December 3, 2015

Math 3012 – Applied Combinatorics Lecture 27

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Reminders

Final Exam Tuesday, December 8, 2015, 8:05 – 10:55am.

Homework 3 Due today! Please turn in during class. Email is ok also.

Complete Chaos – Not on My Watch!

Observation In the discrete world, complete chaos is impossible. Clear patterns exhibiting complete symmetry must emerge.

Ramsey Numbers - Evidence of Uniformity

Definition For positive integers m and n, the Ramsey number $R(m, n)$ is the least positive integer t so that if G is any graph on t vertices, then either G contains a clique of size m or G contains an independent set of size n.

Observation Informally, the assertion that there is a ramsey number $R(m, n)$ can be interpreted as the statement that a very large graph contains either a large clique or a large independent set. You can avoid one of these two conclusions … but not both.

Ramsey Numbers

Definition For positive integers m and n, the Ramsey number $R(m, n)$ is the least positive integer t so that if G is any graph on t vertices, then either G contains a clique of size m or G contains an independent set of size n.

Examples $R(m, n) = R(n, m)$, $R(m, 1) = 1$ for all m, and $R(m, 2) = m$ for all m.

Theorem The Ramsey number $R(m, n)$ exists and satisfies the inequality $R(m, n) \le C(m + n - 2, m - 1)$.

Proof The argument is an easy induction and will be done in class. This will involve showing that

$R(m, n) \le R(m, n - 1) + R(m - 1, n)$ when $m, n \ge 2$.

Small Ramsey Numbers

Example $R(3, 3) = 6$ (to be done in class). In fact, $R(3, n)$ is known exactly for $n \leq 9$. On the other hand, $40 \leq R(3, 10) \leq 42$. For large n, it is now known that $R(3, n) = \Theta(n^2/\log n)$

Example $R(4, 4) = 18$.

Example $43 \leq R(5, 5) \leq 49$.

Example $102 \leq R(6, 6) \leq 165$.

Theorem

$$(1 + o(1))\,(n\sqrt{2}/e)2^{n/2} \leq R(n, n) \leq n^{-(c \log n/\log \log n)}\, 2^{2n}$$

Note Roughly speaking, these inequalities imply:

$$2^{n/2} \leq R(n, n) \leq 2^{2n}$$

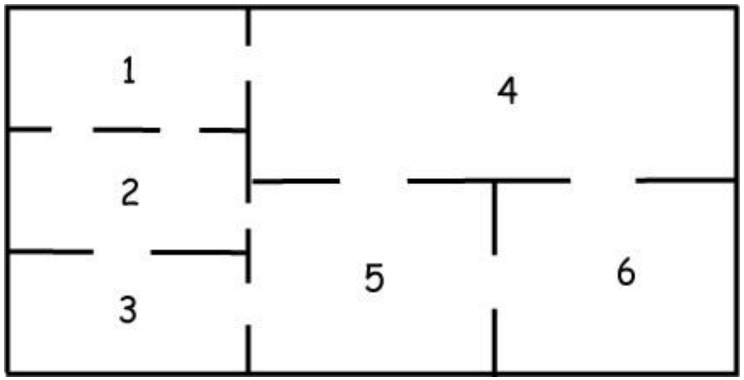
Challenge Worth a PhD

Remark We will explain in class the following inequality, noting that the lower bound requires probability.

$$2^{n/2} \leq R(n, n) \leq 2^{2n}$$

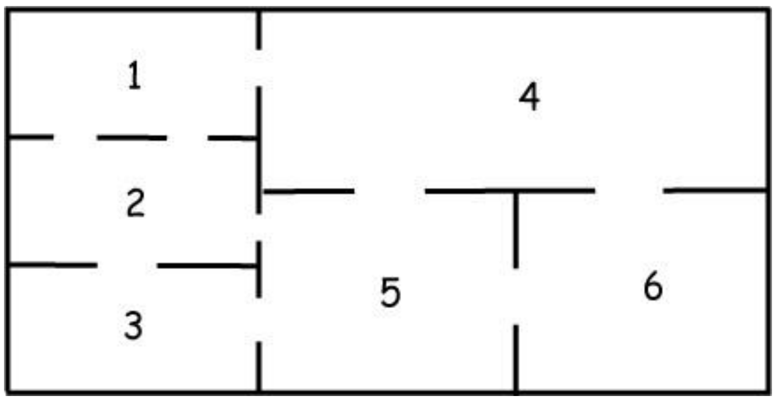
Challenge Move the constant in the exponent in either bound in the inequality given above for $R(n, n)$. You will certainly have a marvelous PhD thesis as a result!

Markov Chains – Examples and Questions



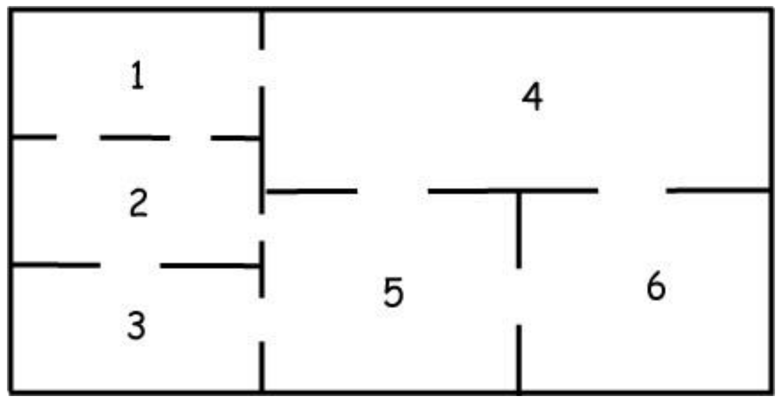
Example We illustrate an enclosure with 6 rooms. Suppose we start in the room in Room 1. We then move from room to room by the following rule: Once an hour, we choose at random one of the doors in the current room and exit to the adjoining room. What is the expected waiting time before we first reach Room 6? In the long term, for Room i, what is the probability p_i that we are in Room i? For which room is p_i maximum?

Markov Chains (2)



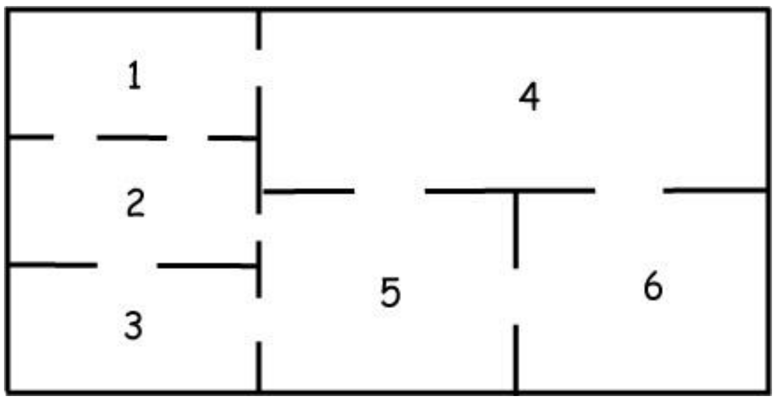
Example How fast does the probability of being in Room i converge to the steady state (long term) probability if we start from a given room. This is the notion of mixing. Some Markov chains mix very rapidly while others mix very slowly. Examples will be given in class.

Absorbing Markov Chains



Example Suppose we start Room 1, but Room 6 has a tiger which will end our journey, what is our expected time of survival.

Absorbing Markov Chains (2)



Example Suppose we start in Room 1. If we enter Room 4, we collect \$200 and the game ends. If we enter Room 6, we collect \$1000 and the game ends. What is the expected value of our winnings?

Example How does the calculation change if we have to pay \$100 for every move?