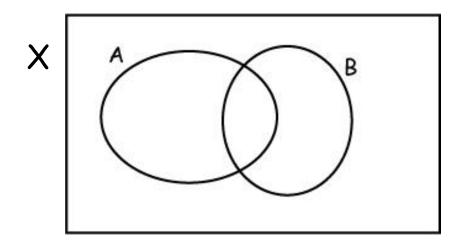# 15 – Inclusion/Exclusion

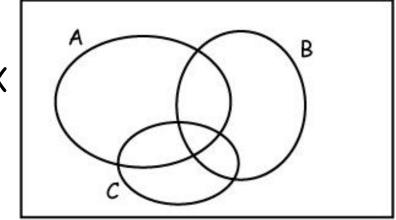## William T. Trotter
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Inclusion/Exclusion – Prelude

Question In the "Venn Diagram" shown below, the universe X contains 23 elements. There are 8 in the set A and 11 in B. If there are 5 in A ∩ B, then how many elements of X belong to neither A nor B?

Inclusion/Exclusion – Prelude (2)

Question In the "Venn Diagram" shown below, the universe X contains 2307 elements.   We want to determine the number of elements of X that don't belong to any of A, B and C. If we know the number of elements in the following sets, can we do this? A, B, C, A $\cap$ B, B $\cap$ C, A $\cap$ C, A $\cap$ B $\cap$ C.

X

Inclusion/Exclusion (1)

Notation Let X be a set of objects and suppose that for every element i in $\{1, 2, \ldots, n\}$, we have a property P_i so that for all x in X, the statement "x satisfies property P_i" is either true or false … but never ambiguous. Then for a subset S of $\{1, 2, \ldots, n\}$, let $N(S)$ be the subset of X consisting of all x in X which satisfy property P_i for all i in S. Note that $N(\emptyset) = X$.

Notation Let N_0 be the subset of X consisting of those objects which satisfy none of the properties.

Inclusion/Exclusion (2)

Theorem   Let X be a set of objects and let P_i be a property for X for each i = 1, 2, …, n. Then:

$$N_0 = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} N(S)$$

Example   When n = 2,

$$N_0 = N(\emptyset) - N(1) - N(2) + N(1).$$

Inclusion/Exclusion (3)

Theorem Let X be a set of objects and let P_i be a property for X for each $i = 1, 2, ..., n$. Then:

$$N_0 = \sum_{S \subseteq \{1,2,...,n\}} (-1)^{|S|} N(S)$$

Example When $n = 3$,

$N_0 = N(\emptyset)$

$\qquad - N(1) - N(2) - N(3)$

$\qquad + N(12) + N(13) + N(23)$

$\qquad - N(123).$

Inclusion/Exclusion (4)

Example When n = 4,

$N_0 = N(\emptyset)$

$\quad\quad - N(1) - N(2) - N(3) - N(4)$

$\quad\quad + N(12) + N(13) + N(14) + N(23) + N(24) + N(34)$

$\quad\quad - N(123) - N(124) - N(134) - N(234)$

$\quad\quad + N(1234).$

Inclusion/Exclusion (5)

Observation   In general, there are 2^n terms in the inclusion/exclusion formula. How can this possibly be of use?

Conclusion   Inclusion/Exclusion may be of value when |N(S)| depends only on |S|. Also, it may be of value when there is some other form of "collapsing" among the exponentially many terms in the formula.

Derangements

Definition A permutation σ of {1, 2, …, n} is called a derangement if σ(i) ≠ i for all i = 1, 2, …, n.

Example 38754126 and 21436587 are derangements but 57314682 and 75318642 are not.

Exercise Write all derangements of {1,2,3,4,5}.

Notation Let d_n denote the number of derangements of {1, 2, …, n}.

Derangements (2)

Inclusion/Exclusion Formula for Derangements

$$d_n = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} N(S)$$

$$= \sum_{0 \le k \le n} (-1)^k C(n,k) (n-k)!$$

Explanation   When S is a subset of $\{1, 2, \ldots, n\}$ and $|S| = k$, $|N(S)| = (n-k)!$ To see this, note that if σ satisfies P_i and i belongs to S, then $\sigma(i) = i$. So the positions corresponding to elements of S are determined, and the other $n-k$ positions are an arbitrary permutation of the remaining elements.

Surjections (1)

Notation   For an integer n, let [n] denote {1, 2, …, n}. Also, let S(n, m) denote the number of surjections from [n] to [m].

Exercise   Determine S(5, 3) by hand.

Surjections (2)

Inclusion/Exclusion Formula for Surjections

$$S(n, m) \; = \; \sum\nolimits_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} \, N(S)$$

$$= \; \sum\nolimits_{0 \le k \le m} (-1)^k \, C(m, k) \, (m - k)^n$$

Explanation When S is a subset of $\{1, 2, \ldots, m\}$ and $|S| = k$, $|N(S)| = (m - k)^n$. To see this, note that if f satisfies P_i and i belongs to S, then i is not in the range of f. In other words, f is an function whose domain is $[n]$ and whose range is a set of size $m - k$.

The Euler φ- function

Notation   For an integer n ≥ 2, let φ(n) denote the number of elements in [n] which are relatively prime to n.

Example φ(12) = 4 since 1, 5, 7 and 11 are relatively prime to 12.

Exercise   Compute φ(144).

Exercise Compute φ(324481700624).

The Euler φ- function

Inclusion/Exclusion Formula for Euler φ-Function

Suppose the prime factors of n are: $p_1, p_2, …, p_k$.

Then

$φ(n) = n (1 – 1/p_1)(1 – 1/p_2) …(1-1/p_k)$

Explanation When m has the common prime factors p_3, p_7 and p_8 with n, then the number of such m is $n/p_3p_7p_8$.

The Euler φ- function

Example Compute φ(324481700624)

Maple reports that

324481700624 = 2^4(109)(727)(255923)

Therefore

φ(324481700624) = 324481700624(1-1/2)(1-1/109)

(1 – 1/727)(1 – 1/255923)

= 2^3(108)(726)(255922)

= 160530657408.