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Math 3012 – Applied Combinatorics Lecture 10

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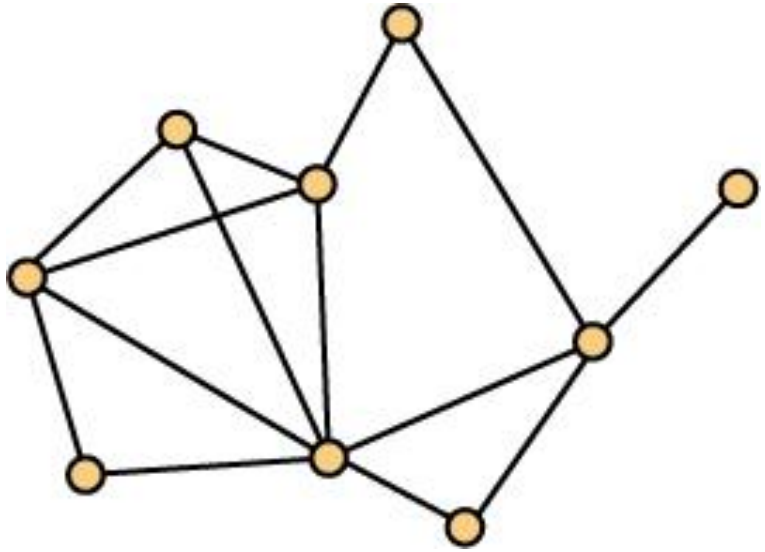
Chromatic Number and Girth

Observation The three constructions we have given for triangle-free graphs with large chromatic number produce graphs with small girth. Although the proof is a bit beyond our scope in this course, here is a historically very important result in applications of probability to combinatorics.

Theorem (Erdős, '59) For every pair (g, t) of positive integers with $g, t \geq 3$, there is a graph G with girth g and chromatic number t.

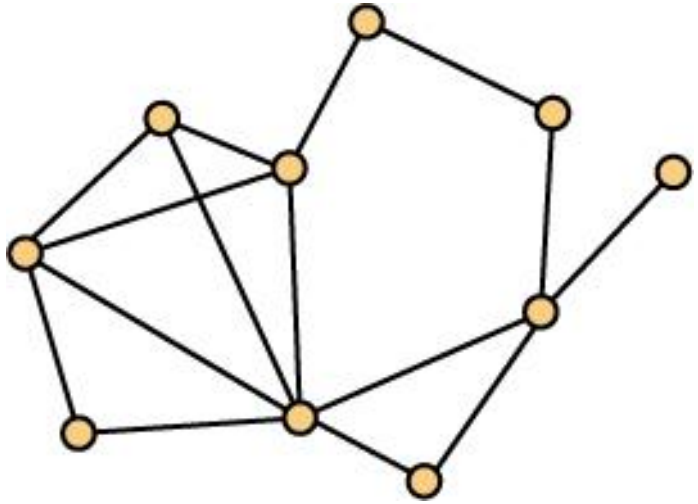
Perfect Graphs

Definition A graph G is perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G. The graph shown below is perfect.



Perfect Graphs and Odd Cycles

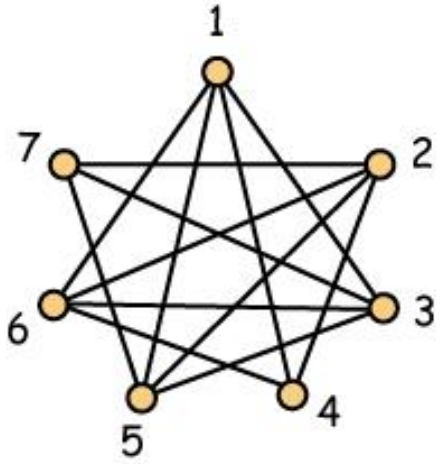
Observation A graph G is not perfect if contains an odd cycle as an induced subgraph. The graph shown below is not perfect. Note that it contains C_5 as an induced subgraph.



The Complement of a Graph

Definition The complement of a graph G, denoted G^c is the graph having the same vertex as G but a pair xy of distinct vertices forms an edge in G^c if and only if it does not form an edge in G.

Observation A graph G is not perfect if its complement contains an odd cycle as an induced subgraph.



Perfect Graphs

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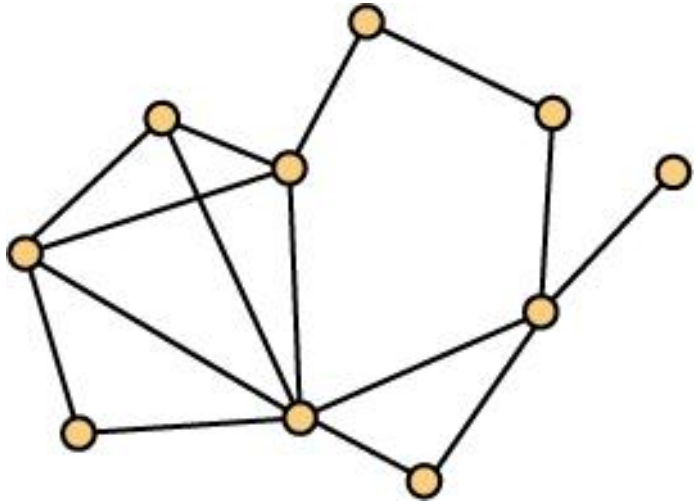
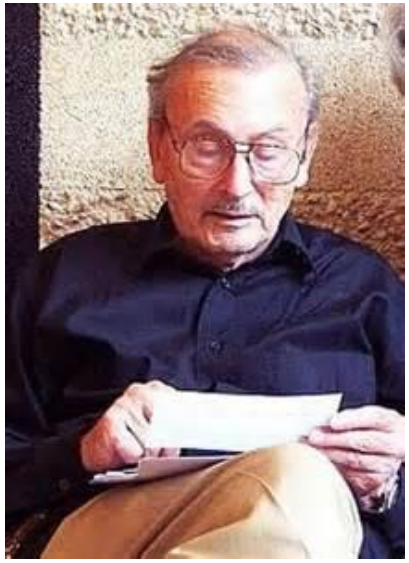


Berge's Perfect Graph Conjecture

Conjecture (Claude Berge, 1961) *A graph G is perfect if and only if neither the graph nor its complement contains an odd cycle as an induced subgraph.*



The Perfect Graph Theorem

Historical Note The following result was proven by Laszlo Lovász in 1972. Lovász has won numerous international prizes, including the 2010 Kyoto Prize (50 million yen ≈ USD 550K), the Wolf Prize, the Fulkerson Prize (twice), the Polya Prize and the Gödel Prize. As a youngster, he won three consecutive gold medals in the Math Olympiad.

Theorem A graph G is perfect if and only if its complement is perfect.

The Strong Perfect Graph Theorem

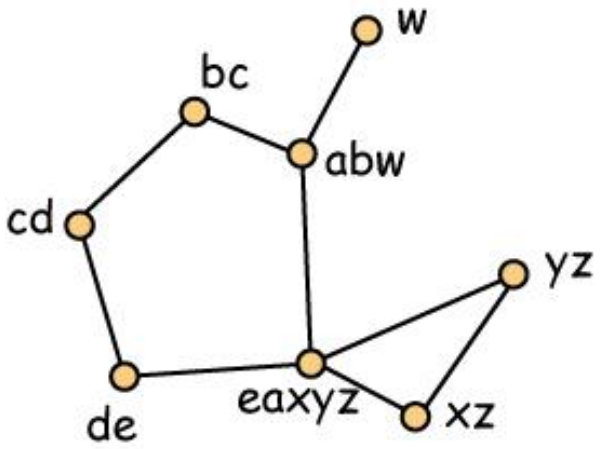
Historical Note The following result is proven in a 178 page paper appeared in the *Annals of Mathematics* in 2006 and won the 2009 Fulkerson Prize and a cash award of $10,000.

Theorem (Chudnovsky, Robertson, Seymour, Thomas) *A graph G is perfect if and only if neither the graph nor its complement contains an odd cycle as an induced subgraph.*



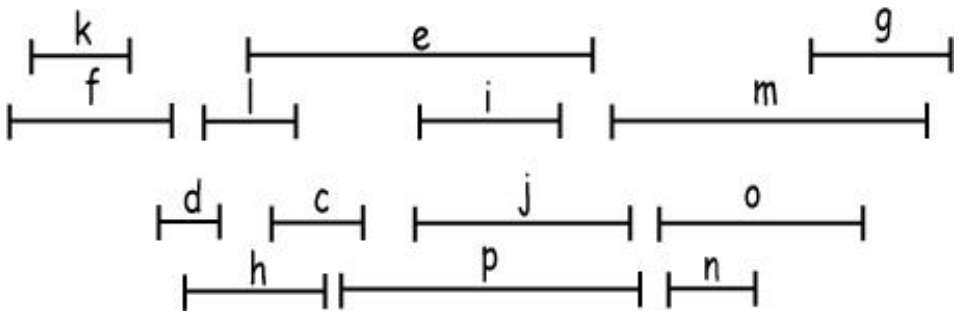

Intersection Graphs

Definition Let $F = \{A_x : x \in X\}$ be a family of sets. We associate with F an intersection graph G where the vertices of G are the elements of X and xy is an edge in G when the sets A_x and A_y intersect.



Interval Graphs

Definition A graph G is called an interval graph when it is the intersection graph of a family of closed intervals of $\mathbf{R}$. For the family shown below, c and p intersect while c and n do not.



Interval Graphs are Perfect

Algorithm Given a representation of an interval graph, apply First Fit (Greedy) and color in the order of left end points.