September 10, 2015

Math 3012 – Applied Combinatorics Lecture 8

William T. Trotter
trotter@math.gatech.edu

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Test 1 and Homework Due Date

Reminder Test 1, Thursday September 17, 2015. Taken here in MRDC 2404. Final listing of material for test will be made via email after class on Thursday, September 10.

Homework Due Date Tuesday, September 15, 2015. Papers will be returned with tests – with a target of Tuesday, September 22, 2015. Scores posted on T-Square.

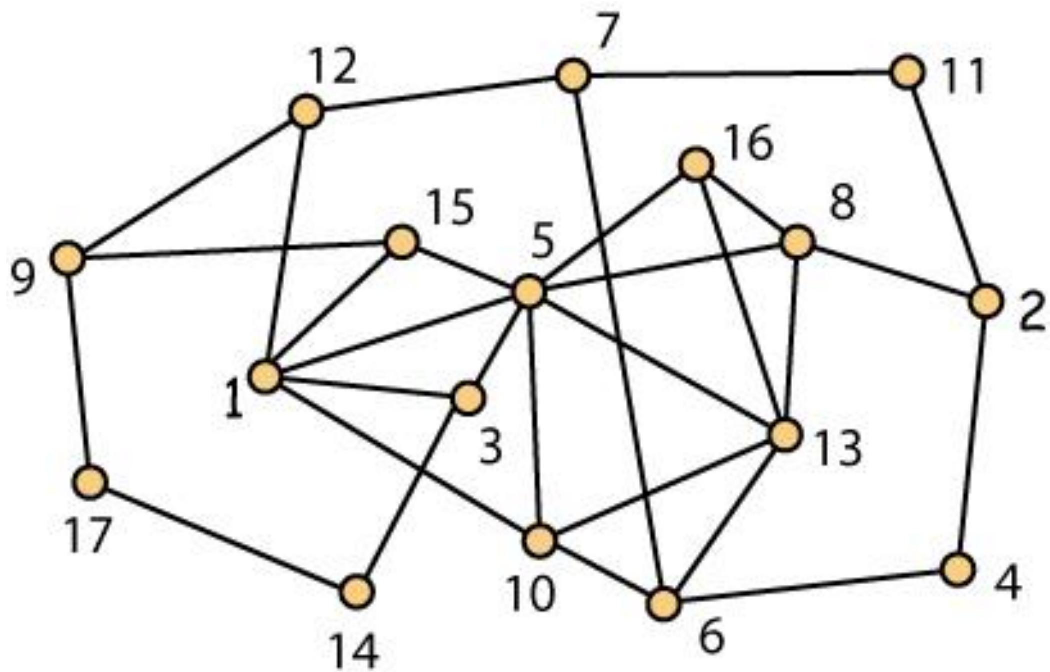
Hamiltonian Paths and Cycles

Definition When G is a graph on $n \geq 3$ vertices, a cycle $C = (x_1, x_2, \ldots, x_n)$ in G is called a Hamiltonian cycle, i.e, the cycle C visits each vertex in G exactly one time and returns to where it started.

Definition When G is a graph on $n \geq 3$ vertices, a path $P = (x_1, x_2, \ldots, x_n)$ in G is called a Hamiltonian path, i.e, the path P visits each vertex in G exactly one time. In contrast to the first definition, we no longer require that the last vertex on the path be adjacent to the first.

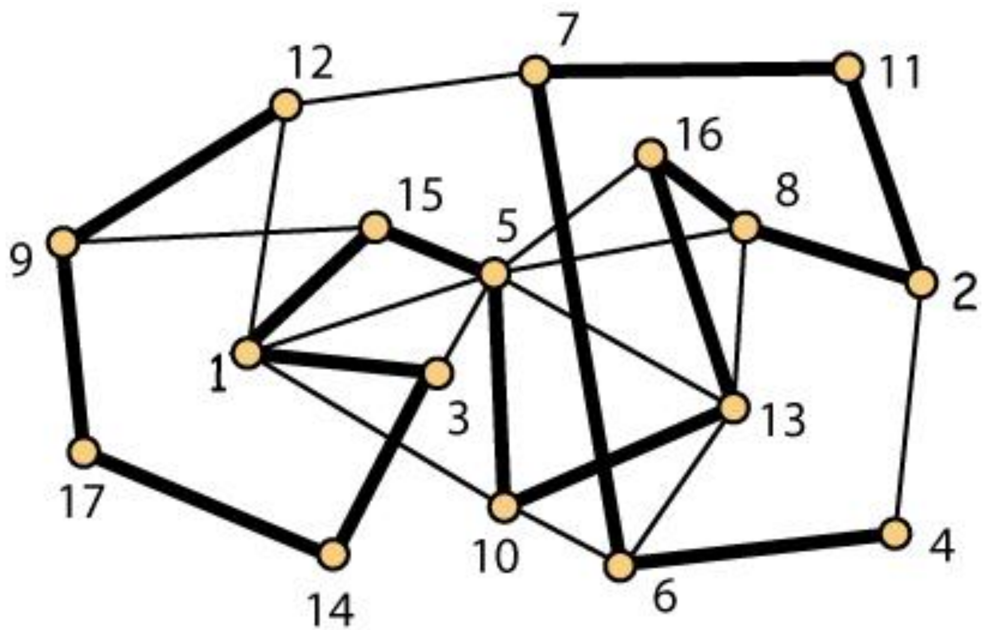
Hamiltonian Paths

Question Does the graph shown below have a Hamiltonian path?



Hamiltonian Paths (2)

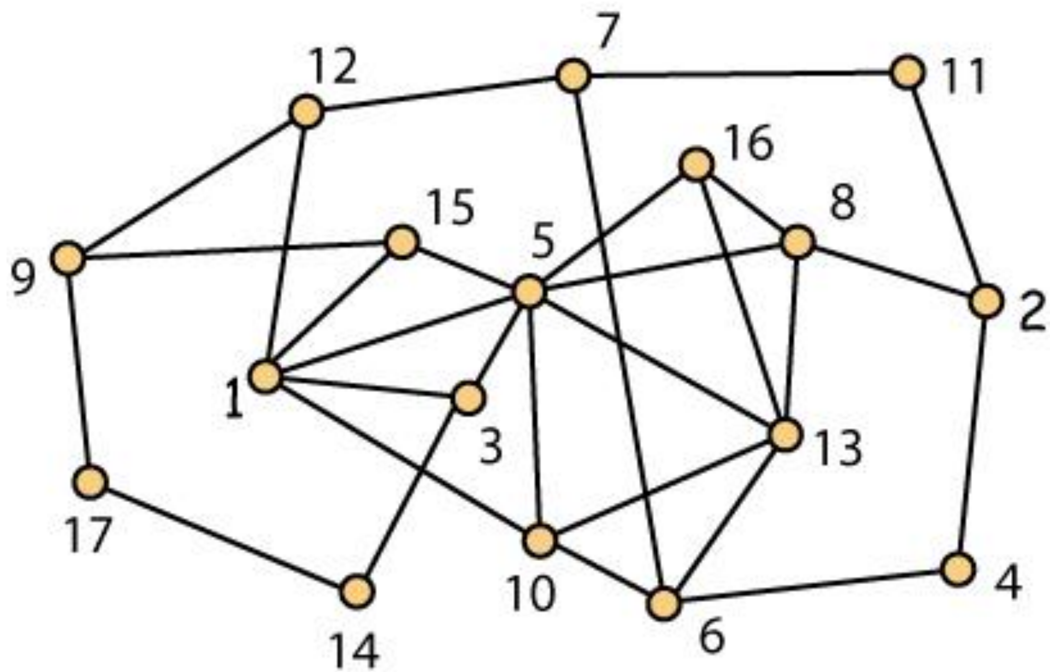
Answer Yes!!



(12, 9, 17, 14, 3, 1, 15, 5, 10, 13, 16, 8, 2, 11, 7, 6, 4)

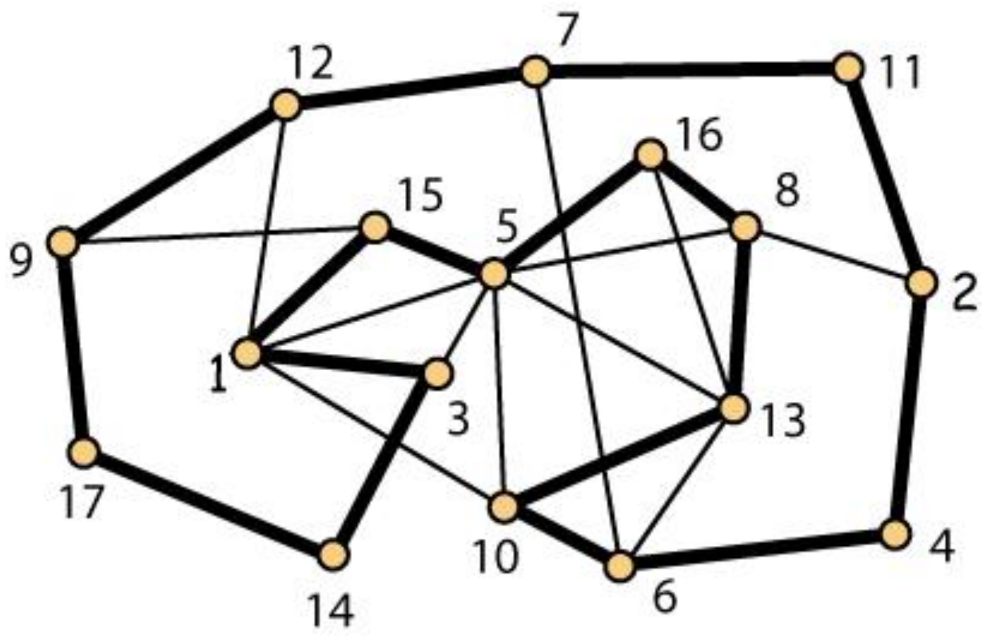
Hamiltonian Cycles

Question Does the graph shown below have a Hamiltonian cycle?



Hamiltonian Cycles (2)

Answer Yes!!



(1, 3, 14, 17, 9, 12, 7, 11, 2, 4, 6, 10, 13, 8, 16, 5, 15)

Certificates for “Yes” Answer

Remark Given a graph G, a “yes” answer to the question: Does G have a Hamiltonian path?” can be validated by providing a certificate in the form of a permutation of the vertex set of G. An impartial referee (computer) can quickly check the essential details. Is every vertex listed exactly once? Are consecutive vertices adjacent in the graph?

Remark An analogous statement applies for Hamiltonion cycles.

Hamiltonian Paths (3)

Question Does this graph have a Hamiltonian path?

```
Graph_data.txt
6
5   2
1   4
6   5
3   6
1   3
4   5
4   6
6   1
```

Answer Yes!!

Certificate (6, 3, 1, 4, 5, 2)

Note The correctness of the answer can be verified quickly by an impartial referee (computer).

Hamiltonian Cycles (3)

Question Does this graph have a Hamiltonian cycle?

```
Graph_data.txt
6
5   2
1   4
6   5
3   6
1   3
4   5
4   6
6   2
```

Answer Yes!!

Certificate (6, 3, 1, 4, 5, 2)

Note The correctness of the answer can be verified quickly by an impartial referee (computer).

Certificates for "No" Answer

Remark Given a graph G, there does not seem to be a way to provide a certificate to validate a "no" answer to the question: Does G have a Hamiltonian cycle?" To be more precise, there does not seem to be a way to provide an impartial referee (computer) with information which can be effectively checked and will satisfy the referee that your answer is correct.

There Are Exceptions

Question Does this graph have a Hamiltonian cycle?

```
Graph_data.txt
6
5   2
1   4
6   5
3   6
1   3
4   5
4   6
6   1
```

Answer No!!

Certificate Vertex 2 has degree 1. If a graph has a Hamiltonian cycle, every vertex has degree at least 2.

Note The correctness of the answer can be verified quickly by an impartial referee (computer).

Certificates for "No" Answer

Remark Given a graph G, there does not seem to be a way to provide a certificate to validate a "no" answer to the question: Does G have a Hamiltonian cycle?" To be more precise, there does not seem to be a way to provide an impartial referee (computer) with information which can be effectively checked and will satisfy the referee that your answer is correct, at least not in general. This does not preclude there being a justification for a "no" answer in *some* cases.

Computational Complexity

A Very Informal Perspective The class **P** consists of all "yes-no" questions for which the answer can be determined using an algorithm which is provably correct and has a running time which is polynomial in the input size.

Examples

1. Given a list of n numbers, is 2388643 in the list?
2. Given a list of n numbers, can you find three distinct numbers a, b and c in the list so that $a + b = c$?
3. Given a graph G, does it have an Euler circuit?

Computational Complexity (2)

A Very Informal Perspective The class **NP** consists of all "yes-no" questions for which there is a certificate for a "yes" answer whose correctness can be verified with an algorithm whose running time is polynomial in the input size. *Any* question in **P** is also in **NP**.

Examples

1. Given a list of n numbers, is there a fair division?
2. Given a graph G, is there a clique whose size is at least $n/2$?
3. Given a graph G, does it have a Hamiltonian cycle?

Computational Complexity (3)

Observation As we have already noted, any problem which is in **P** is also in **NP**, but no one knows whether the converse statement is true or not. The current reward for settling this question:

$$\mathbf{P} = \mathbf{NP}\,?$$

Stands at $1, 000, 000 USD.

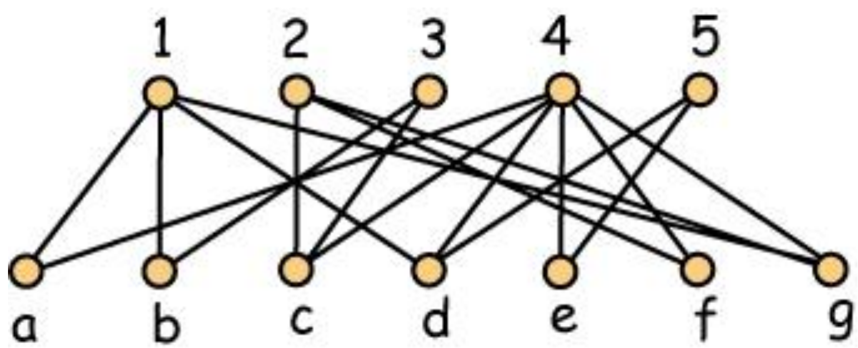
http://www.claymath.org/millennium-problems

Revisiting Euler Circuits

Remark Given a graph G, a "no" answer to the question: Does G have an Euler circuit?" can be validated by providing a certificate. Now this certificate is one of the following. Either the graph is not connected, so the referee is told of two specific vertices for which the graph does not contain a path between them. On the other hand, if the graph is connected, then the referee is told that there is vertex of odd degree.

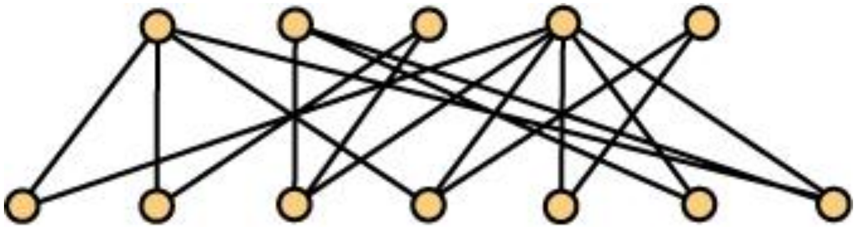
Bipartite Graphs

Definition A bipartite graph is a triple (A, B, E) where A and B are disjoint finite sets and E is a collection of 2-element sets, each of which contains one element of A and one element of B. In the bipartite graph shown below, $A = \{a, b, c, d, e, f, g\}$ and $B = \{1, 2, 3, 4, 5\}$



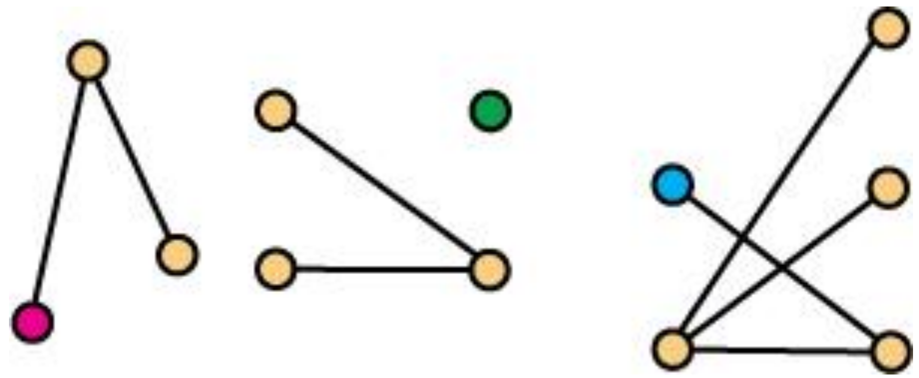
Unlabelled Bipartite Graphs

Caution In a discussion of unlabelled bipartite graphs, care has to be exercised regarding which elements belong to A and which belong to B. The potential for confusion is minor when the graph is connected.



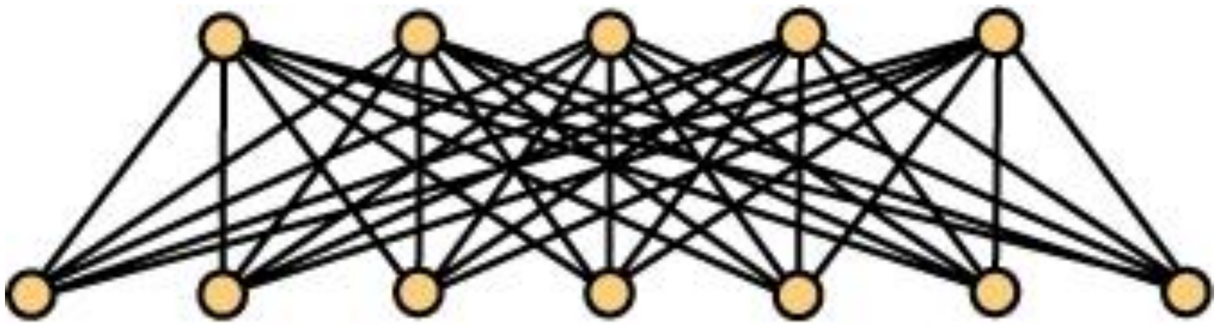
Unlabelled Bipartite Graphs

Caution But there are real problems when the graph is disconnected. For example, consider the red, blue and green points in the graph shown below. Which side are they on?



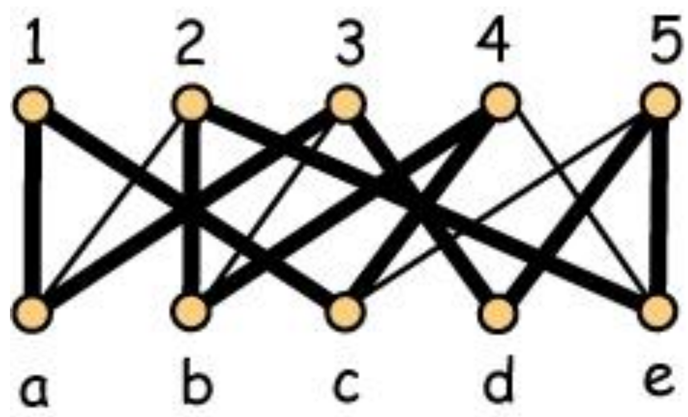
Complete Bipartite Graphs

Definition For $m, n \geq 1$, the complete bipartite graph $K_{m,n}$ has $m + n$ vertices, with m on one side and n on the other. There are mn edges in $K_{m,n}$, i.e., each vertex on one side is adjacent to every vertex on the other. Here is a drawing of $K_{7,5}$.



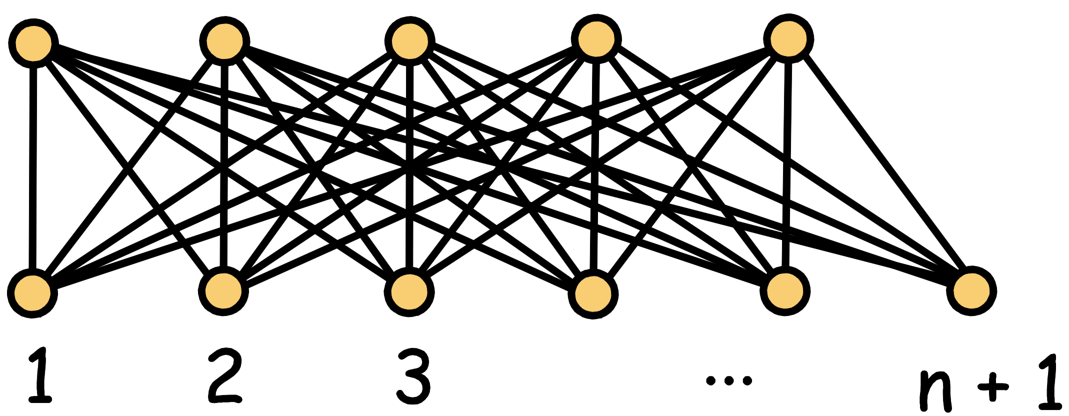
Hamiltonian Cycles in Bipartite Graphs

Observation If a bipartite graph $G = (A, B, E)$ has a Hamiltonian cycle, then it is connected and $|A| = |B|$.



Hamiltonian Cycles in Bipartite Graphs (2)

Observation In particular, the complete bipartite graph $K_{n,\, n+1}$ does not have a Hamiltonian cycle, even though every vertex is adjacent to (nearly) half the other vertices.



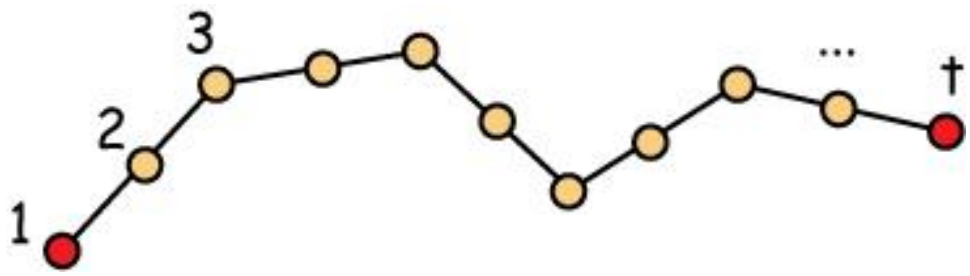
Dirac's Theorem

Theorem If G is a graph on n vertices and every vertex in G has at least $n/2$ neighbors, then G has a Hamiltonian cycle.

Note The complete bipartite graph $K_{n,\, n+1}$ has $2n + 1$ vertices but the vertices in the larger part have only n neighbors and $n < (2n + 1)/2$.

An Algorithm to Find a Hamiltonian Cycle

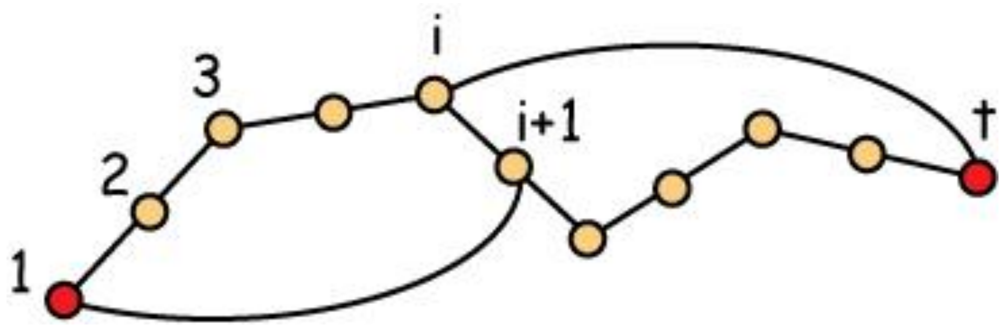
Initialization: Build Long Path



Note We may assume that all the neighbors of the end (red) vertices are on the path; otherwise we get a longer path. This implies $t > 1 + n/2$.

A Two-Phase Algorithm

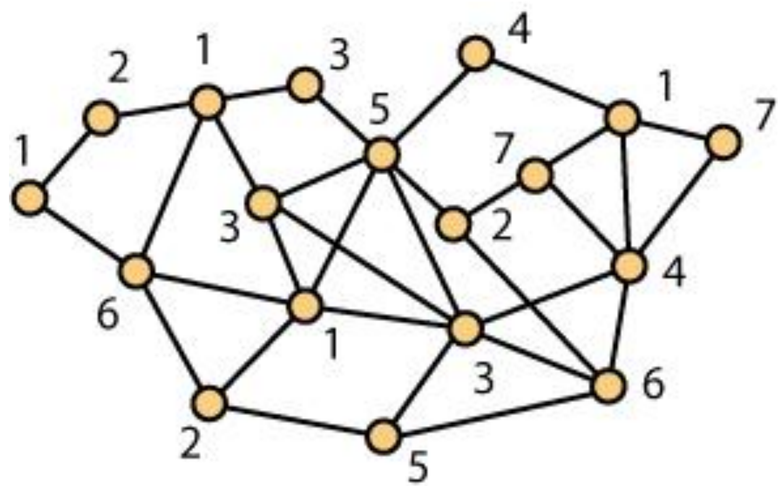
Phase 1 – Turn long path into cycle of same size



Note Using the pigeon-hole principle, there are consecutive vertices i and $i+1$ on the path with $\{1, i+1\}$ and $\{i, t\}$ as edges in G.

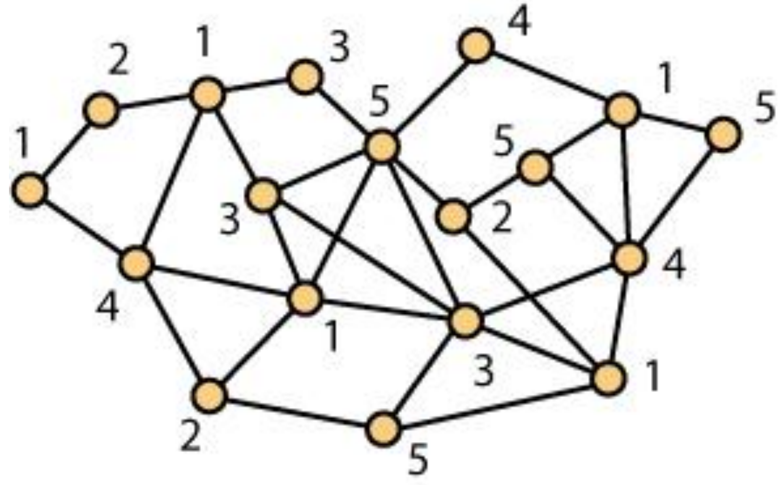
Chromatic Number

Definition A t-coloring of a graph G is an assignment of integers (colors) from $\{1, 2, \ldots, t\}$ to the vertices of G so that adjacent vertices are assigned distinct colors. We show a 7-coloring of the graph below.



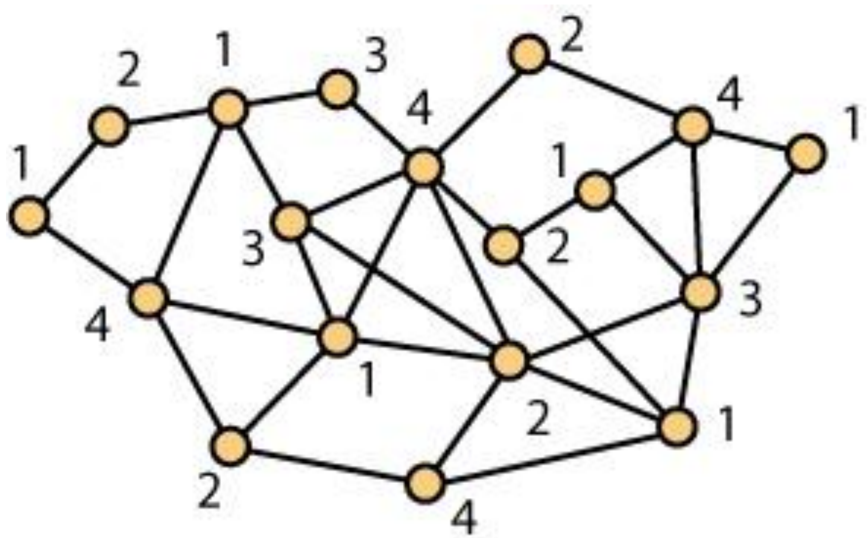
Chromatic Number (2)

Optimization Problems Given a graph G, what is the least t so that G has a t-coloring? This integer is called the chromatic number of G and is denoted $\chi(G)$. The coloring below is the same graph but now we illustrate a 5-coloring, so $\chi(G) \leq 5$.



Chromatic Number (3)

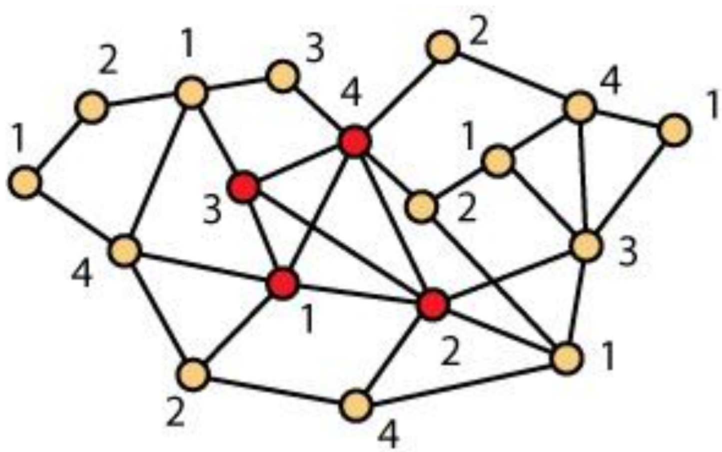
Optimization Problems The coloring below is the same graph but now we illustrate a 4-coloring, so $\chi(G) \leq 4$.



Maximum Clique Size

Definition Given a graph G, the maximum clique size of G, denoted $\omega(G)$, is the largest integer k for which G contains a clique (complete subgraph) of size k.

Trivial Lower Bound $\chi(G) \geq \omega(G)$ so in this case, we know $\chi(G) = \omega(G) = 4$.



Maximum Clique Size

Observation When $n \geq 2$, the odd cycle C_{2n+1} satisfies $\chi(C_{2n+1}) = 3$ and $\omega(C_{2n+1}) = 2$ so the inequality

$$\chi(G) \geq \omega(G)$$

need not be tight. In our next lecture, we will investigate this inequality in greater detail.

Computing Chromatic Number

Computational Complexity Detail Given a graph G and an integer t, the yes-no question: "Is $\chi(G) \leq t$?" belongs to the class **NP**.

Explanation It is obvious that a "yes" answer has a certificate that can be checked very efficiently. The certificate is just the assignment of colors to vertices.

Chromatic Number – A Special Case

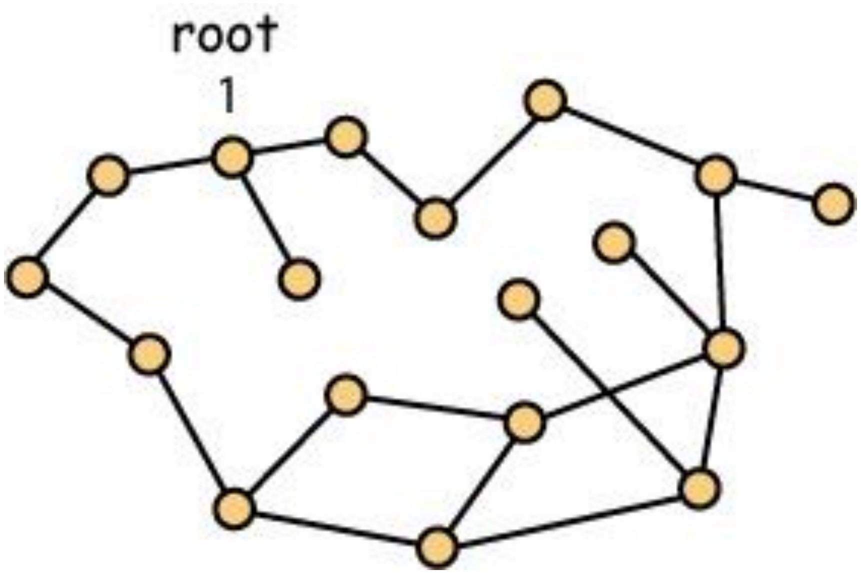
Computational Complexity Detail Given a graph G and an integer t, the yes-no question: "Is $\chi(G) \leq 2$?" belongs to the class **P**.

Basic Idea It is easy to see that $\chi(G) \geq 3$ when G contains an odd cycle. The algorithm we present will show that $\chi(G) \leq 2$ if and only if G does not contain an odd cycle. CS students will recognize that the algorithm uses "breadth-first" search. We will revisit this concept in greater detail later in the course.

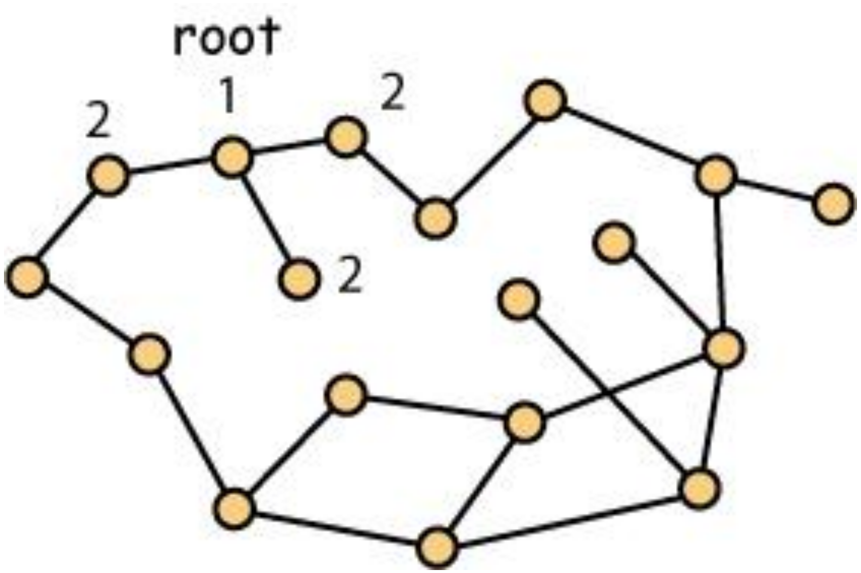
Chromatic Number – A Special Case (2)

Algorithm Choose an arbitrary vertex x and color it 1. Then find all uncolored vertices that are neighbors of colored vertices and color them with 2. Pause to check if you have an edge among the vertices colored 2. If yes, there is a triangle, so $\chi(G) \geq 3$ and the answer is "no". If no, find all uncolored neighbors of colored neighbors and color them 1. Pause to see if there are any edges among the vertices just colored. If yes, there is 5-cycle in G and the answer is "no". If yes, continue, alternating colors 1 and 2. Either the graph will be eventually 2-colored or we will find an odd cycle.

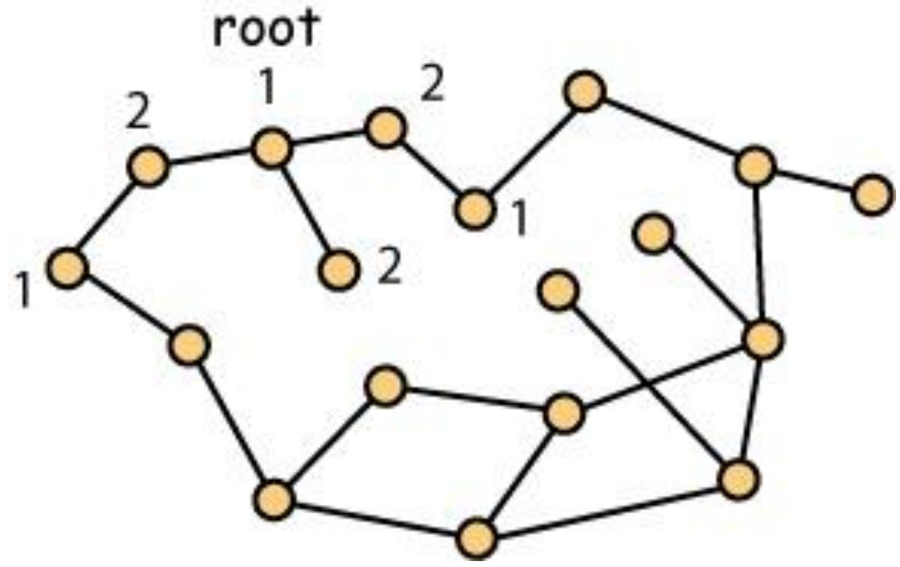
Applying the Algorithm (1)



Applying the Algorithm (2)

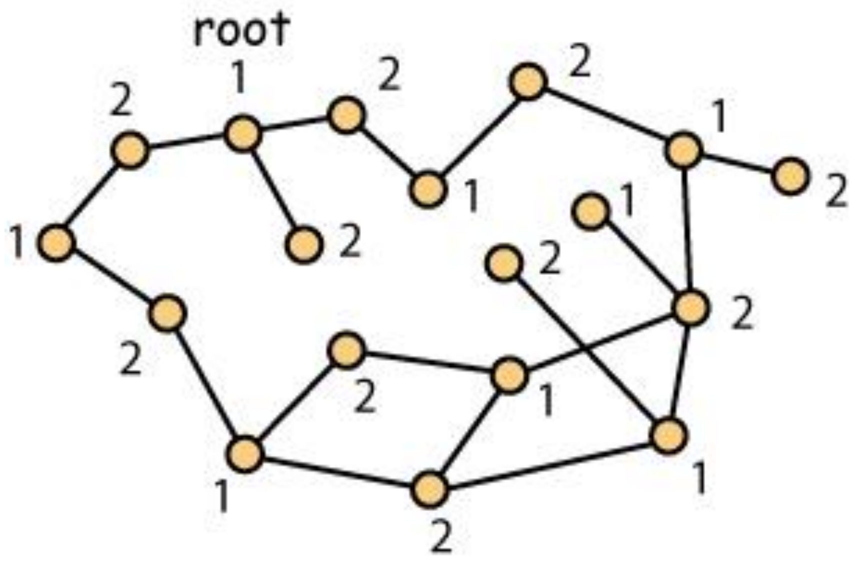


Applying the Algorithm (3)



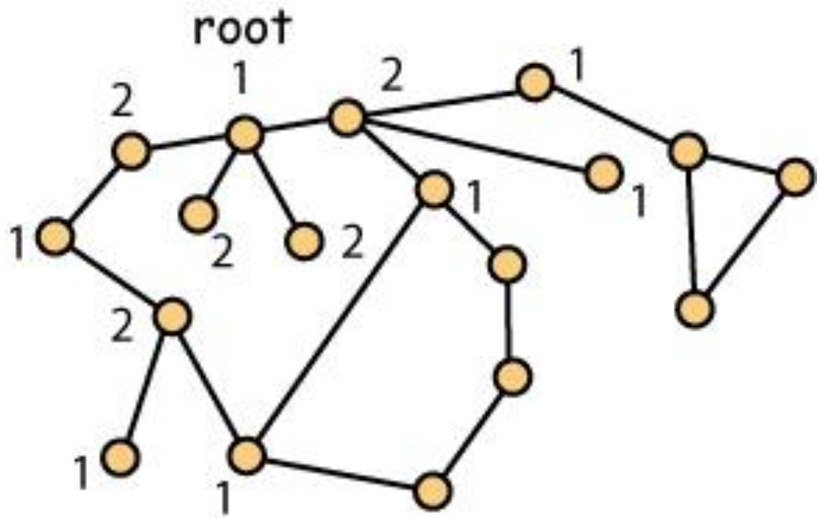
Applying the Algorithm (4)

Observation After several more steps, the algorithm halts with a 2-coloring of G.



Applying the Algorithm (5)

Observation Here's an example (using a different graph) of how the algorithm will detect an odd cycle.



Another Way to Earn a Million Bucks!!

Computational Complexity Question Given a graph G and an integer t, the yes-no question: "Is $\chi(G) \leq 3$?" belongs to the class **NP**. Does it also belong to **P**?

Remark As was stated explicitly in our lectures, I am not encouraging Math 3012 students to ponder on this question, as the greatest minds in the world have spent enormous amounts of time on it without success. However, it does represent just how challenging the delightful world of discrete mathematics can be.