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5 - Graph Theory Basics

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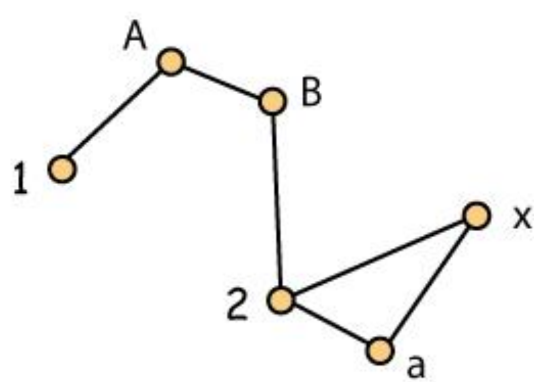
Basic Definitions

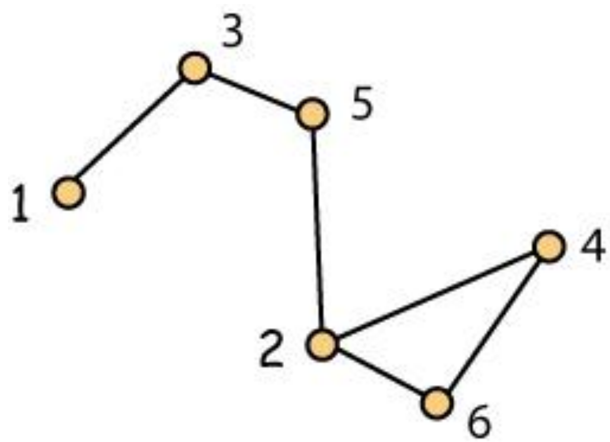
Definition A graph G is a pair (V, E) where V is a finite set and E is a set of 2-element subsets of V. The set V is called the vertex set of G and the set E is called the edge set of G.

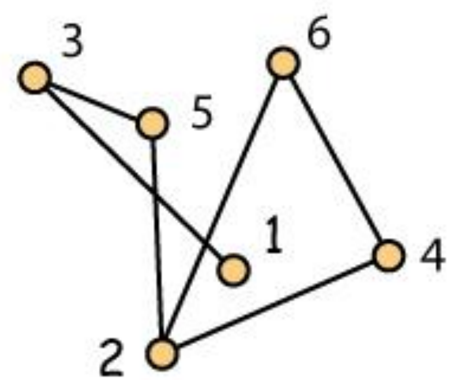
Example $G = (V, E)$ where $V = \{1, 2, A, x, B, a\}$ and $E = \{ \{1, A\}, \{2, x\}, \{x, a\}, \{A, B\}, \{B, 2\}, \{2, a\}\}$.



Its All About Adjacency

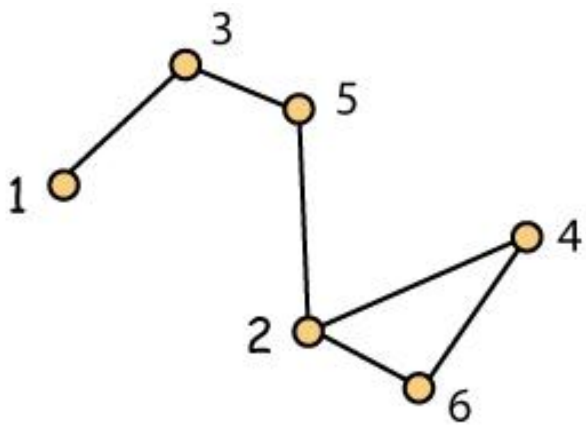
Comment We show below two drawings of the same graph whose vertex set is {1, 2, 3, 4, 5, 6}.

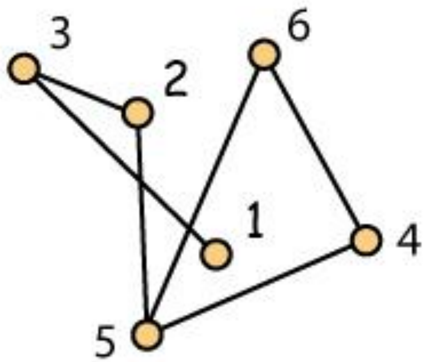




Its All About Adjacency (2)

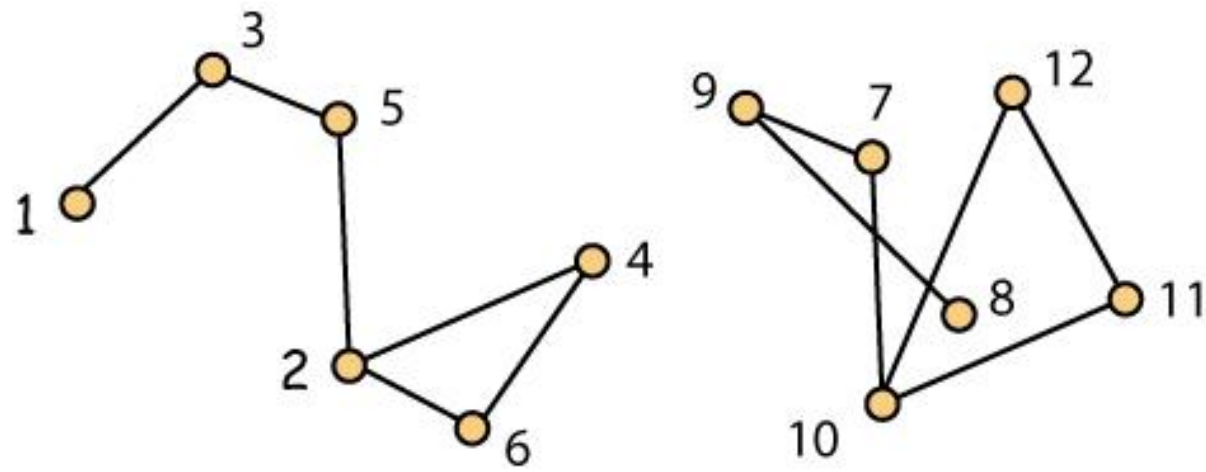
Comment We show below two drawings of graphs, each having vertex set {1, 2, 3, 4, 5, 6}, but now they represent different graphs.





Its All About Adjacency (3)

Question Is this a drawing of one graph whose vertex set is {1, 2, 3, …, 12} or do we have drawings of two graphs, one with vertex set {1, 2, 3, 4, 5, 6} and the other {7, 8, 9, 10, 11, 12}?



Answer Depends on the meaning of V in the pair (V, E).

Notation and Terminology

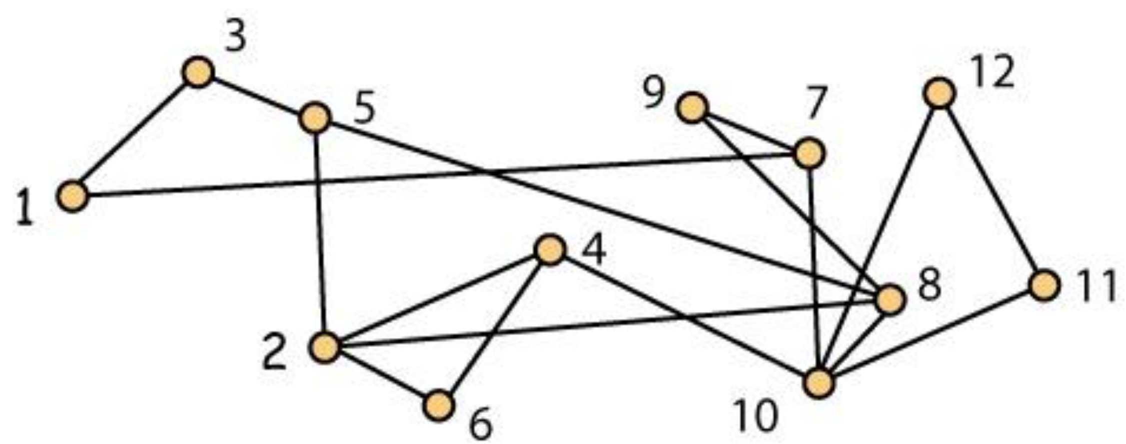
1. Vertices are also called nodes, points, locations, stations, etc.
2. Edges are also called arcs, lines, links, pipes, connectors, etc.
3. Remember that mathematicians are *selectively* lazy so when there is no confusion, an edge {x, y} will be denoted as xy. This can create some confusion when vertices are positive integers as how would one interpret a comment such as "consider the edge 2786".

Notation and Terminology (2)

1. When xy is an edge in G, we say x and y are adjacent in G. Alternatively, we say they are neighbors in G.
2. In a graph G, the set of all neighbors of a vertex x is denoted $N_G(x)$. And when the graph G is fixed in the discussion, this is typically abbreviated to just $N(x)$.
3. The integer $|N_G(x)|$ is called the degree of x in G, and is denoted $\deg_G(x)$. Again, when the graph is fixed, this is shortened to $\deg(x)$.

Notation and Terminology (3)

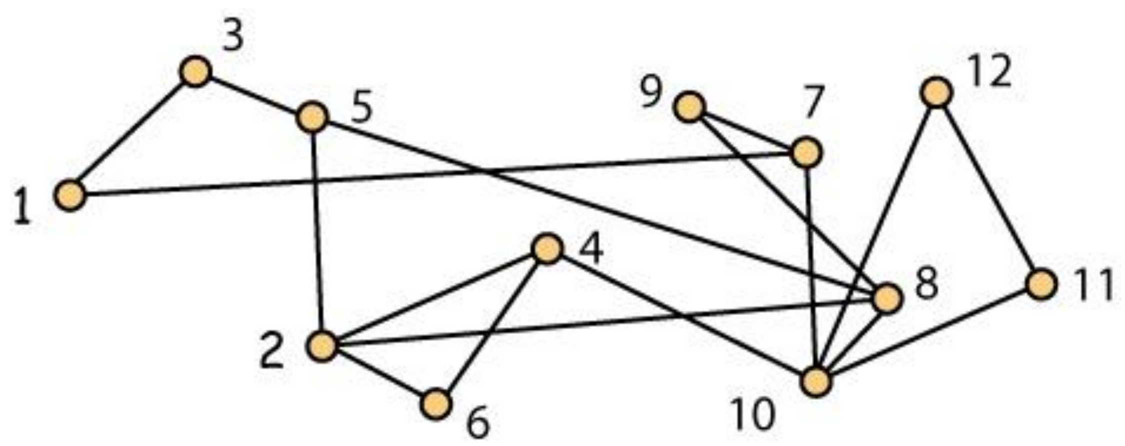
Example A graph with vertex set {1, 2, 3, …, 12}.



Questions Are 8 and 11 neighbors? What is deg(8)?

First Theorem in Graph Theory

Example Let $G = (V, E)$ be a graph and let q be the number of edges in G. Then $\sum_{x \in V} deg_G(x) = 2q$



Exercise Verify this theorem for the graph illustrated above.

Carlos and Dave

Overheard in Conversation Dave said that he was working with a graph and carefully counted all the degrees and said here is full listing of all the values:

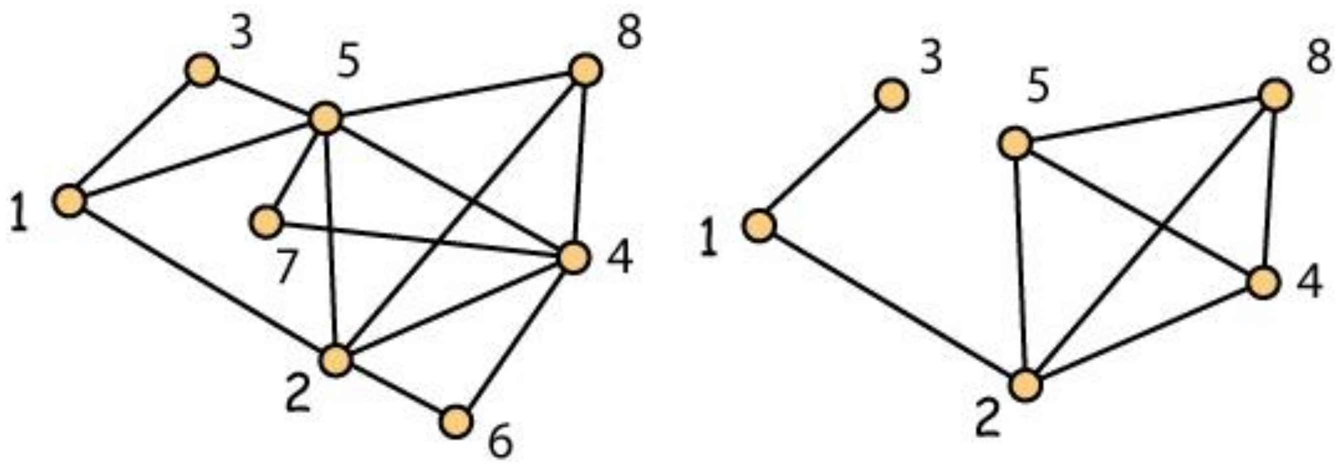
16, 13, 12, 18, 16, 22, 11, 16, 14, 10, 8, 12, 14, 16, 8, 7, 10, 20, 12, 14, 16, 8, 6, 6, 8, 4, 8, 6, 6, 6, 6, 8, 10, 5, 8, 8, 6, 6, 6, 6, 3, 6, 4, 8, 8, 8, 4, 8, 10, 12

Carlos remarked gently "Perhaps you should check your work."

The Notion of a Subgraph

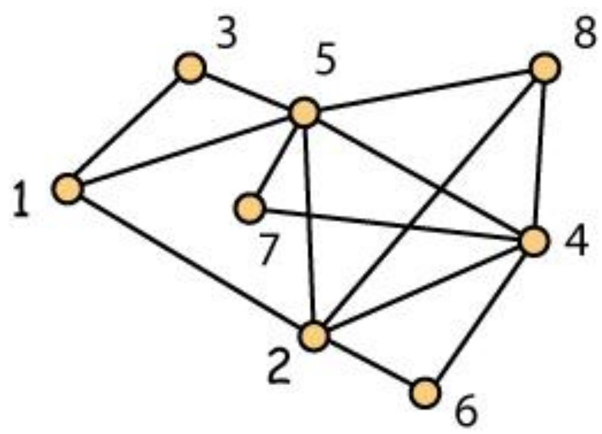
Definition A graph $G' = (V', E')$ is a subgraph of a graph $G = (V, E)$ when V' is contained in V and E' is contained in E.

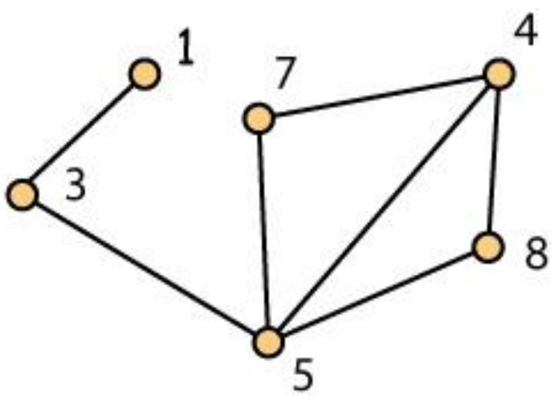
Example On the left, we show a graph with vertex set $\{1, 2, \ldots, 8\}$. The graph on the right is a subgraph.



The Notion of a Subgraph

Question We show a graph G with vertex set $\{1, 2, \ldots, 8\}$ on the left. Is the graph on the right a subgraph?

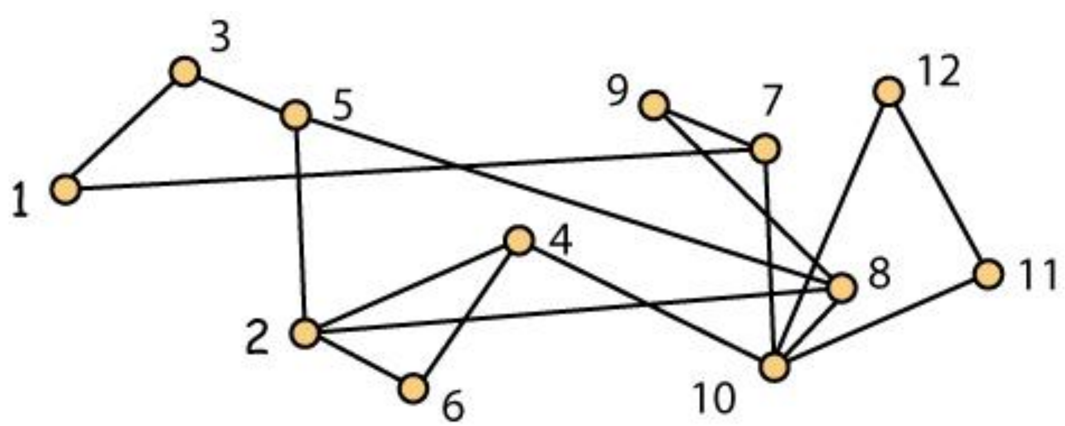




Paths in Graphs

Definition Let $G = (V, E)$ be a graph. When $n \geq 1$, a sequence $P = (x_1, x_2, \ldots, x_n)$ of n distinct vertices in G is called a path from x_1 to x_n in G if x_i is adjacent to x_{i+1} in G whenever $1 \leq i < n$.

Example In the graph shown, (7, 9, 8, 5, 2, 6, 4) is a path from 7 to 4.



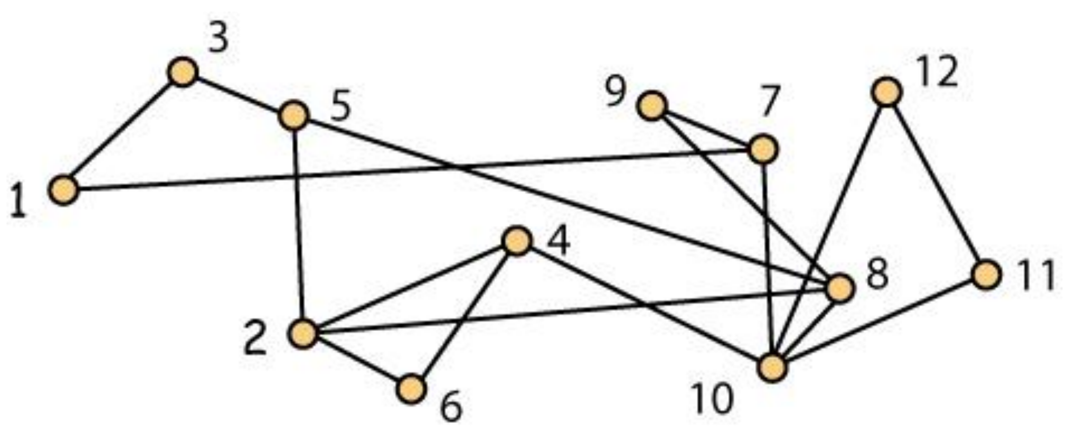
Size of Paths

Convention Many authors measure how big a path is in terms of the number of edges, so they will say that a path (a, b, c, d, e) from a to e has length 4. In particular, they would say that when x and y are neighbors, the path (x, y) has length 1. Other authors prefer to measure paths in terms of the number of vertices, so they would say that the path (a, b, c, d, e) has size 5. We prefer the second option, so we will always talk about paths of a certain size and this will count the number of vertices and not the number of edges.

Connected Graphs

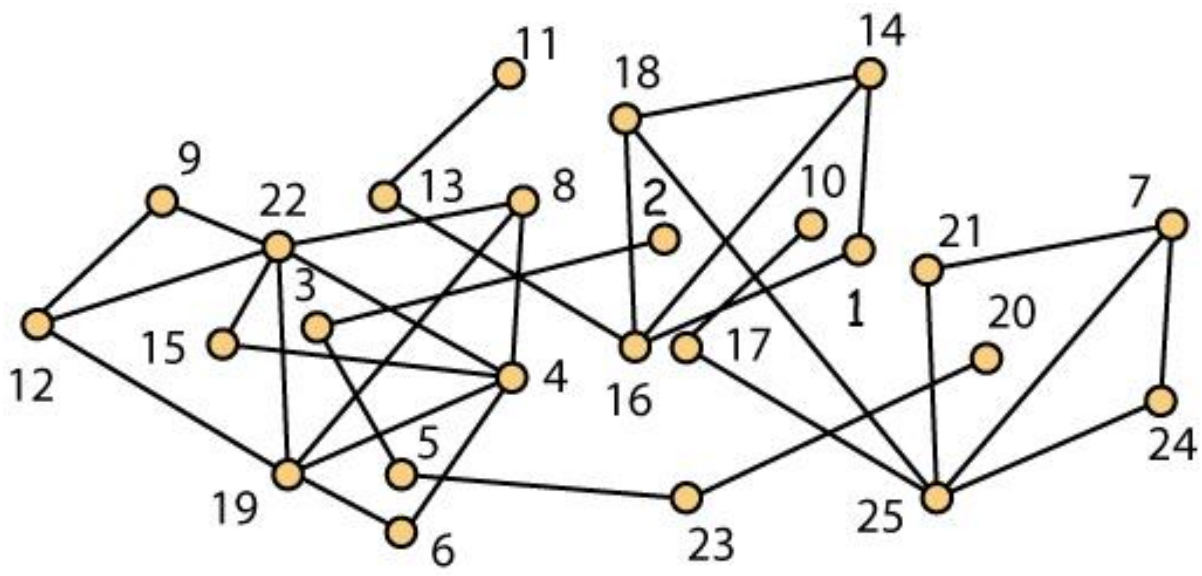
Definition Let $G = (V, E)$ be a graph. We say G is connected if for all x, y in V with $x \neq y$, there is a path from x to y in G.

Example The graph shown below is connected.



Connected Graphs (2)

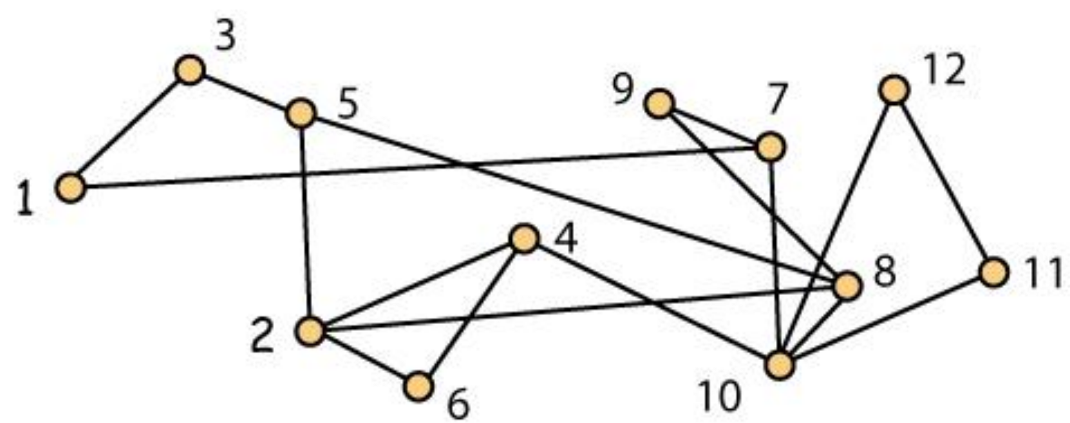
Definition Let $G = (V, E)$ be a disconnected graph. A subgraph $H = (V', E')$ of G is called a component of G if H is connected and any subgraph of G which contains H properly is disconnected. Is this graph connected? If not, how many components does it have?



Cycles in Graphs

Definition Let $G = (V, E)$ be a graph. When $n \geq 3$, a sequence $P = (x_1, x_2, \ldots, x_n)$ of n distinct vertices in G is called a cycle of size n in G if x_i is adjacent to x_{i+1} in G whenever $1 \leq i < n$ and x_n is adjacent to x_1 in G.

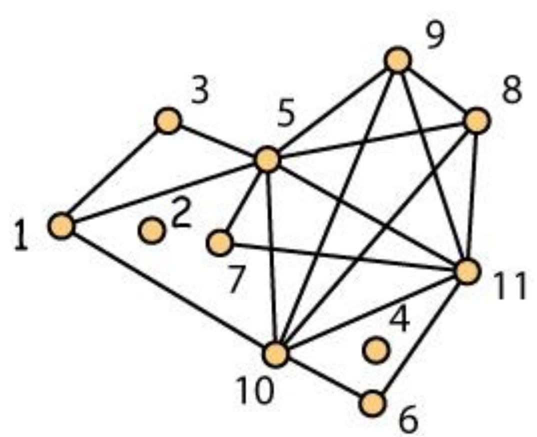
Example In the graph shown, (5, 8, 9, 7, 1, 3) is a cycle of size 6.



Loose Points in Graphs

Definition A vertex x in a graph G is called a loose point (also an isolated point) if it has no neighbors, i.e., $\deg_G(x) = 0$.

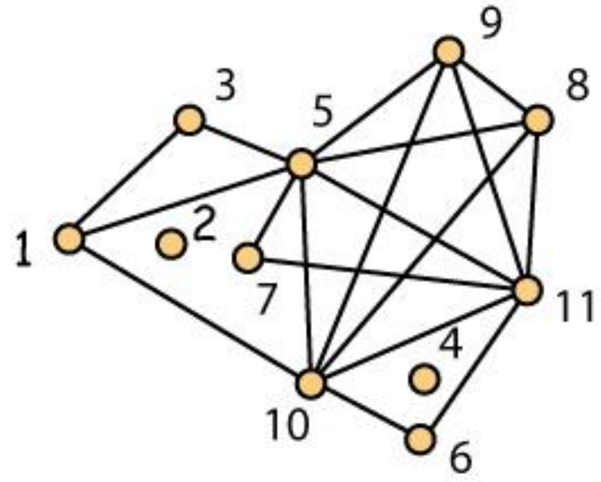
Example Below we show a graph with vertex set {1, 2, …, 11}. In this graph, vertices 2 and 4 are loose points.



Cliques in Graphs

Definition Let $G = (V, E)$ be a graph. When $n \geq 1$, a set S of vertices in G is called a clique if any two distinct vertices in S are adjacent in G.

Example In this graph, the subsets $\{2\}$, $\{6, 10\}$, $\{1, 3, 5\}$ and $\{5, 8, 9, 10, 11\}$ are cliques. There are many more.



Xing and Zori

Overheard in Conversations Xing is a very good programmer and remarked to Zori that he could easily detect whether a large graph was connected and if it was disconnected whether it had any loose vertices. Zori was not impressed as she couldn't see any reason why anybody would care about either issue. Still, moderately annoyed with Xing's enthusiasm, she asked him about a problem she had read about on the web: Could he tell whether a graph on $2n$ vertices had a clique of size n. Xing hadn't thought about it … but now that he was challenged, he said he thought he could. Hmmm …

Questions for Thought

Challenges or Not? Given a graph $G = (V, E)$ with $|V| = n$, which of the following problems is easy and which is hard?

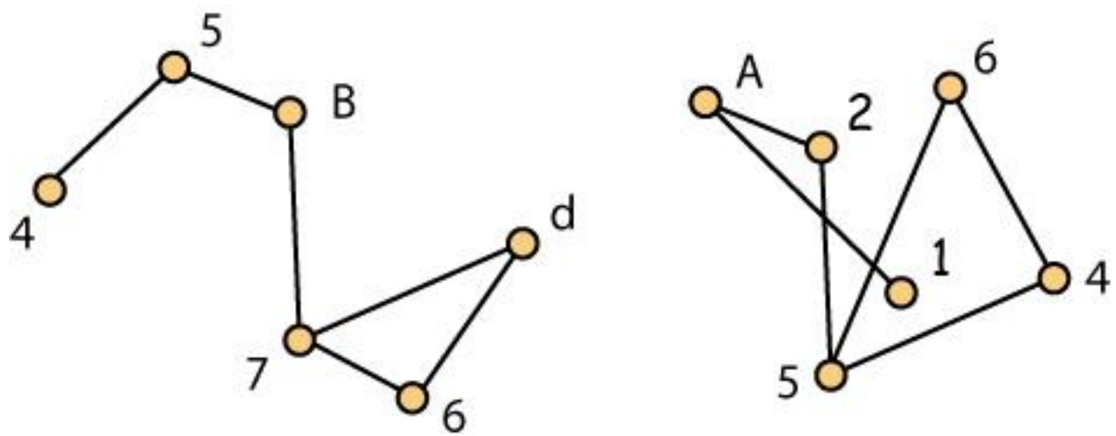
1. Is G connected?
2. Does G have a path of size at least $n/2$?
3. Does G have a cycle of size at least $n/2$?
4. Does G have a clique of size at least $n/2$?

Also Suppose Alice and Bob are arguing about the correct answers to these questions for a graph with $n = 10{,}000$. Would you rather defend a "yes" answer or a "no" answer.

Isomorphic Graphs

Definition Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic when there is a bijection $f\colon V_1 \to V_2$ so that $\{x, y\}$ is an edge in G_1 if and only if $\{f(x), f(y)\}$ is an edge in G_2.

Exercise Show that the two graphs shown below are isomorphic.

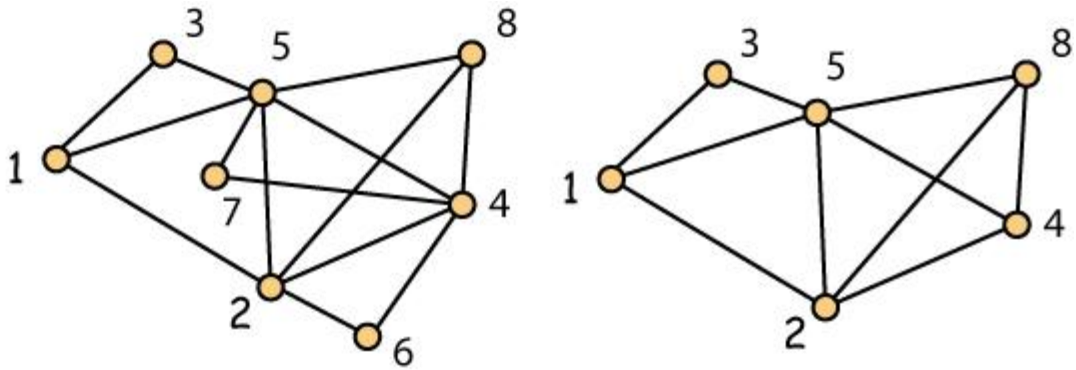


Another Question for Thought

Challenge or Not? Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is it hard to tell whether they are isomorphic? If Yolanda says "yes" and Bob says "no", who has the easier task to convince an impartial referee?

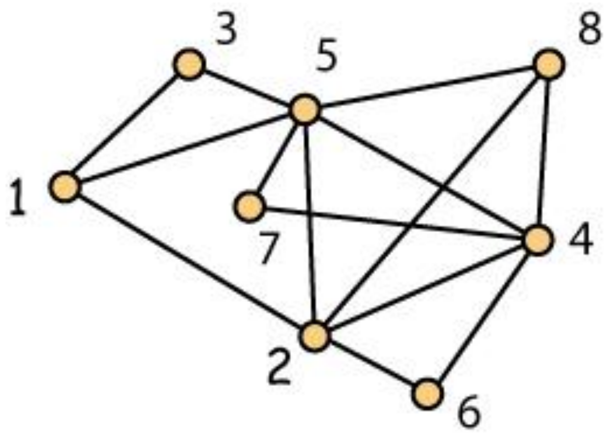
Induced Subgraphs

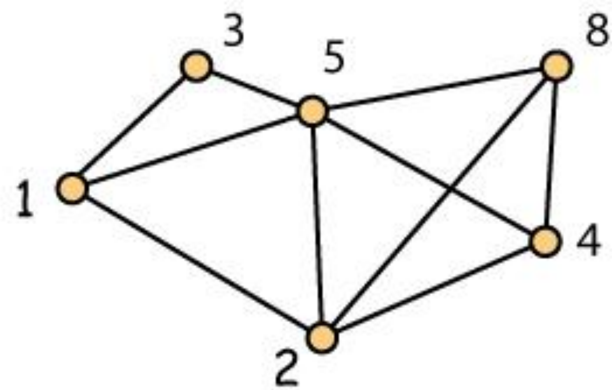
Definition A graph $H = (V', E')$ is an induced subgraph of a graph $G = (V, E)$ if $V' \subseteq V$ and xy is an edge in H whenever x and y are distinct vertices in V' and xy is an edge in G. In the drawing below, the graph on the right is an induced subgraph of the graph on the left.



Induced Subgraphs (2)

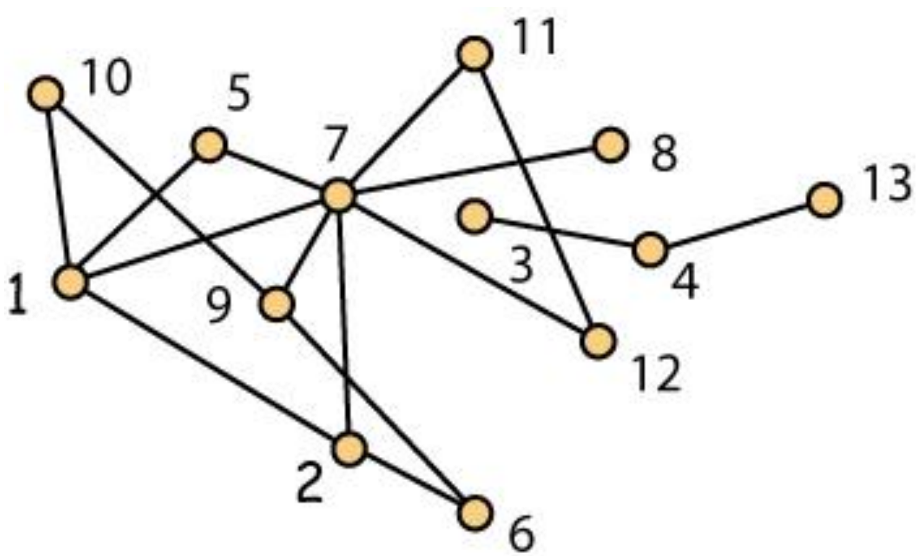
Remark When $G = (V, E)$ is a graph, an induced subgraph of G is determined entirely by its vertex set, so for example, the induced subgraph on the right can be denoted as $G - \{6, 7\}$.





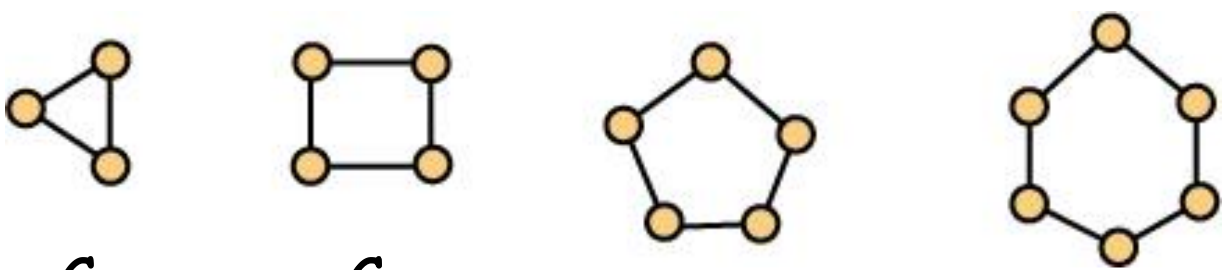
Cut Vertices

Definition A vertex x in a graph G is called a cut vertex of G if the induced subgraph $G - x$ has more components than G. In the graph shown below, 4 and 7 are cut vertices.



Special Classes of Graphs

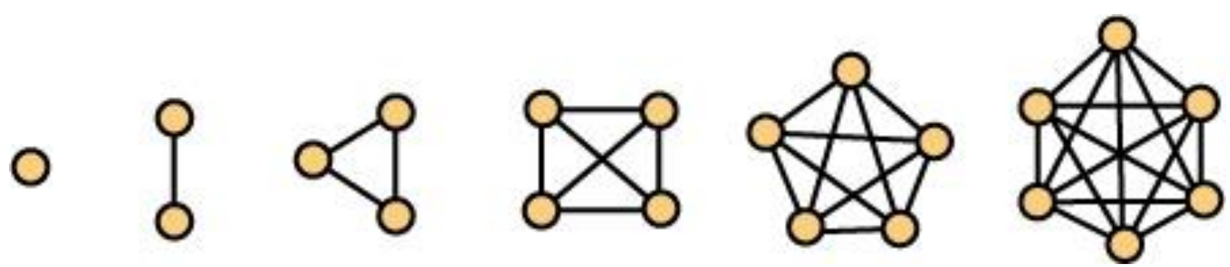
Definition For $n \geq 3$, C_n denotes a cycle on n vertices. Here are C_3, C_4, C_5 and C_6.



C_3 C_4 C_5 C_6

Special Classes of Graphs (2)

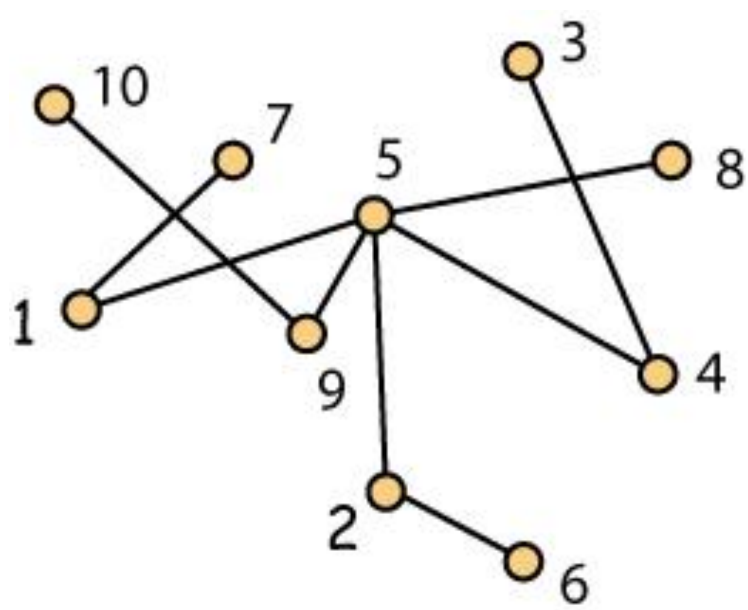
Definition For $n \geq 1$, K_n denotes a complete graph (also called a clique) on n vertices. Here are K_1, K_2, K_3, K_4, K_5 and K_6.



K_1 K_2 K_3 K_4 K_5 K_6

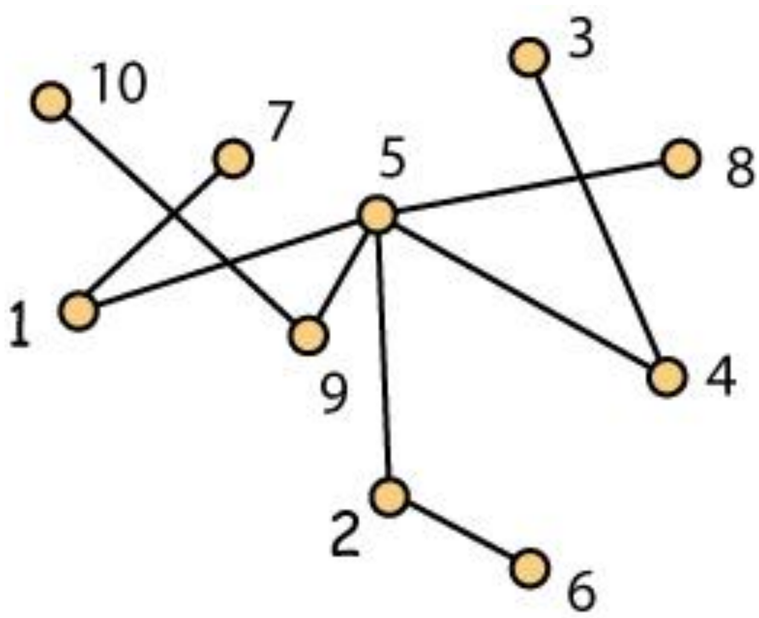
Special Classes of Graphs (3)

Definition A graph G on n vertices is a tree if G is connected and contains no cycles.



Properties of Trees

Definition When T is a tree, a vertex of degree 1 is called a leaf. This tree has five leaves: 3, 6, 7, 8, 10. The other vertices are cut vertices.



Properties of Trees (2)

Theorem When T is a tree on n vertices and $n \geq 2$, then T has at least two leaves.

Proof Induction on n. Obviously true when $n = 2$. Assume valid when T is a tree with at least 2 and at most k vertices for some $k \geq 2$. Then let T be a tree with $k + 1$ vertices. Then $k + 1 \geq 3$, so if T does not have 3 leaves, it has a cut vertex x. It follows that if C is a component of $G - x$, then $C + x$ is a tree and has at least 2 leaves. One of these is distinct from x and is therefore a leaf in T.

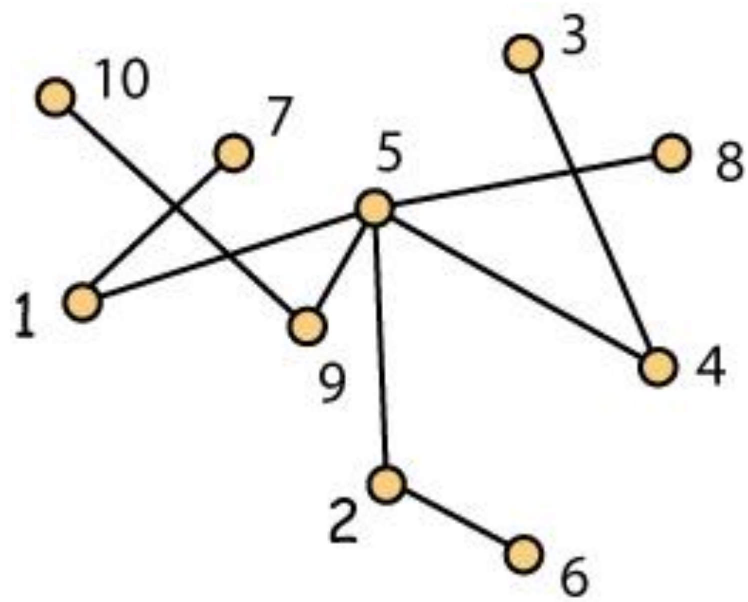
Properties of Trees (3)

Theorem When T is a tree on n vertices, T has $n - 1$ edges.

Proof Induction on n. True when $n = 1$. Now assume valid when $n = k$ for some integer $k \geq 1$. Then let T be a tree on $k + 1$ vertices. Choose a leaf x (there are at least two from which to choose). Then $\deg(x) = 1$ while the tree $T - x$ has k vertices and $k - 1$ edges. Therefore T has $(k - 1) + 1 = k$ edges.

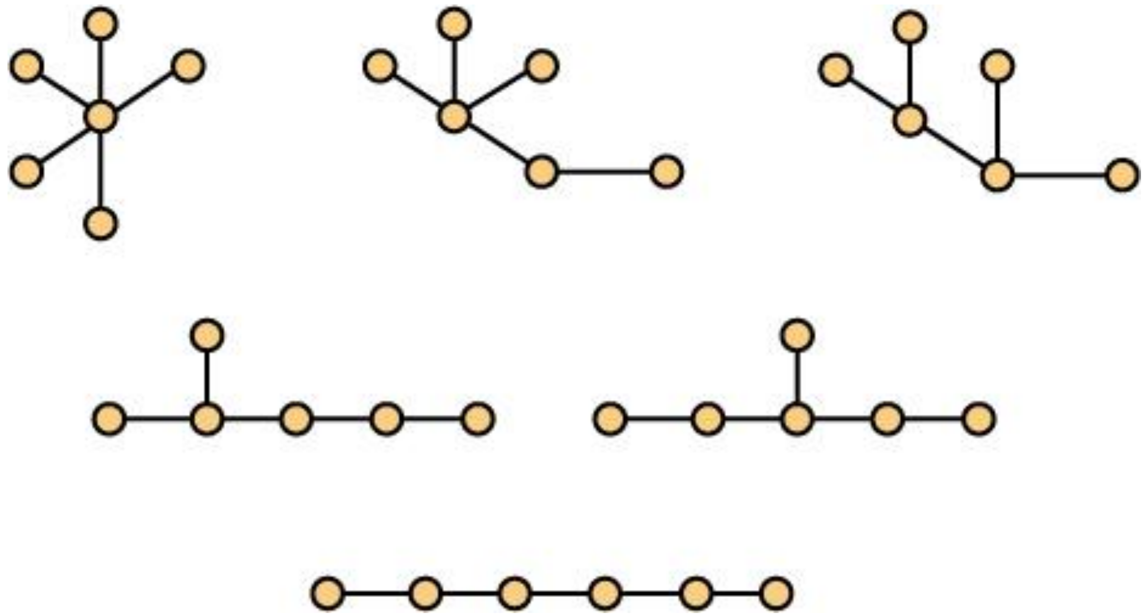
Paths and Trees

Theorem When T is a tree, T is a path unless it has more than two leaves.



Counting Trees

Exercise Explain why there are 6 unlabelled trees on 6 vertices. They are shown below.



The Unlabelled Trees on 6 Vertices

Exercise Show that when $1 \leq n \leq 6$, the number of trees with vertex set $\{1, 2, \ldots, n\}$ is n^{n-2}. Actually, we did the work when $1 \leq n \leq 5$ in class, so all you really have to do is the case $n = 6$.

Remark Later in the course, we will show that this is true for all $n \geq 1$.