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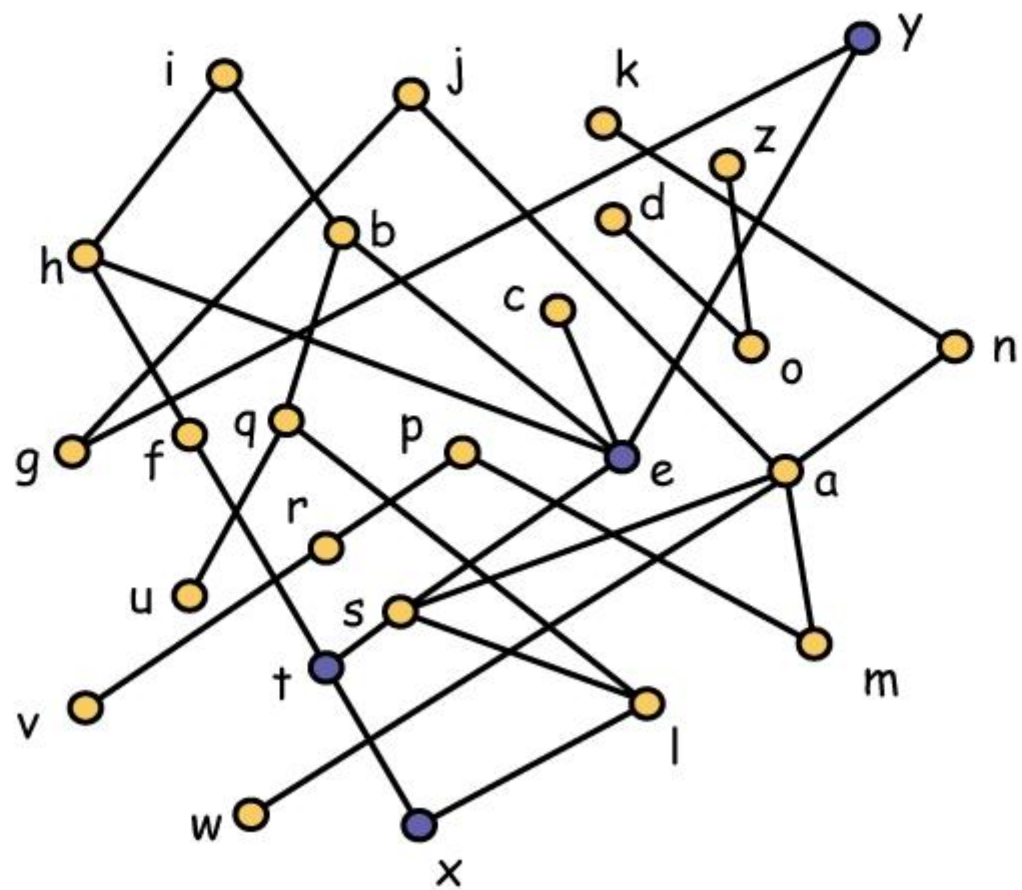
11 – Chain and Antichain Partitions

William T. Trotter
trotter@math.gatech.edu

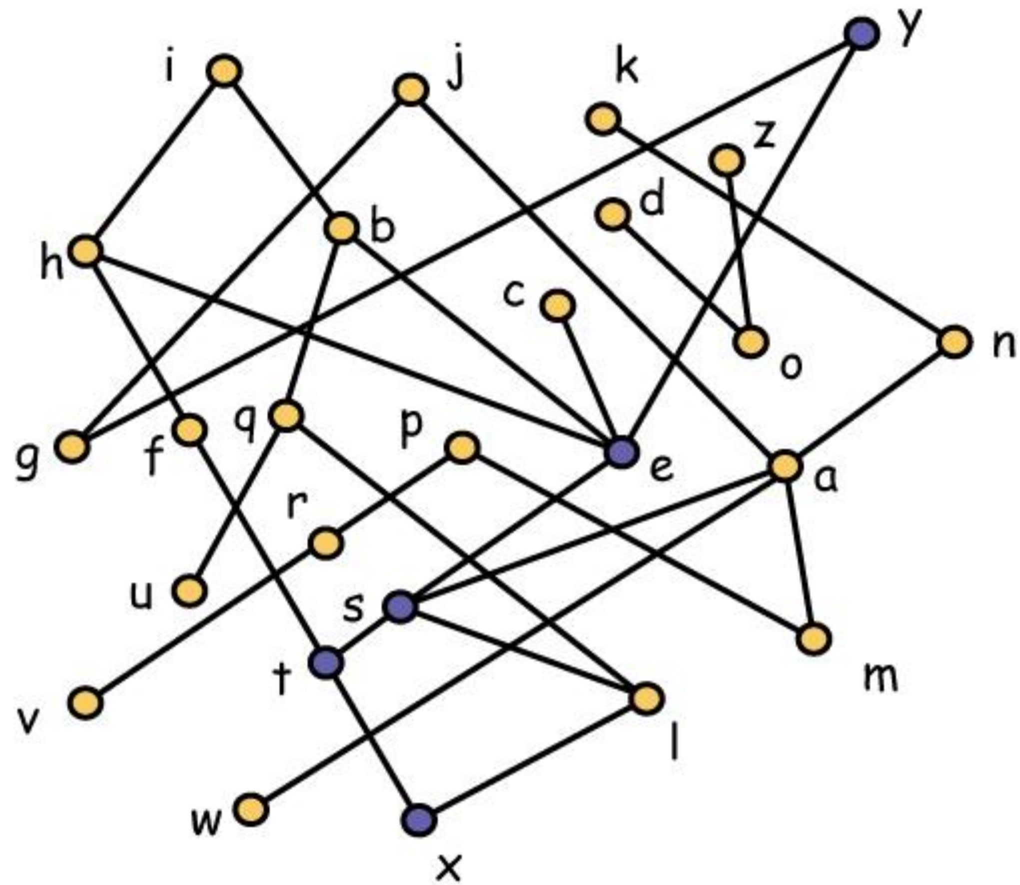
A Chain of Size 4

Definition *A* **chain** is a subset in which every pair is comparable.



A Maximal Chain of Size 5

Definition A chain is **maximal** when no superset is also a chain.



Height of a Poset

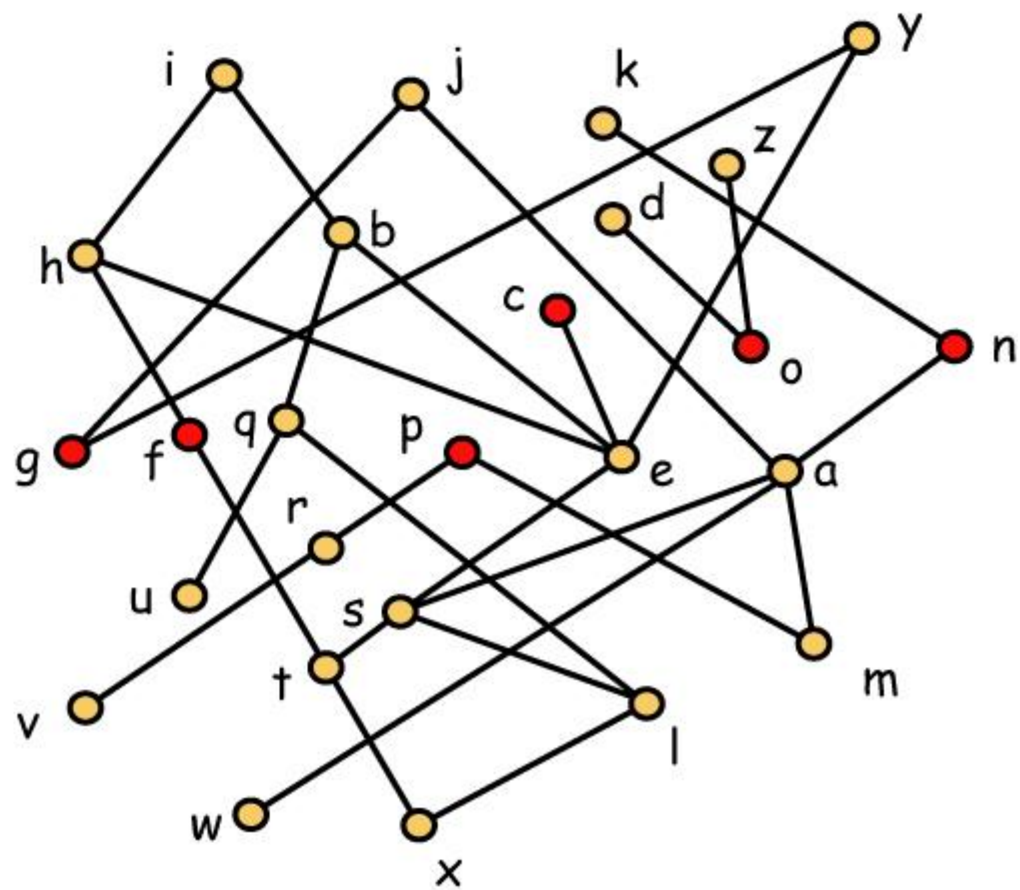
Definition The **height** of a poset P is the maximum size of a chain in P.

Proposition To partition a poset P of height h into antichains, at least h antichains are required.

Question How hard is it to find the height of a poset and the minimum size of a partition of the poset into antichains?

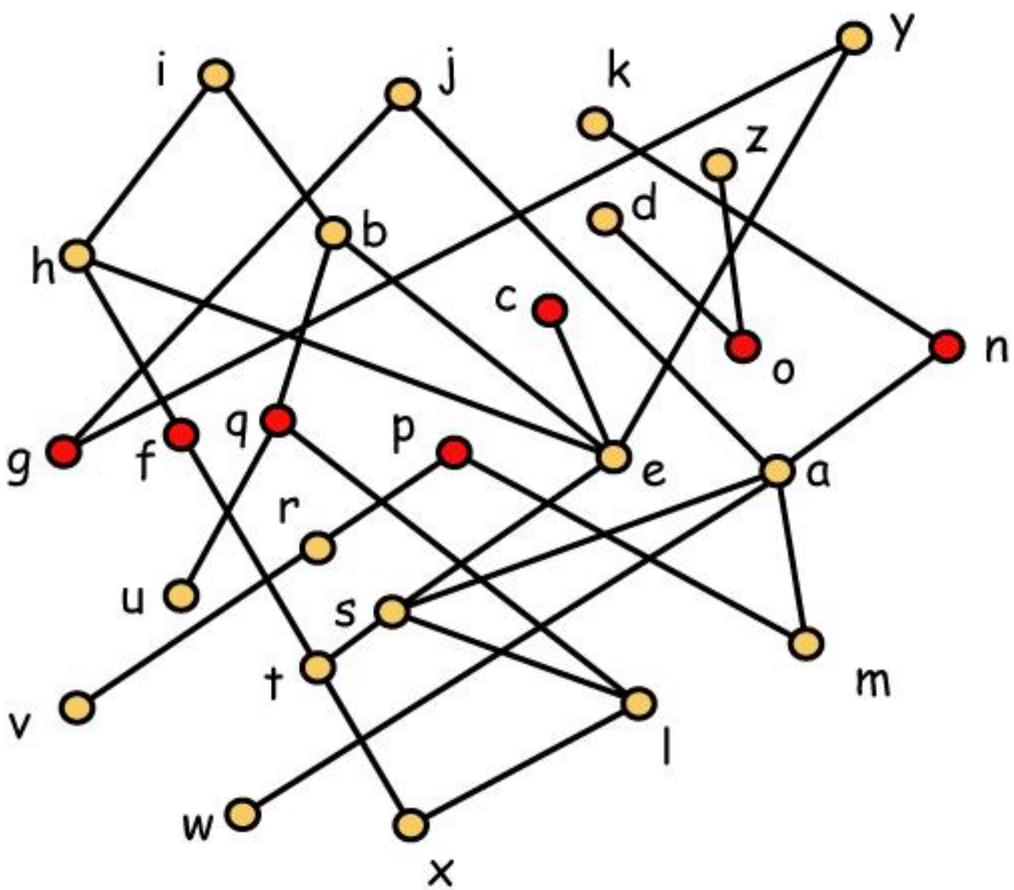
An Antichain of Size 6

Definition A subset is an **antichain** when every pair is incomparable.



A Maximal Antichain of Size 7

Definition An antichain is **maximal** when no superset is an antichain.



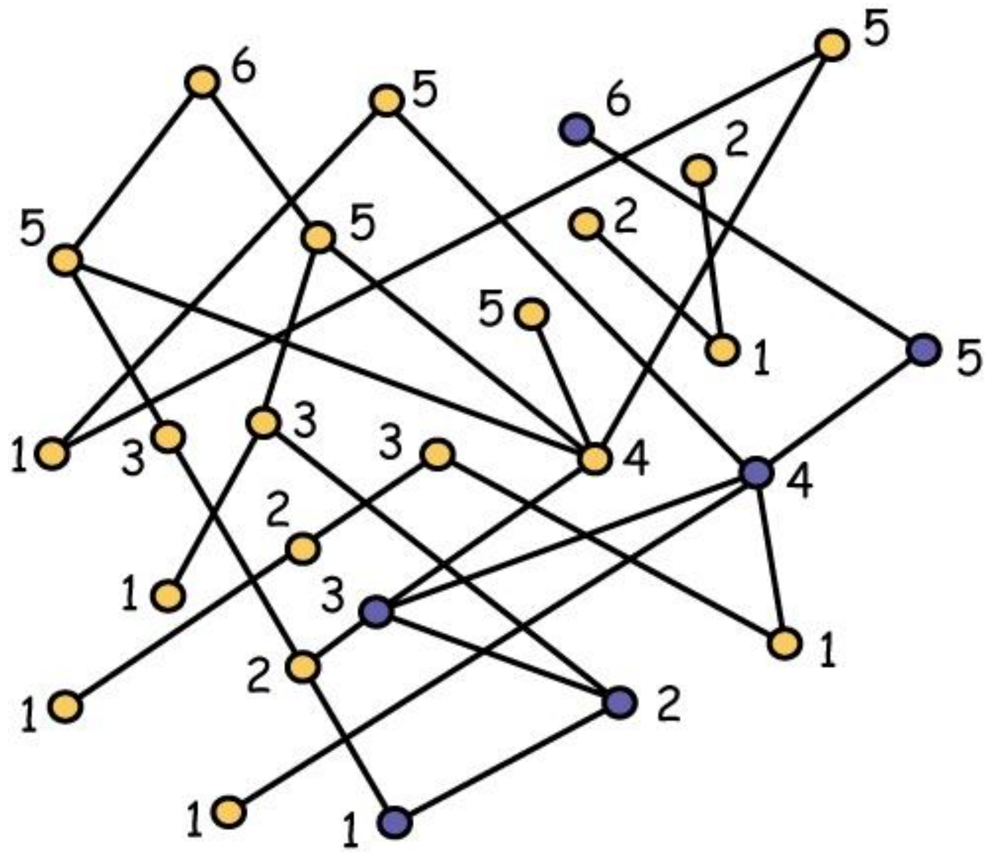
Width of a Poset

Definition The **width** of a poset P is the maximum size of an antichain in P.

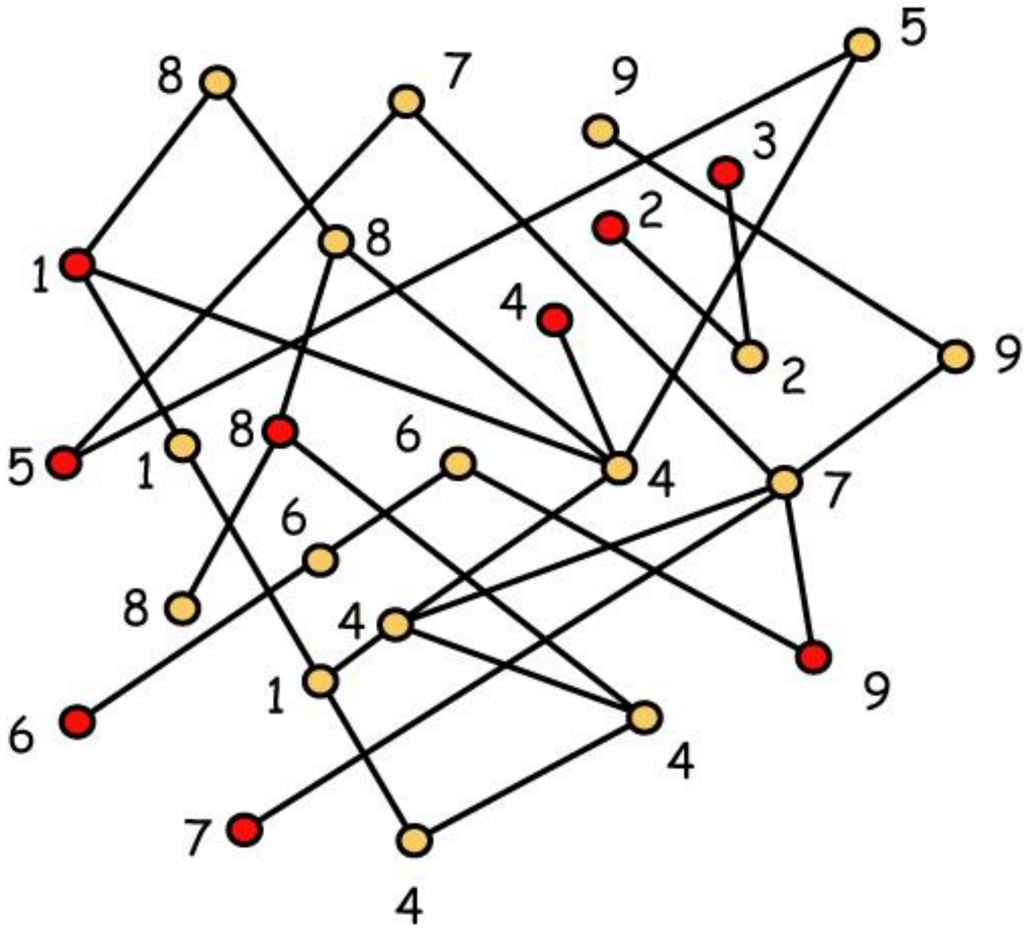
Proposition To partition a poset P of width w into chains, at least w chains are required.

Question How hard is it to find the width of a poset and the minimum size of a partition of the poset into chains?

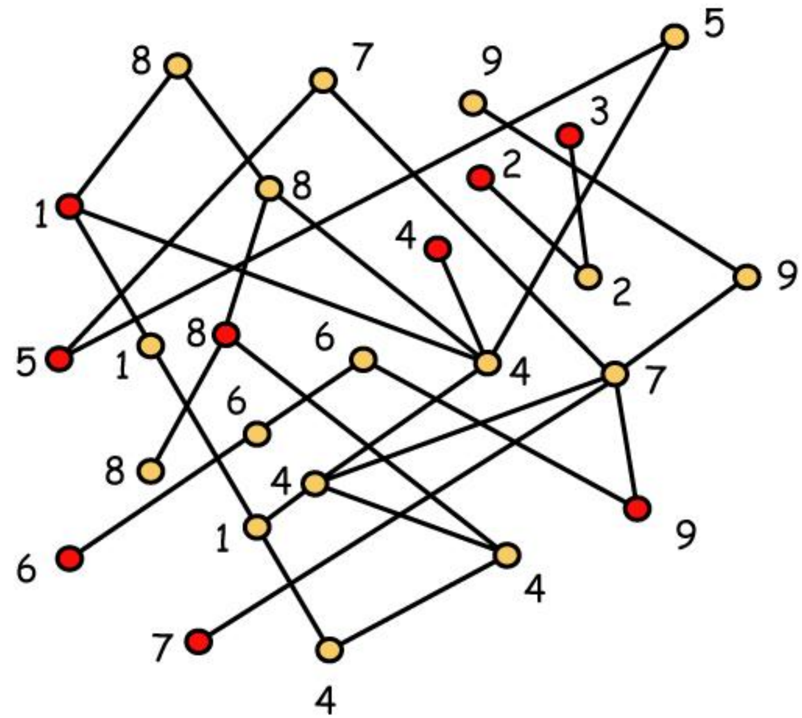
Height = 6



Width = 9



Dilworth's Theorem



Theorem (1950) A poset of width w can be partitioned into w chains.

Proofs of Dilworth's Theorem

Fulkerson (1954) Used bipartite matching algorithm (network flows) to find minimum chain partition and maximum antichain simultaneously. We will study this right at the end of the course.

Gallai/Milgram (1960) Path decompositions in oriented graphs.

Perles (1963) Simple induction depending on whether there is a maximum antichain A with $U(A)$ and $D(A)$ non-empty. This is the proof found in most combinatorics textbooks.

The Proof of Dilworth's Theorem (1)

Proof True when width $w = 1$ and thus when $|P| = 1$. Assume valid when $|P| \leq k$. Then consider a poset P with $|P| = k + 1$.

For each maximal antichain A, let $D(A) = \{x : x < a$ for some a in $A\}$, and $U(A) = \{x : x > a$ for some a in $A\}$. Evidently, $P = A \cup D(A) \cup U(A)$ is a partition into pairwise disjoint sets.

The Proof of Dilworth's Theorem (2)

Case 1 There exists a maximum antichain A with both $D(A)$ and $U(A)$ non-empty.

Label the elements of A as $a_1, a_2, \ldots, a_w$. Then apply the inductive hypothesis to $A \cup D(A)$, which has at most k points, since $U(A)$ is non-empty. WLOG, we obtain a chain partition $C_1, C_2, \ldots, C_w$ of $A \cup D(A)$ with a_i the greatest element of C_i for each $i = 1, 2, \ldots, w$.

The Proof of Dilworth's Theorem (3)

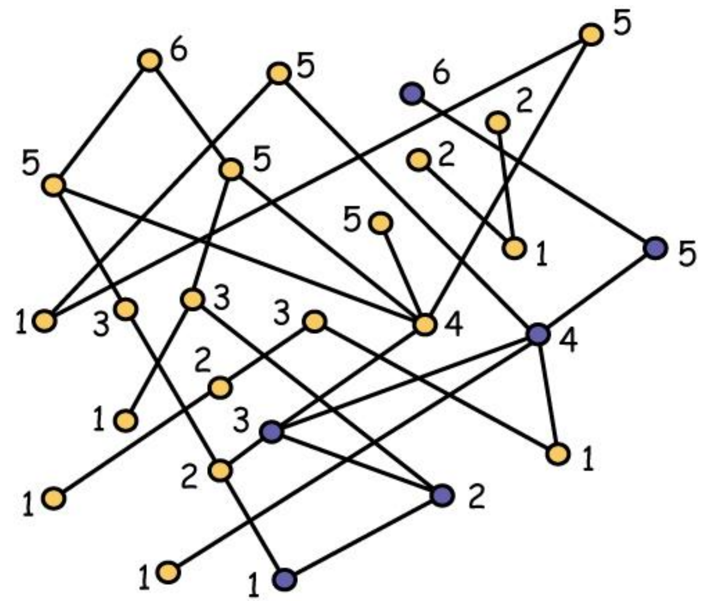
Then apply the inductive hypothesis to $A \cup U(A)$. WLOG, we obtain a chain partition $C'_1, C'_2, \ldots, C'_w$ with a_i the least element of C'_i for each i. Then $C_i \cup C'_i$ is a chain for each $i = 1, 2, \ldots, w$ and these w chains cover P.

Case 2 For every maximum antichain A, at least one of $D(A)$ and $U(A)$ is empty.

Choose a maximal element y. Then choose a minimal element x with $x \leq y$ in P. Note that we allow $x = y$. Regardless, $C = \{x, y\}$ is a chain – of either one or two points - and the width of $P - C$ is $w - 1$. Partition $P - C$ into $w - 1$ chains, and then add chain C to obtain the desired chain partition of P.

Dilworth's Theorem – Dual Form

Theorem A poset of height h can be partitioned into h antichains.



Basic Idea for the Proof Recursively strip off the minimal elements.

Details for the Proof of Dual Dilworth

Proof For each i, let, A_i consist of those elements x from P for which the longest chain in P with x as its largest element has i elements. Evidently, each A_i is an antichain. Furthermore, the number of non-empty antichains in the resulting partition is just h, the height of P. Also, a chain C of size h can be easily found using back-tracking, starting from any element of A_h.

Algorithm A_1 is just the set of minimal elements of P. Thereafter, A_{i+1} is just the set of minimal elements of the poset resulting from the removal of $A_1, A_2, \ldots, A_i$.

Historical Notes on Dilworth's Theorem

1. Gallai & Milgram published their work in 1960, but they had the result much earlier (in the late 1940's) before Dilworth's theorem was published. Also, the Gallai-Milgram theorem is stronger than Dilworth's theorem.
2. But Dilworth knew the chain partitioning theorem much earlier too, so it remains historically accurate to attribute the result to Dilworth.
3. Sweeping generalizations of Dilworth's theorem were obtained in 1976 by Greene and Kleitman.

Historical Notes on Dual Dilworth

1. Dilworth, Fulkerson, Gallai & Milgram and many others also knew the dual form of Dilworth's theorem in the 1940's, but evidently, all of them considered the result too trivial to write down.
2. However, the dual form of Dilworth's theorem was published in 1971 by Mirsky in a one page paper. Today most researchers just refer to the dual result as "dual Dilworth" and don't make an attribution.
3. A powerful (and very non-trivial) extension of dual Dilworth was published by Greene in 1976.