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22 - Solving the Dilworth Problem

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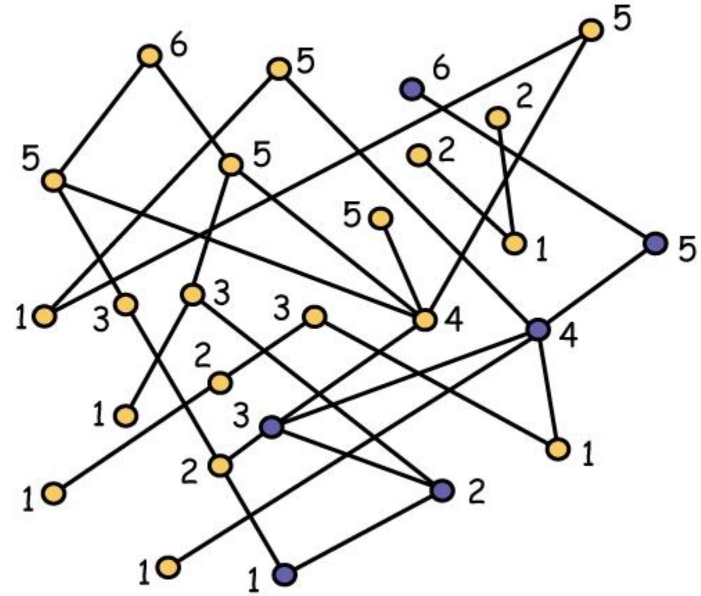
Height of a Poset

Definition The **height** of a poset P is the maximum size of a chain in P.

Proposition To partition a poset P of height h into antichains, at least h antichains are required.

Dilworth's Theorem – Dual Form



Theorem A poset of height h can be partitioned into h antichains.

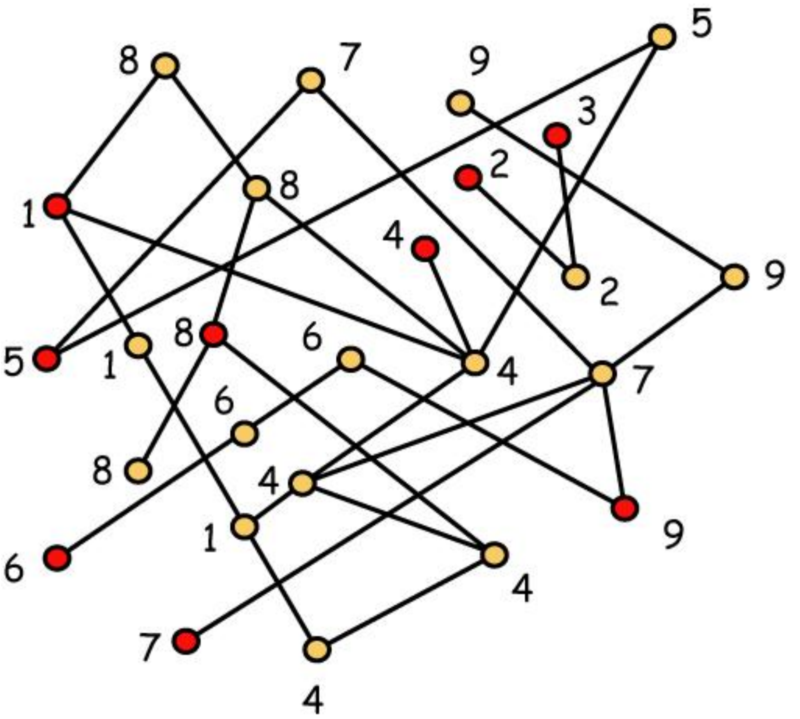
Proof As illustrated in the figure, we recursively strip off the minimal elements.

Width of a Poset

Definition The **width** of a poset P is the maximum size of an antichain in P.

Proposition To partition a poset P of width w into chains, at least w chains are required.

Dilworth's Theorem

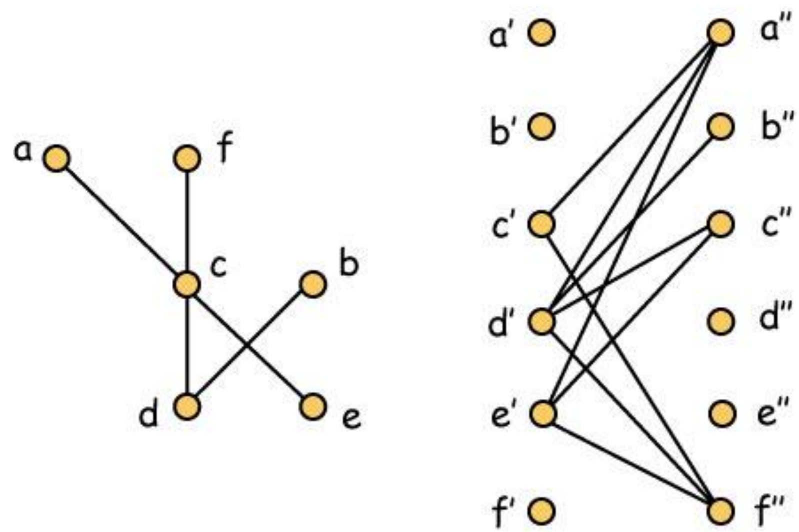


Theorem (1950) *A* poset of width w can be partitioned into w chains.

Algorithms for Dilworth's Theorem

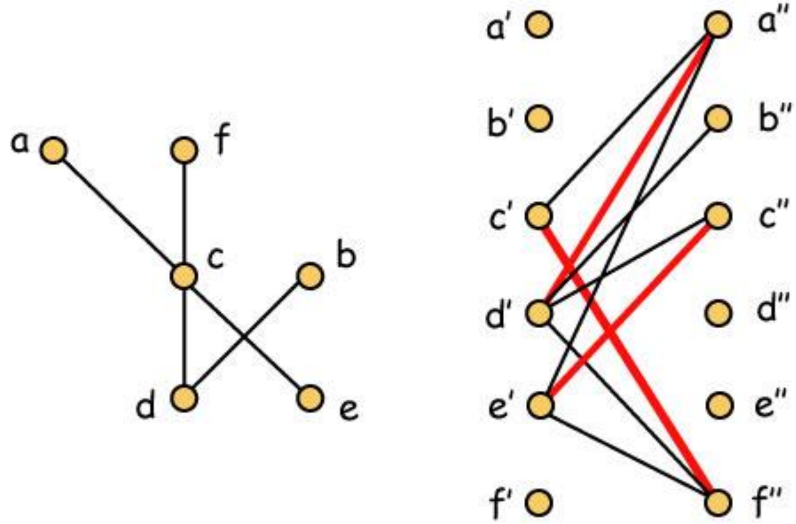
Reminder The proof we presented earlier in our class only showed that the partition of P into w chains exists. However, it (apparently) did not give an effective procedure for finding an antichain of size w or a partition of P into w chains. So our goal will be to use network flows and bipartite matchings in particular to solve the Dilworth problem.

Posets to Bipartite Graphs



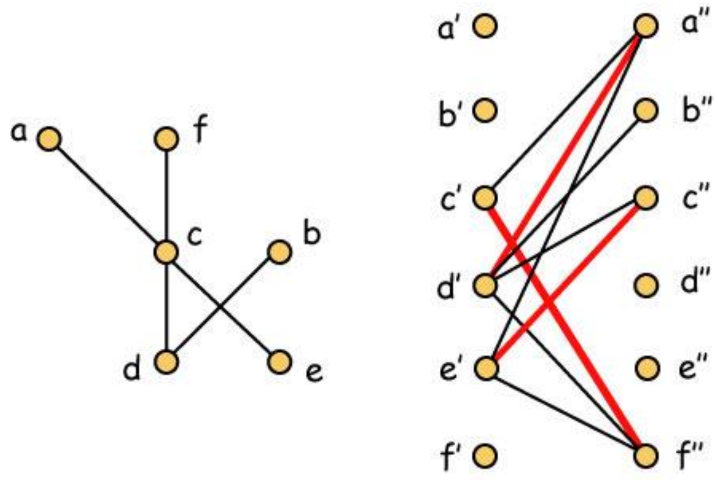
Remark For each x in P, the bipartite graph contains two points labeled x′ and x″ respectively. The edges in the graph have the form x′y″ where x < y in P.

Posets to Bipartite Graphs



Remark A matching in G determines a chain partition of P with x immediately below y in a chain when $x'y''$ is one of the matching edges. This matching corresponds to the chain partition: $C_1 = \{e < c < f\}$ $C_2 = \{d < a\}$ and $C_3 = \{b\}$. If the matching is maximum, then the chain partition is a Dilworth partition of P, i.e, it uses w chains where $w = \text{width}(P)$.

Finding a Maximum Antichain



Remark When the Ford-Fulkerson labelling algorithm halts, for each chain C_i in the partition, there is a point x_i in C_i so that x_i' is labeled and x_i'' is not. These points form an antichain. In chain $C_1 = \{e < c < f\}$, take f. In chain $C_2 = \{d < a\}$, take a and $C_3 = \{b\}$, take b (the only choice). Note that {f, a, b} is a 3-element antichain. DONE!!!