| - 8}{3} - \frac{|f'| - 4}{3} \\ &= \phi(G, a)\end{aligned}$$

As G does not satisfy the theorem, it is not exceptional, and hence neither is G'. Furthermore, the neighbours of v are not adjacent to a by Claim 2.2.2, and so by the induction hypothesis

$$\begin{aligned}E &= E' + 3 \\ &\leq 3V' - 9 + \phi(G', a) + 3 \\ &= 3V - 9 + \phi(G, a),\end{aligned}$$

a contradiction. □

We now show an upper bound on the degree of a by a simple discharging argument. Start by assigning a charge of one to each vertex in $N(a)$, and for each $v \in N(a)$ distribute its charge equally to its incident faces in G°. Then the sum of the charges of the faces of G° is equal to $d(a)$.

By Claim 2.2.4, each $v \in N(a)$ is incident to at least three faces, so it gives at most $1/3$ charge to each incident face. By Claim 2.2.2, each face $f \in \mathcal{F}$ is incident to at most $\lfloor |f|/2 \rfloor$ neighbours of a. Thus the final charge of face f is at most $\lfloor |f|/2 \rfloor / 3$, and

$$d(a) \leq \sum_{f \in \mathcal{F}} \frac{\lfloor |f|/2 \rfloor}{3}$$

Since $\lfloor k/2 \rfloor \leq k - 2$ for all $k \geq 3$,

$$d(a) \leq \sum_{f \in \mathcal{F}} \frac{|f| - 2}{3} \tag{2}$$

The remainder of the proof follows from arithmetic using Euler's formula. Let F° denote the number of faces of G°. By the handshaking lemma, we have $2E^\circ = \sum_{f \in \mathcal{F}} |f|$. Since $F^\circ = \sum_{f \in \mathcal{F}} 1$, by Euler's formula:

$$\begin{aligned}8 &\leq 4V^\circ - 4E^\circ + 4F^\circ \\ &= 4V^\circ - 2E^\circ - \sum_{f \in \mathcal{F}} (|f| - 4)\end{aligned}$$

Rearranging, we have

$$E^\circ \leq 2V^\circ - 4 - \sum_{f \in \mathcal{F}} \frac{|f| - 4}{2} \tag{3}$$

Similarly, we have $3F^\circ \le 2E^\circ \le 2V^\circ + 2F^\circ - 4$, which gives

$$F^\circ \le 2V^\circ - 4. \tag{4}$$

Putting (2), (3), and (4) together, we have

$$\begin{aligned}
E &= E^\circ + d(a)\\
&\le V^\circ + F^\circ - 2 + \sum_{f\in\mathcal{F}} \frac{|f|-2}{3}\\
&= V^\circ + \frac{1}{3}F^\circ + \frac{2}{3}E^\circ - 2\\
&\le V^\circ + \frac{1}{3}(2V^\circ - 4) + \frac{2}{3}\left(2V^\circ - 4 - \sum_{f\in\mathcal{F}} \frac{|f|-4}{2}\right) - 2\\
&= 3V^\circ - 6 - \sum_{f\in\mathcal{F}} \frac{|f|-4}{3}\\
&= 3V - 9 + \phi(G,a),
\end{aligned}$$

a contradiction. □

3 Proof of Theorem 1.10

We prove the following slightly more general statement from which Theorem 1.10 follows:

Theorem 3.1. *Let $p \ge 4$ be an integer. Suppose that no graph G with $|E(G)| > (p-2)|V(G)| - \binom{p-1}{2}$ can be obtained by contracting $\max\{p-4, 2\}$ edges from a triangle-free graph on at least $2p-3$ vertices with no K_p-minor. Then every triangle-free graph on $V \ge 2p-5$ vertices with no K_p-minor has at most $(p-2)V - (p-2)^2$ edges.*

Let us first show that Theorem 3.1 implies Theorem 1.10:

Proof of Theorem 1.10, assuming Theorem 3.1. For $p = 2, 3$ Theorem 1.10 is easy. For $p = 4, 5, 6, 7$, it follows directly from Theorems 1.1 and 3.1, as there are no graphs G on at least $p-1$ vertices with no K_p-minor and strictly more than $(p-2)|V(G)| - \binom{p-1}{2}$ edges.

For $p = 8$, by Theorem 1.2, a graph G on at least seven vertices with no K_8-minor and strictly more than $6|V(G)| - 21$ edges is a $(K_{2,2,2,2,2,2}, 5)$-cockade. It is easy to see that, given any four vertices of a $(K_{2,2,2,2,2,2}, 5)$-cockade, one can always find a triangle disjoint from those four vertices. Thus a $(K_{2,2,2,2,2,2}, 5)$-cockade cannot be obtained by contracting four edges from a triangle-free graph, and the result follows by Theorem 3.1.

For $p = 9$, by Theorem 1.3, a graph G on at least eight vertices with no K_9-minor and strictly more than $7|V(G)| - 28$ edges is either a $(K_{1,2,2,2,2,2,2}, 6)$-cockade or isomorphic to $K_{2,2,2,3,3}$. Again it is easy to verify that, given any five vertices of such a graph, one can always find a triangle disjoint from those five vertices. Therefore neither a $(K_{1,2,2,2,2,2,2}, 6)$-cockade nor $K_{2,2,2,3,3}$ can be obtained by contracting five edges from a triangle-free graph, and the result follows by Theorem 3.1. □

Let us remark that the same argument shows that Conjecture 1.4 and Theorem 3.1 imply that Theorem 1.10 holds for $p = 10$, formally as follows:

Theorem 3.2. *If Conjecture 1.4 holds, then every triangle-free graph on $V \geq 15$ vertices with no K_{10}-minor has at most $8V - 64$ edges.*

3.1 Proof of Theorem 3.1

Let $p \geq 4$ be an integer and let G be a counterexample with $|V(G)|$ minimum. Let $V = |V(G)|$ and $E = |E(G)|$. We prove by a series of claims that G is a complete bipartite graph. This leads to a contradiction: suppose G is isomorphic to $K_{n,V-n}$ with $n \leq V/2$. If $n \geq p-1$, then G contains a K_p-minor, and if $n \leq p-2$, then $E = n(V-n) \leq (p-2)(V-(p-2))$ as $V \geq 2p-5$.

Claim 3.2.1. *The graph G has at least $2p-3$ vertices.*

Proof. If $V \leq 2p-4$, then by Mantel's theorem [7] $E \leq (p-2)(V-p+2)$, contrary to G being a counterexample. □

Claim 3.2.2. $\delta(G) > p-2$

Proof. Let v be a vertex of G of minimum degree, and let $G' = G - v$. Then $|E(G')| = E - \delta(G)$ and $|V(G')| = V - 1$. Since G is a minimal counterexample and $V > 2p-3$ by Claim 3.2.1,

$$\begin{aligned}(p-2)V - (p-2)^2 &< E \\ &= |E(G')| + \delta(G) \\ &\leq (p-2)|V(G')| - (p-2)^2 + \delta(G) \\ &= (p-2)V - (p-2)^2 + \delta(G) - (p-2),\end{aligned}$$

and so $p-2 < \delta(G)$, as desired. □

Claim 3.2.3. *For $1 \leq k \leq p-2$, given any set of k disjoint edges $\{e_1, \dots, e_k\}$ in G, we can find another edge disjoint from each e_i, $1 \leq i \leq k$.*

Proof. Let $e_1, \dots, e_k$ be given, where $e_i = x_i y_i$. By Claim 3.2.1 there exists a vertex v not equal to any x_i, y_i. Since G is triangle-free, v can be adjacent to at most one of $\{x_i, y_i\}$ for each i. Since $deg(v) \geq \delta(G) > p-2 \geq k$ by Claim 3.2.2, there is an edge incident with v disjoint from each e_i, as desired. □

Claim 3.2.4. *Let e_1, e_2 be any two disjoint edges of G. Then $G[e_1 \cup e_2]$ forms a 4-cycle.*

Proof. Since G is triangle-free, there are at most two edges between e_1 and e_2, and if there are two edges, $G[e_1 \cup e_2]$ forms a 4-cycle. Suppose for a contradiction that there is at most one edge between e_1 and e_2. Let $k = \max\{p-4, 2\}$. By Claim 3.2.3, we can find pairwise disjoint edges $e_3, \dots, e_k$, each disjoint from both e_1 and e_2. Let G' be the graph obtained by contracting all edges $e_1, \dots, e_k$ and let ℓ denote the number of parallel edges identified. Then $|E(G')| = E - k - \ell$, $|V(G')| = V - k$, and $|E(G')| \leq (p-2)|V(G')| - \binom{p-1}{2}$ by hypothesis as G' is obtained from the graph G by contracting k edges. Since there are $\binom{k}{2}$ pairs of edges in $\{e_1, \dots, e_k\}$ and there is at most one edge between e_1 and e_2, we have $\ell \leq \binom{k}{2} - 1$. If $p \leq 5$, then let $\epsilon = 1$; otherwise let $\epsilon = 0$.

Then

$$\begin{aligned}
E &= |E(G')| + \ell + k \\
&\le \left((p-2)|V(G')| - \binom{p-1}{2}\right) + \left(\binom{k}{2} - 1\right) + k \\
&= (p-2)(V-k) - \frac{(p-1)(p-2) - k(k-1)}{2} + k - 1 \\
&= (p-2)V - (p-2)^2 - \epsilon \\
&\le (p-2)V - (p-2)^2,
\end{aligned}$$

a contradiction since G is a counterexample. □

Claim 3.2.5. *G is a complete bipartite graph.*

Proof. Let $e = xy$ be an edge and let $v \in V(G) \setminus \{x, y\}$. By Claims 3.2.2 and 3.2.4, v is adjacent to either x or y, but not both as G is triangle-free. Thus $V(G) \setminus \{x, y\}$ can be partitioned into two disjoint sets $X' \cup Y'$ where every vertex in X' is adjacent to y and every vertex in Y' is adjacent to x. Since G is triangle-free, there are no edges between vertices of X' and between vertices of Y'. Thus G is bipartite with bipartition $X \cup Y$, where $X = X' \cup \{x\}$ and $Y = Y' \cup \{y\}$. Moreover, for any $x' \in X'$ and $y' \in Y'$, the two edges xy' and $x'y$ induce a 4-cycle by Claim 3.2.4. Therefore x' is adjacent to y', completing the proof of the claim. □

References

[1] B. Bollobás, P. A. Catlin and P. Erdös, Hadwiger's conjecture is true for almost every graph, *Europ. J. Combin.* **1** (1980), 195–199.

[2] W. Fernandez de la Vega, On the maximum density of graphs which have no subtractions to K^s, *Discrete Math.* **46** (1983), 109–110.

[3] L. Jorgensen, Contraction to K_8, *J. Graph Theory* **18** (1994), 431–448.

[4] A. V. Kostochka, The minimum Hadwiger number for graphs with a given mean degree of vertices, *Metody Diskret. Analiz.* **38** (1982), 37—58 (in Russian).

[5] A. V. Kostochka, A lower bound for the product of the Hadwiger numbers of a graph and its complement, *Combinatorial analysis* **8** (1989), 50—62 (in Russian).

[6] W. Mader, Homomorphiesätze für Graphen, *Math. Ann.* **178** (1968), 154—168.

[7] W. Mantel, Problem 28 (Solution by H. Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh and W. A. Wythoff), *Wiskundige Opgaven* **10** (1907), 60—61.

[8] R. McCarty and R. Thomas, The extremal function for bipartite linklessly embeddable graphs, `arXiv:1708.08439`.

[9] N. Robertson, P. D. Seymour and R. Thomas, Sach's linkless embedding conjecture, *J. Combin. Theory Ser. B* **64** (1995), 185–227.

[10] Z. Song and R. Thomas, The extremal function for K_9 minors, *J. Combin. Theory Ser. B* **96** (2006), 240–252.

[11] R. Thomas and D. Zhu, unpublished.

This material is based upon work supported by the National Science Foundation. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.