October 1, 2015

Math 3012 – Applied Combinatorics Lecture 13

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Partially Ordered Sets (Posets)

Definition A partially ordered set (also called a poset) is a set P equipped with a binary relation $\leq$ satisfying the following three properties:

- If $x \in P$, then $x \leq x$ in P (reflexive property).
- If $x, y, z \in P$, $x \leq y$ in P and $y \leq x$ in P, then $x = y$ (antisymmetric property).
- If $x, y, z \in P$, $x \leq y$ in P and $y \leq z$ in P, then $x \leq z$ in P (transitive property).

Examples of Posets

Notation When P is a poset, $x < y$ in P means $x \leq y$ in P and $x \neq y$. Also, $y > x$ in P means the same as $x < y$ in P. Similarly, $x \leq y$ in P means the same as $y \geq x$ in P.

Example When P is a collection of sets, set $x \leq y$ in P when x is a subset of y. In this poset $\{2, 5\} < \{2, 5, 7, 8\}$ and $\{5, 8, 9\} \geq \{5, 8, 9\}$.

Example When P is a set of positive integers, set $x \leq y$ in P when x divides y without remainder. In this poset, $15 < 105$ and $12 < 48$. But 17 is not less than $1{,}000{,}000{,}000$.

Linear Orders

Observation The familiar binary relation $\leq$ on number systems like **Z** (integers), **Q** (rationals) and **R** (reals) is a partial order. However, in each of these three cases, the binary relation $\leq$ satisfies a fourth condition:

For all x, y, either $x \leq y$ in P or $y \leq x$ in P.

Definition Partial orders satisfying this additional condition are called linear orders or total orders.

Covers in a Poset

Definition When x and y are distinct points in a poset P, we say that x is covered by y in P when $x < y$ in P and there is no point z with $x < z < y$ in P. Alternatively, we may say that y covers x in P.

Example With inclusion, $\{2, 5\}$ is covered by $\{2, 5, 7\}$ but $\{4, 6, 7\}$ is not covered by $\{4, 6, 7, 9, 11, 12\}$

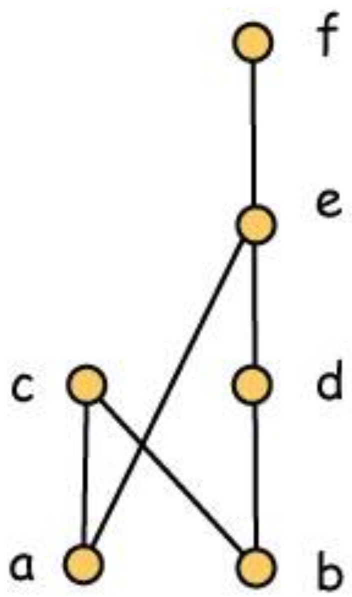
Example With division, 15 is covered by 105, but 14 is not covered by 84.

Cover Graphs and Order Diagrams

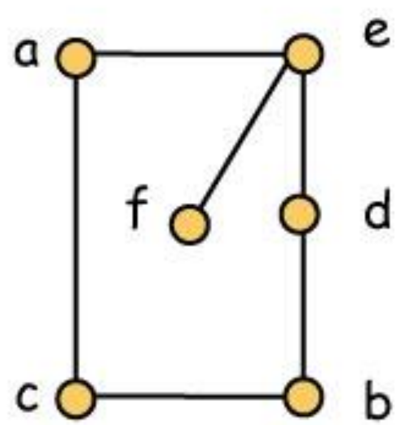
Definition When P is a poset, we associate with P a graph G called the cover graph of P. The vertices of G are the points of P. When x and y are distinct points in P, they are adjacent in G when one of x and y covers the other in P.

Definition When G is the cover graph of a poset P, a drawing of G in the plane (traditionally with straight line segments for edges) is called an order diagram (or Hasse diagram) if y is higher in the plane than x whenever y covers x in P.

Order Diagrams and Cover Graphs

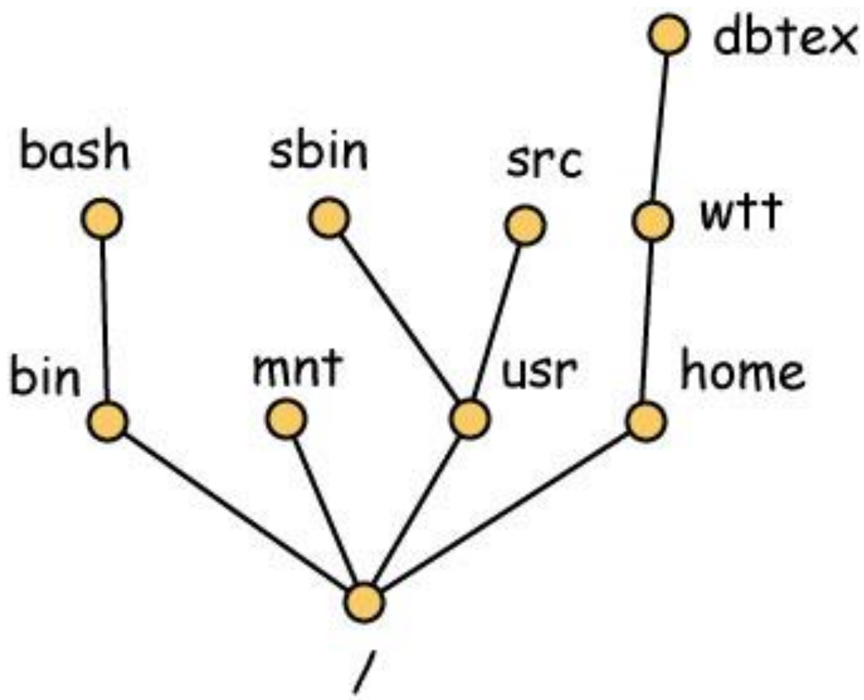


Order Diagram

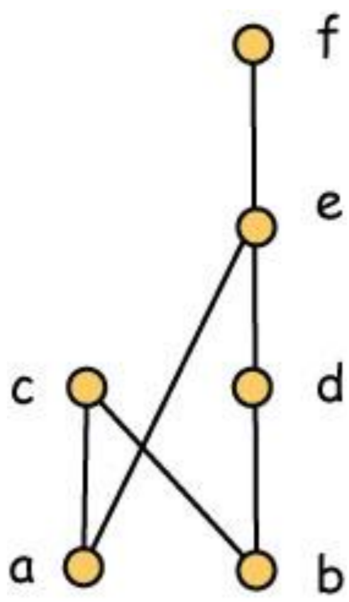


Cover Graph

Posets are Everywhere!!



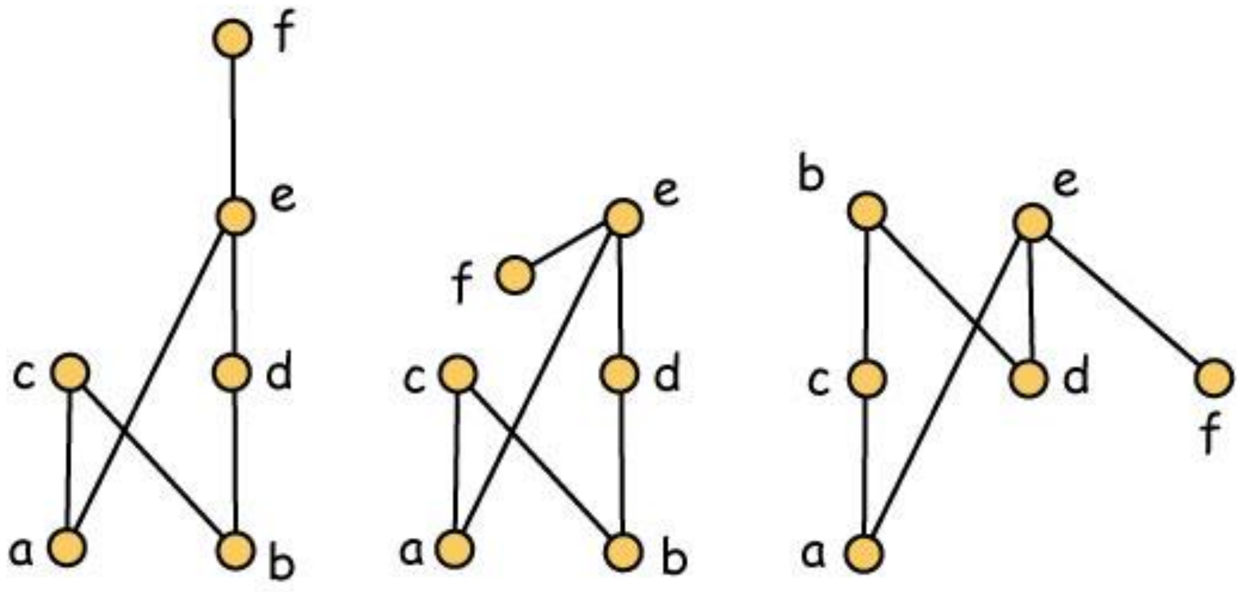
Order Diagrams and Binary Relations



Exercise What is the binary relation for the poset shown on the left?

Exercise Draw an order diagram for the poset whose ground set is {1, 2, 3, 4, 5, 6, 7} with partial order {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (2, 5), (3, 4), (3, 6), (3, 2), (3, 5), (4, 5), (7, 1)}

Three Posets with the Same Cover Graph



Exercise How many posets altogether have the same cover graph as these three?

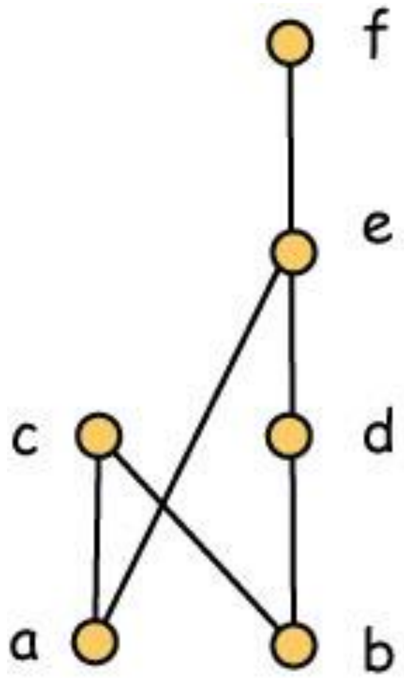
Comparable and Incomparable Points

Definition When P is poset, we say that two distinct points x and y of P are comparable when either $x < y$ in P or $x > y$ in P. When x and y are not comparable, they are said to be incomparable. A partially ordered set (or poset) P is a set equipped with a binary relation $\leq$ which is reflexive, antisymmetric and transitive.

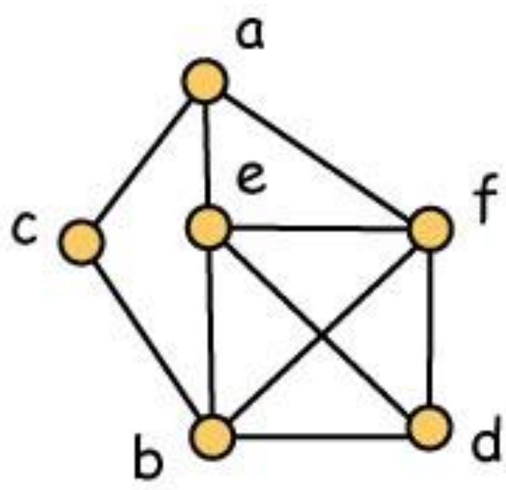
Comparability and Incomparability Graphs

Definition When P is poset, we can associate with P two graphs. One is called comparability graph of P and the other is the incomparability graph of P. Both graphs have the elements of P as their vertex set. In the comparability graph, distinct elements x and y of P are adjacent when they are comparable in P. Analogously, x and y are adjacent in the incomparability graph when they are incomparable in P.

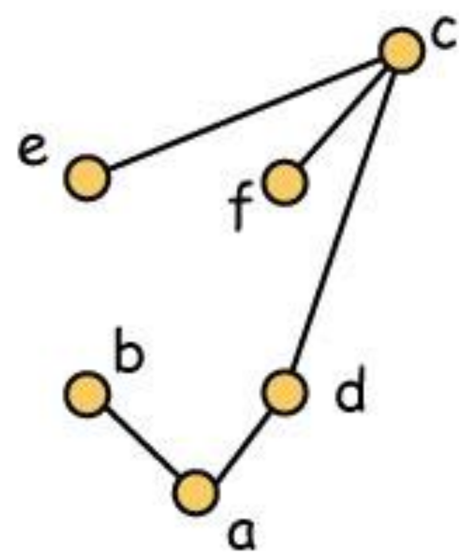
Comparability and Incomparability Graphs



Poset



Comparability Graph



Incomparability Graph

Alternate Definition

Definition A **poset** P is a set equipped with a binary relation $<$ which is irreflexive and transitive. For example:

- A family of closed intervals of $\mathbf{R}$ with $[a, b] < [c, d]$ if and only if $b < c$ in $\mathbf{R}$.

Note To avoid operator overloading confusion, we write $x <_P y$ in P. When there is no ambiguity, we just write $x < y$.

Maximal and Minimal Points

Definition An element x of a poset P is said to be a maximal point of P when there is no point y of P with $y > x$ in P.

Definition An element w of P is called a minimal point of P when there is no point z in P with $z < w$ in P.

A Concrete Example

Example Let $X = \{1,2,3,4,5,6\}$ and $P = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(6,1),(6,4),(1,4),(6,5),(3,4),(6,2)\}$.

Then

6 and 3 are minimal elements.

$2, 4$ and 5 are maximal elements.

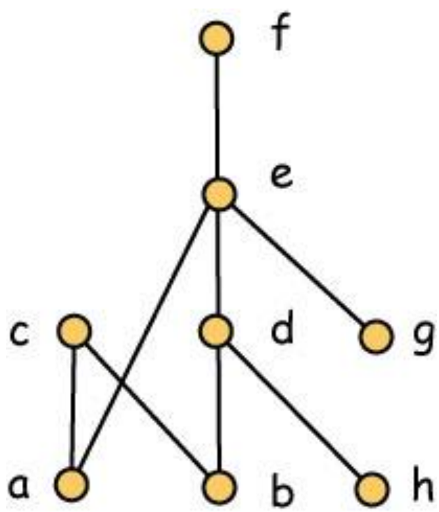
4 is comparable to 6.

2 is incomparable to 3.

1 covers 6 and 3 is covered by 5.

$4 > 6$ but 4 does not cover 6, since $6 < 1 < 4$.

Another Concrete Example



Example

c and f are maximal elements.

a, b, g and h are minimal elements.

a is comparable to f.

c is incomparable to h.

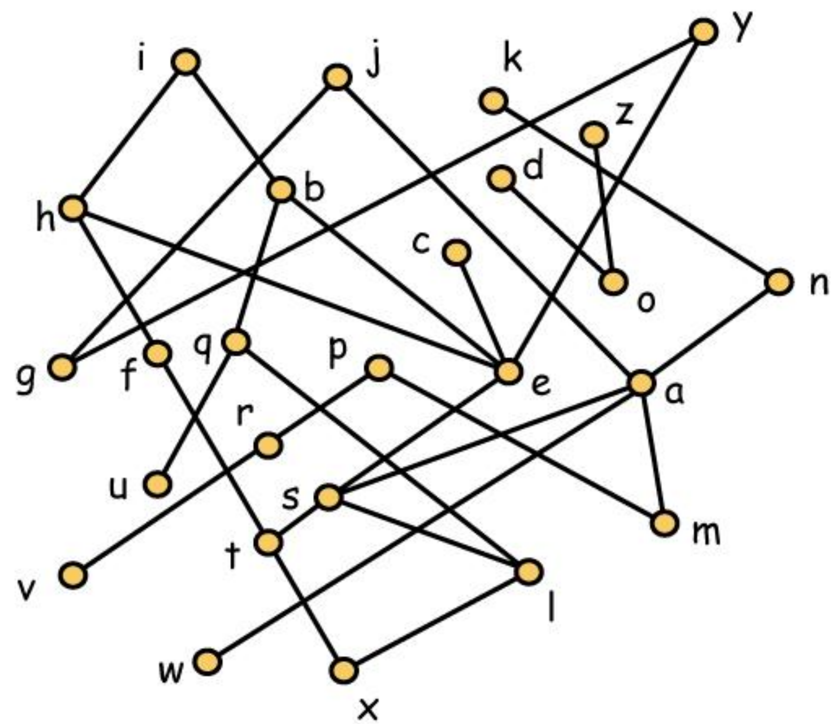
e covers a and h is covered by d.

e > h but e does not cover h.

Diagram for a Poset on 26 points

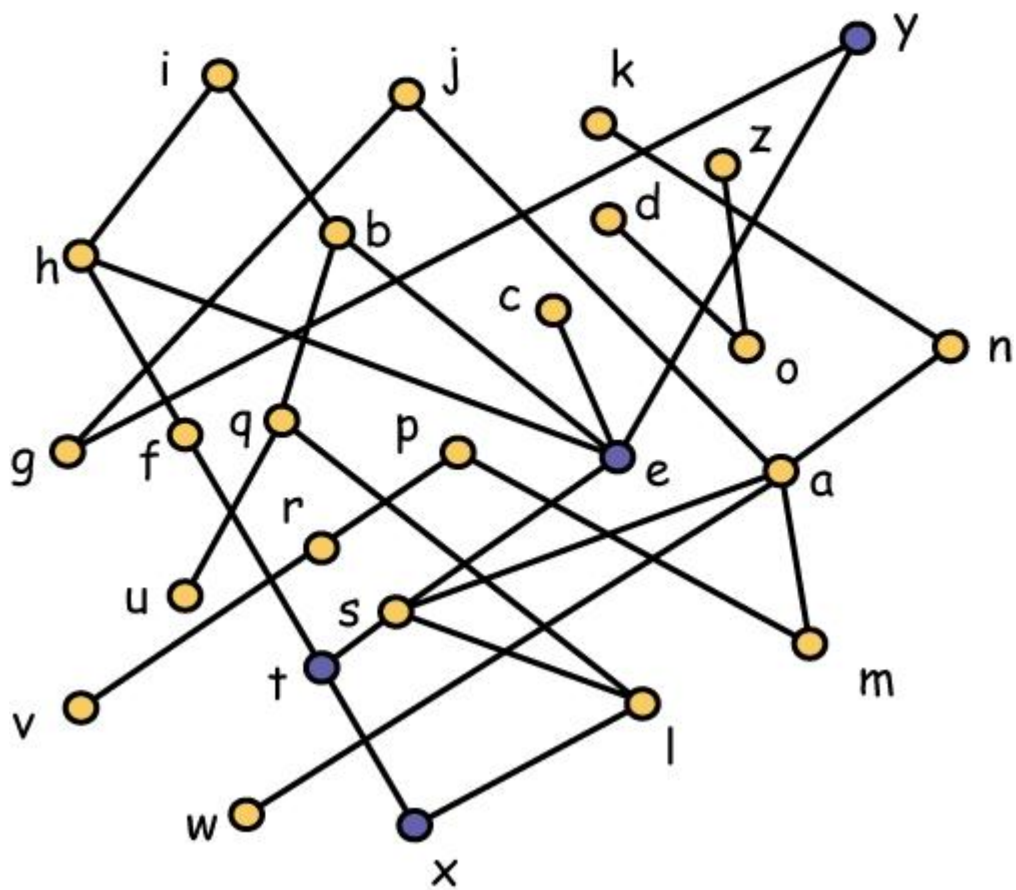
Terminology:

- b < i and s < y.
- j covers a.
- b > e and k > w.
- s and y are comparable.
- j and p are incomparable.
- c is a maximal element.
- u is a minimal element.



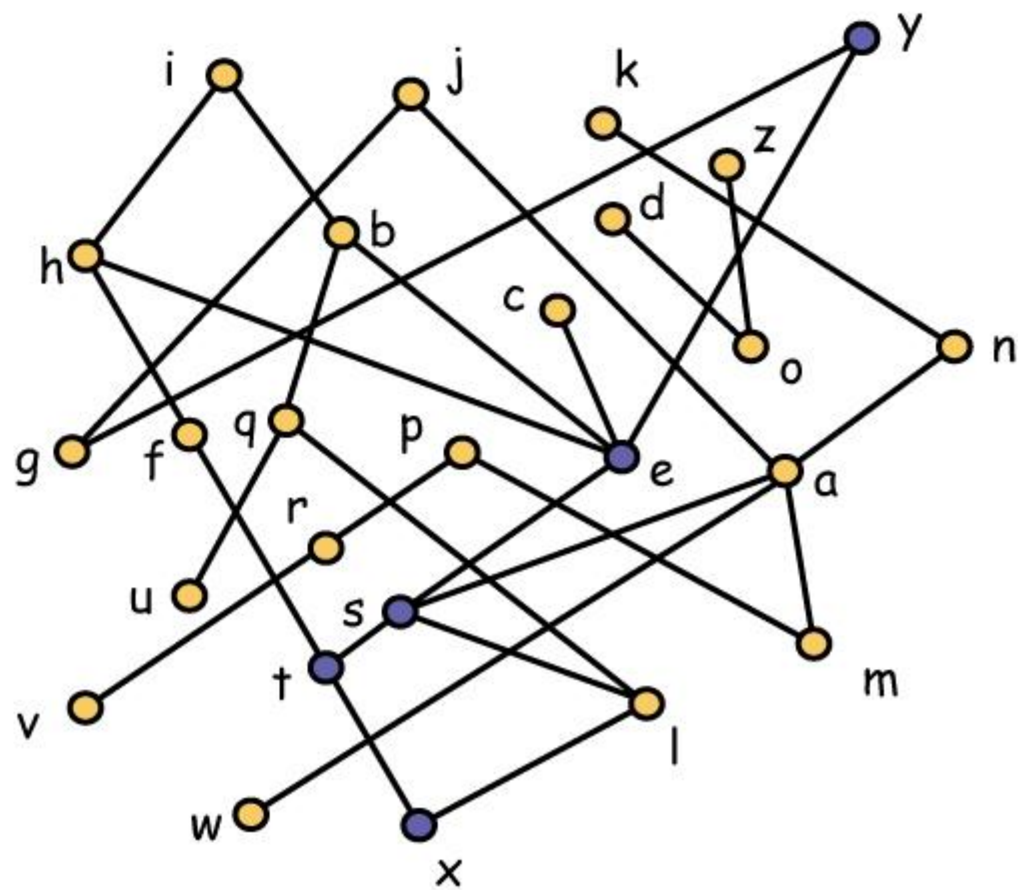
A Chain of Size 4

Definition A **chain** is a subset in which every pair is comparable.



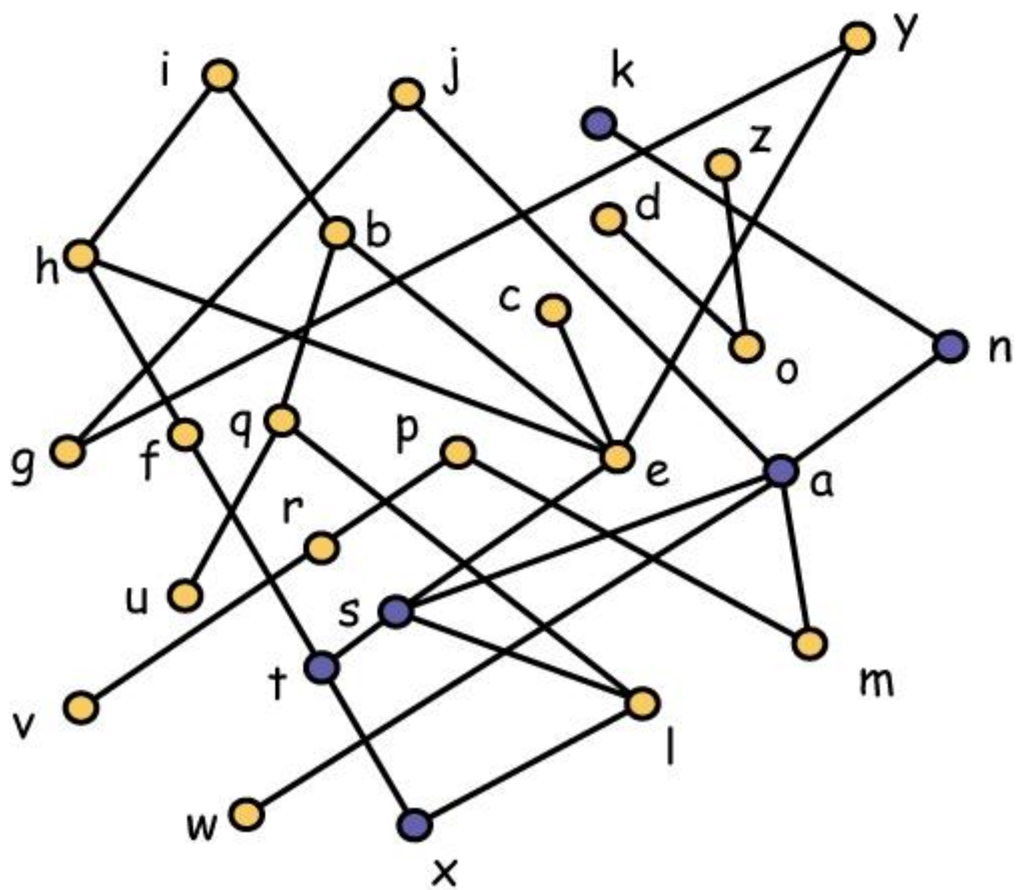
A Maximal Chain of Size 5

Definition A chain is **maximal** when no superset is also a chain.



A Maximal Chain of Size 6

Definition A chain is **maximal** when no superset is also a chain.



Height of a Poset

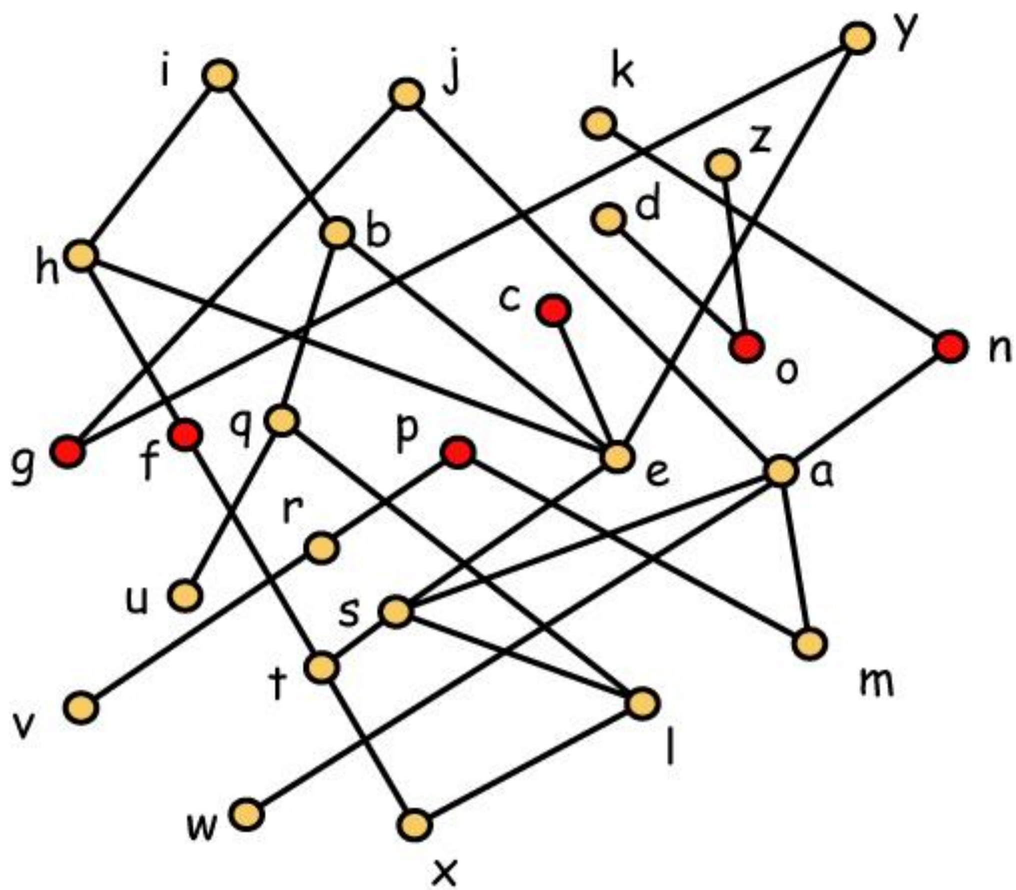
Definition The **height** of a poset P is the maximum size of a chain in P.

Proposition To partition a poset P of height h into antichains, at least h antichains are required.

Question How hard is it to find the height of a poset and the minimum size of a partition of the poset into antichains?

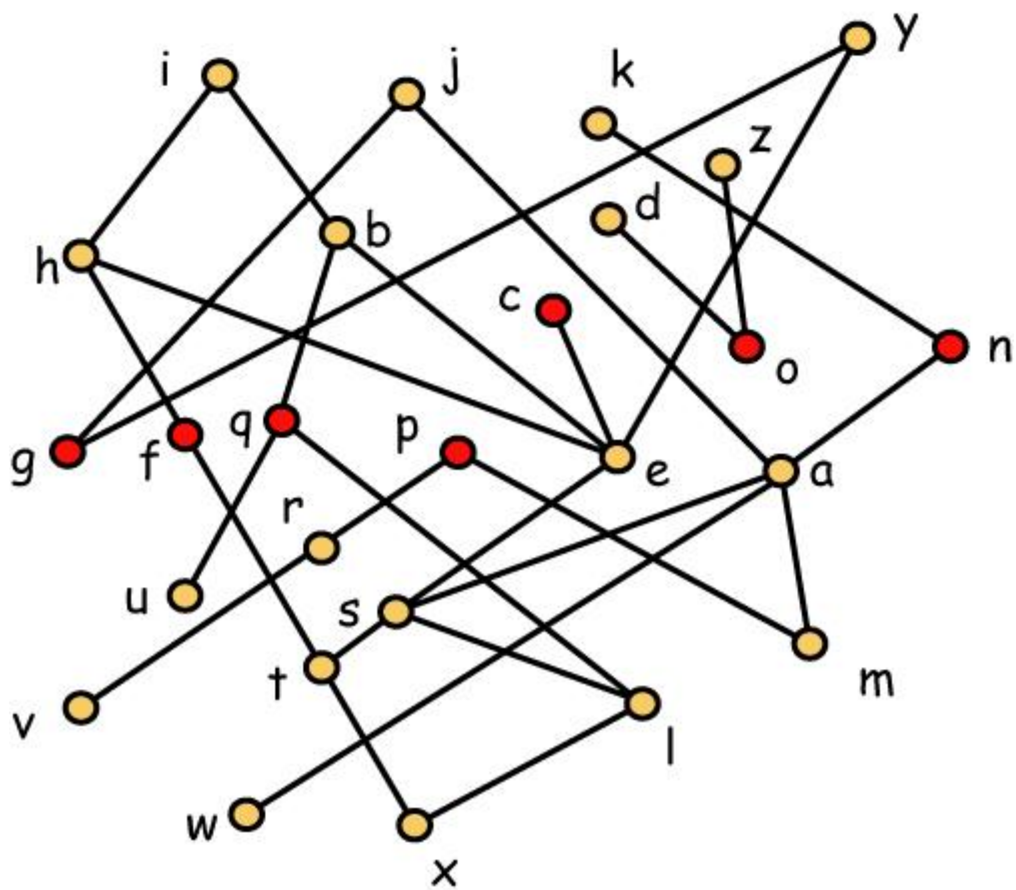
An Antichain of Size 6

Definition A subset is an **antichain** when every pair is incomparable.



A Maximal Antichain of Size 7

Definition An antichain is **maximal** when no superset is an antichain.



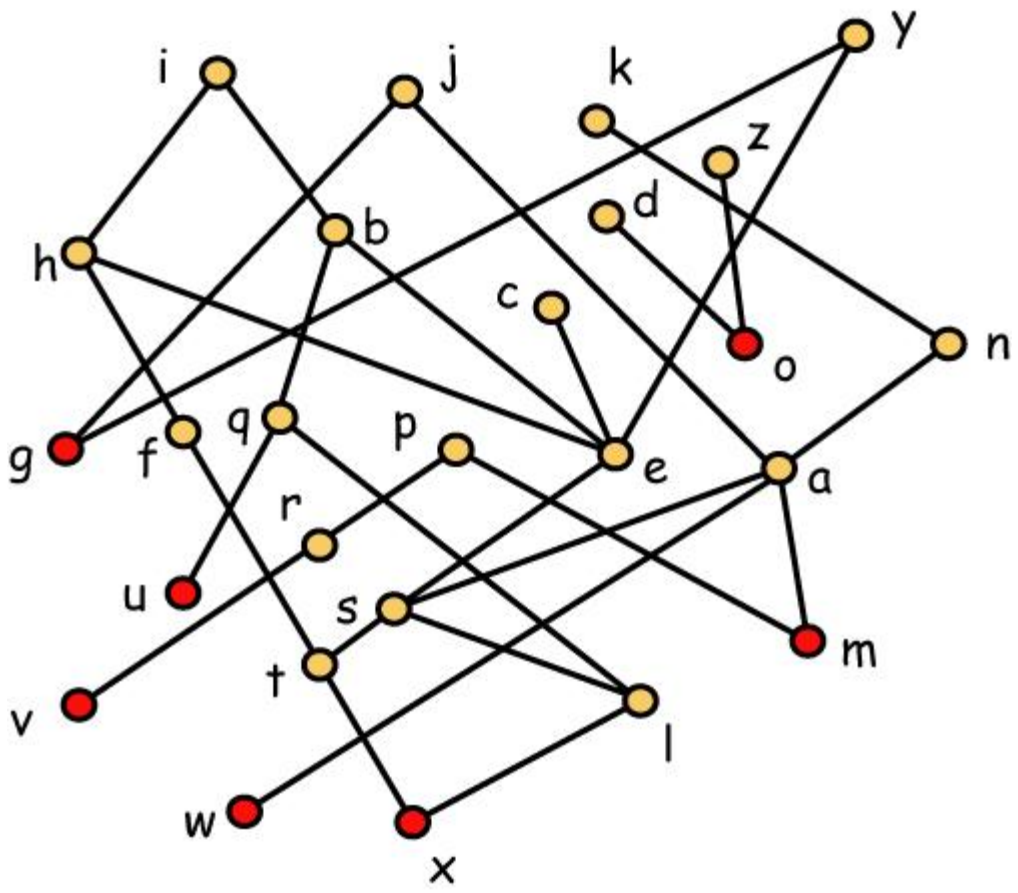
Width of a Poset

Definition The **width** of a poset P is the maximum size of an antichain in P.

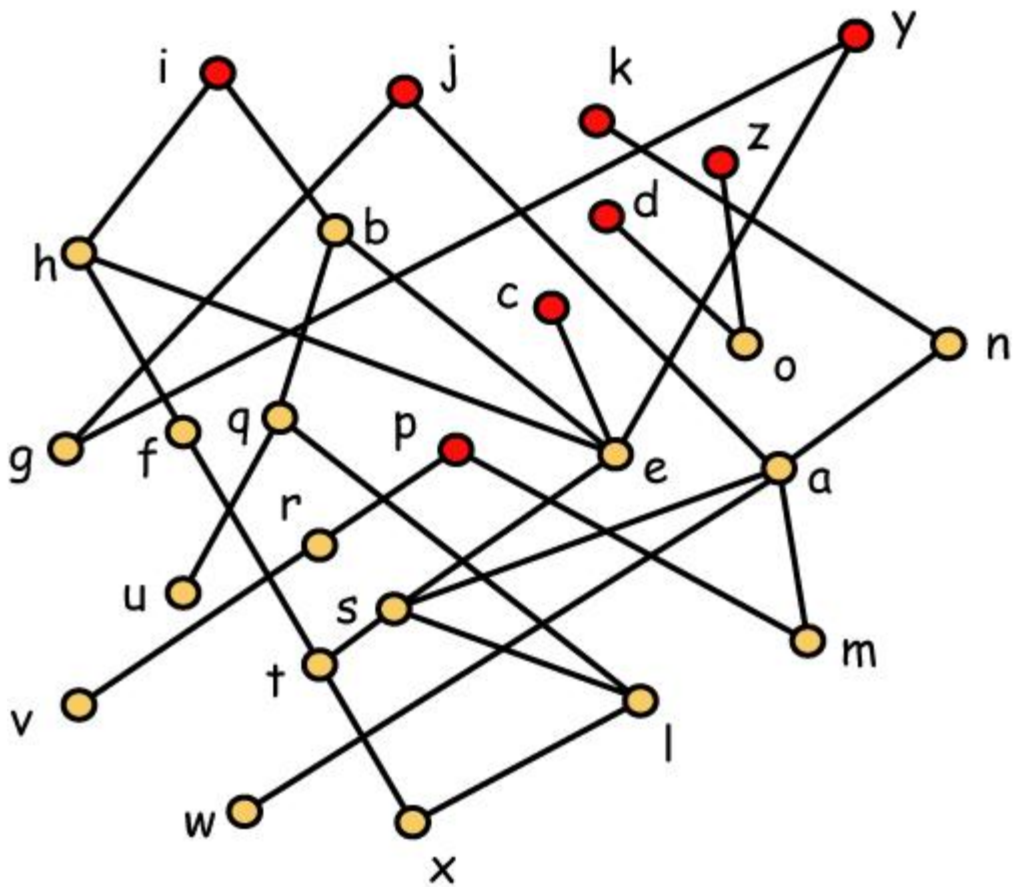
Proposition To partition a poset P of width w into chains, at least w chains are required.

Question How hard is it to find the width of a poset and the minimum size of a partition of the poset into chains?

There are 7 Minimal Elements



There are 8 Maximal Elements



Width ≥ 9