

September 15, 2015



# Math 3012 - Applied Combinatorics Lecture 9

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# Test 1

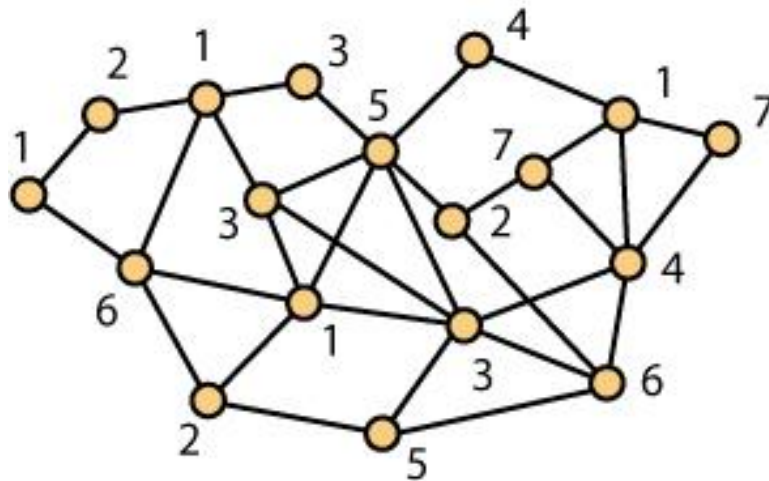
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**Reminder** Test 1, Thursday September 17, 2015. Taken here in MRDC 2404. Final listing of material for test was made via email after class on Thursday, September 10. Check the email archive on T-Square if you can't find your copy.

**Reminder** Don't miss the test! Check my strict but very clear policy on excused absences.

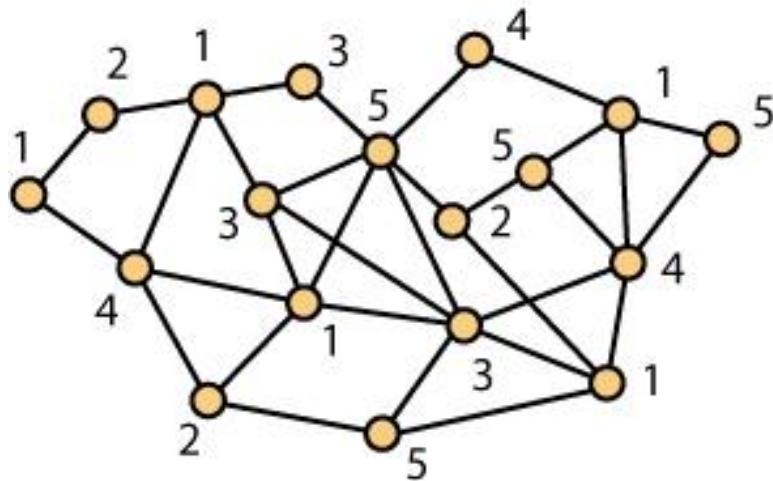
# Chromatic Number

**Definition** A  **$t$ -coloring** of a graph  $G$  is an assignment of integers (colors) from  $\{1, 2, \dots, t\}$  to the vertices of  $G$  so that adjacent vertices are assigned distinct colors. We show a 7-coloring of the graph below.



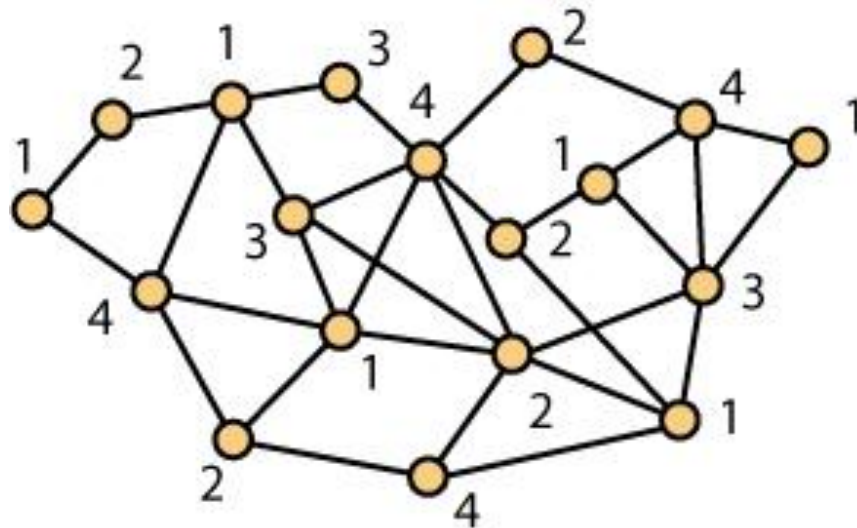
# Chromatic Number (2)

**Optimization Problems** Given a graph  $G$ , what is the least  $t$  so that  $G$  has a  $t$ -coloring? This integer is called the **chromatic number** of  $G$  and is denoted  $\chi(G)$ . The coloring below is the same graph but now we illustrate a 5-coloring, so  $\chi(G) \leq 5$ .



# Chromatic Number (3)

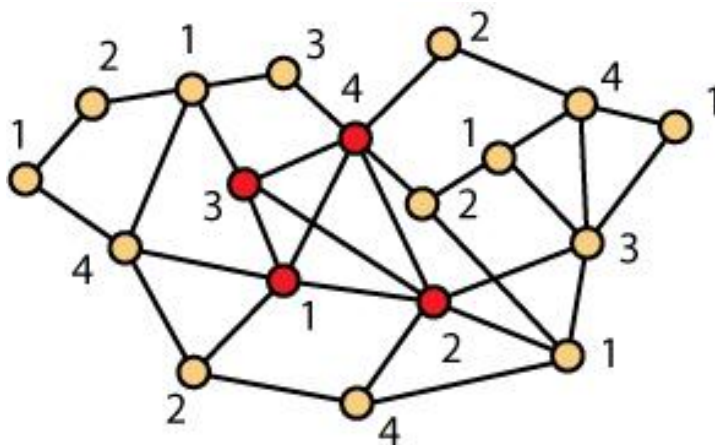
**Optimization Problems** The coloring below is the same graph but now we illustrate a 4-coloring, so  $\chi(G) \leq 4$ .



# Maximum Clique Size

**Definition** Given a graph  $G$ , the maximum clique size of  $G$ , denoted  $\omega(G)$ , is the largest integer  $k$  for which  $G$  contains a clique (complete subgraph) of size  $k$ .

**Trivial Lower Bound**  $\chi(G) \geq \omega(G)$  so in this case, we know  $\chi(G) = \omega(G) = 4$ .



# Maximum Clique Size

**Observation** When  $n \geq 2$ , the odd cycle  $C_{2n+1}$  satisfies  $\chi(C_{2n+1}) = 3$  and  $\omega(C_{2n+1}) = 2$  so the inequality

$$\chi(G) \geq \omega(G)$$

need not be tight. In the remainder of this lecture, we explore this inequality in greater depth, as it is of interest to understand conditions that make this inequality tight, and it is of interest to understand how badly it can fail.

# The Inequality Can Fail Arbitrarily

**Note** In today's class, we will give three different explanations for the following result.

**Theorem** For every  $t \geq 3$ , there is a graph  $G$  with  $\chi(G) = t$  and  $\omega(G) = 2$ .

**Note** A clique of size 3 is also called a **triangle**. Graphs with  $\omega(G) \leq 2$  are said to be **triangle-free**. So the fact can be rephrased as asserting that there are triangle-free graphs with arbitrarily large chromatic number.



# A Construction Using the Pigeon-Hole Principle

**Basic Idea** Proceed by induction. When  $t = 3$ , take  $G$  as the odd cycle  $C_5$ . Now suppose that for some  $t \geq 3$ , we have a triangle-free graph  $G$  with  $\chi(G) = t$ . Here's how we build a new triangle-free graph whose chromatic number is  $t + 1$ . Suppose  $G$  has  $m$  vertices labelled  $x_1, x_2, \dots, x_m$ .

Start with a "large" independent set  $Y$ . For each  $m$ -element subset  $\{y_1, y_2, \dots, y_m\}$  of  $Y$ , attach a copy of  $G$  with  $x_i$  adjacent to  $y_i$  for each  $i = 1, 2, \dots, m$ . This works if  $Y$  has size at least  $t(m - 1) + 1$  by the Pigeon-Hole principle.

# The Mycielski Construction

**Basic Idea** Proceed by induction. When  $t = 3$ , take  $G$  as the odd cycle  $C_5$ . Now suppose that for some  $t \geq 3$ , we have a triangle-free graph  $G$  with  $\chi(G) = t$ . Here's how we build a new triangle-free graph whose chromatic number is  $t + 1$ .

Start with a copy of  $G$ . Then add an independent set  $Y$  containing a "mate"  $y_x$  for every vertex  $x$  of  $G$ . The mate  $y_x$  has exactly the same neighbors in  $G$  as does  $x$ .

Then add one new vertex  $x_0$  which is adjacent to every vertex in  $Y$  but to none of the vertices in  $G$ .

# Shift Graphs

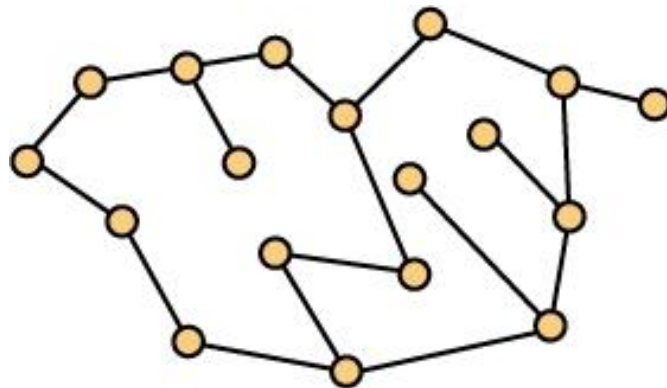
**Definition** When  $n \geq 2$ , the shift graph  $S_n$  has  $C(n, 2)$  vertices and these are the 2-element subsets of  $\{1, 2, \dots, n\}$ . For each 3-element subset  $\{i, j, k\}$  of  $\{1, 2, \dots, n\}$ , with  $i < j < k$ , the vertex  $\{i, j\}$  is adjacent to the vertex  $\{j, k\}$  in  $S_n$ .

**Theorem** For every  $n \geq 2$ , the chromatic number of the shift graph  $S_n$  is the least positive integer  $t$  so that  $2^t \geq n$ .

# The Girth of a Graph

**Definition** A graph containing no cycles is called a **forest**. In a forest, every component is a tree. So a tree is a forest. We say that the **girth** of a forest is infinite.

**Definition** When  $G$  is not a forest, we define the **girth** of  $G$  as the size of the smallest cycle in  $G$ . The graph shown below has girth 8.



# Chromatic Number and Girth

**Observation** The three constructions we have given for triangle-free graphs with large chromatic number produce graphs with small girth. Although the proof is a bit beyond our scope in this course, here is a historically very important result in applications of probability to combinatorics.

**Theorem** (Erdős, '59) For every pair  $(g, t)$  of positive integers with  $g, t \geq 3$ , there is a graph  $G$  with girth  $g$  and chromatic number  $t$ .