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6 - Euler Circuits and Hamiltonian Cycles

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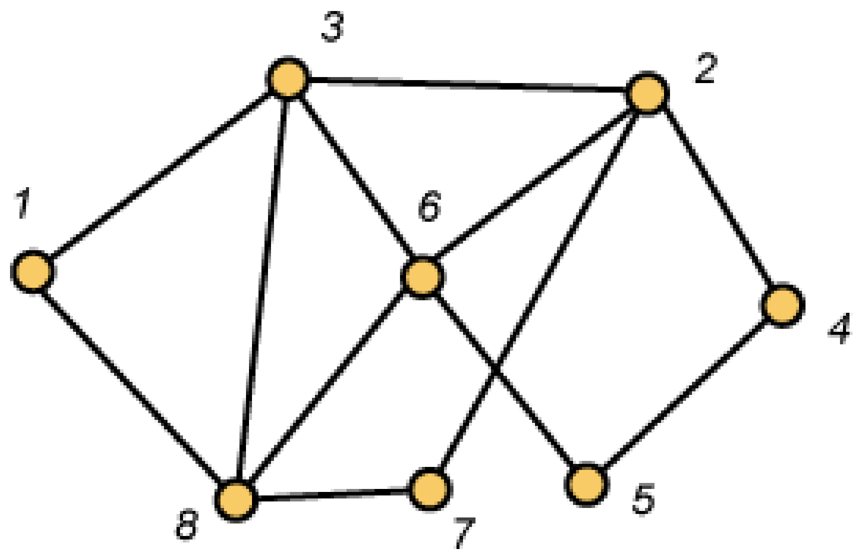
Euler Trails and Circuits

Definition A trail $(x_1, x_2, x_3, \ldots, x_t)$ in a graph G is called an Euler trail in G if for every edge e of G, there is a unique i with $1 \leq i < t$ so that $e = x_i x_{i+1}$.

Definition A circuit $(x_1, x_2, x_3, \ldots, x_t)$ *in a graph* G *is called an Euler circuit if for every edge* e *in* G, *there is a unique* i *with* $1 \leq i \leq t$ so that $e = x_i x_{i+1}$.
Note that in this definition, we intend that $x_t x_{t+1} = x_t x_1$.

Euler Circuits in Graphs



Here is an Euler circuit for this graph:

(1,8,3,6,8,7,2,4,5,6,2,3,1)

Euler's Theorem

Theorem A non-trivial connected graph G has an Euler circuit if and only if every vertex has even degree.

Theorem A non-trivial connected graph has an Euler trail if and only if there are exactly two vertices of odd degree.

Algorithm for Euler Circuits

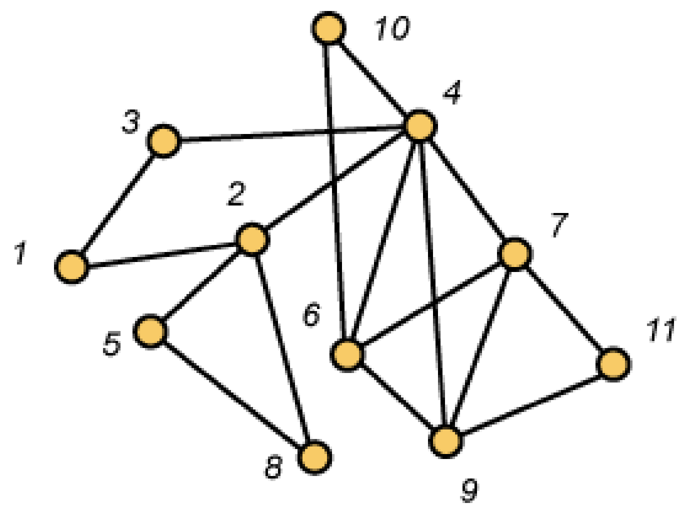
1. Choose a root vertex r and start with the trivial partial circuit (r).

2. Given a partial circuit $(r = x_0, x_1, \dots, x_t = r)$ that traverses some but not all of the edges of G containing r, remove these edges from G. Let i be the least integer for which x_i is incident with one of the remaining edges. Form a greedy partial circuit among the remaining edges of the form $(x_i = y_0, y_1, \dots, y_s = x_i)$.

3. Expand the original circuit by setting

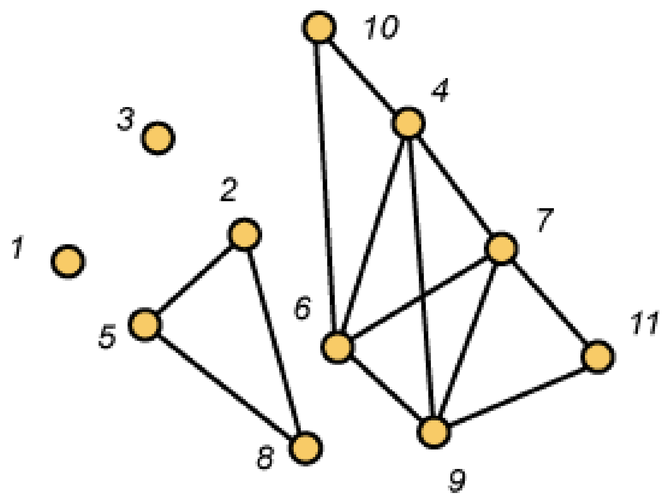
$$r = (x_0, x_1, \dots, x_{i-1}, x_i = y_0, y_1, \dots, y_s = x_i, x_{i+1}, \dots, x_t = r)$$

An Example



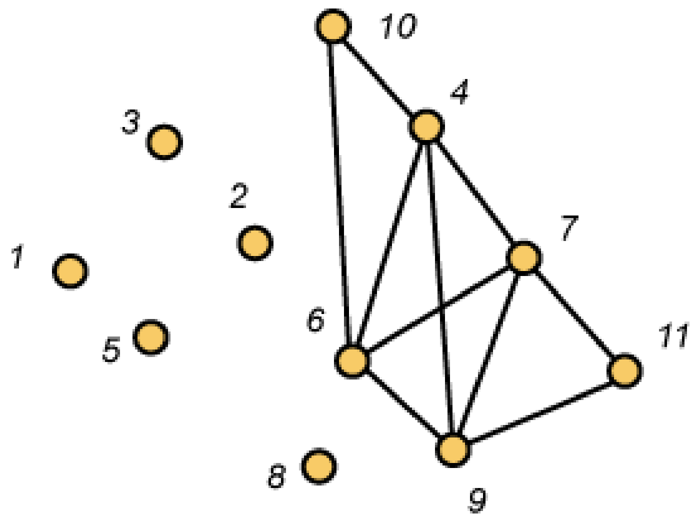
Start with the trivial circuit (1). Then the greedy algorithm yields the partial circuit (1,2,4,3,1).

Remove Edges and Continue



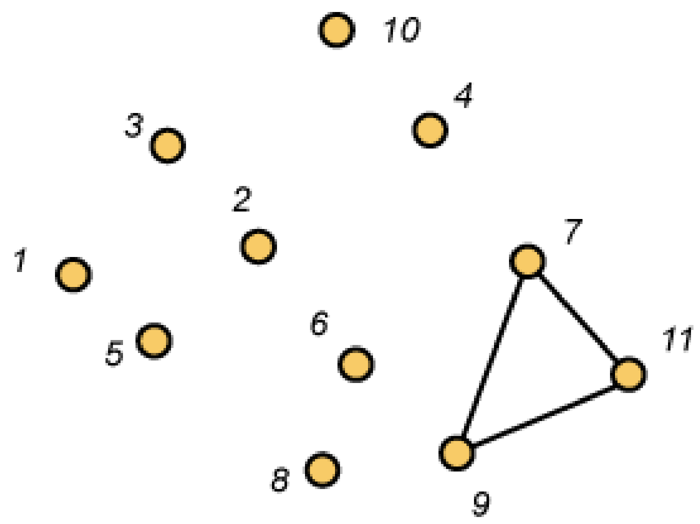
Start with the partial circuit (1,2,4,3,1). First vertex incident with an edge remaining is 2. A greedy approach yields (2,5,8,2). Expanding, we get the new partial circuit (1,2,5,8,2,4,3,1)

Remove Edges and Continue



Start with the partial circuit (1,2,5,8,2,4,3,1). First vertex incident with an edge remaining is 4. A greedy approach yields (4,6,7,4,9,6,10,4). Expanding, we get the new partial circuit (1,2,5,8,2,4,6,7,4,9,6,10,4,3,1)

Remove Edges and Continue



Start with the partial circuit (1,2,5,8,2,4,6,7,4,9,6,10,4,3,1) First vertex incident with an edge remaining is 7. A greedy approach yields (7,9,11,7). Expanding, we get the new partial circuit (1,2,5,8,2,4,6,7,9,11,7,4,9,6,10,4,3,1). This exhausts the edges and we have an euler circuit.

Interpreting Halting Conditions

Remark Suppose any loop halts with a starting vertex x and a terminating vertex y which is distinct from x. The conclusion is that y has odd degree. If we are searching for an Euler circuit, there isn't one. End of story. But if we are willing to accept an Euler trail, start over with y as root.

Remark If we halt with another odd pair, then there's not even an Euler trail.

Remark If we halt, there are unvisited edges and there's no place to start the next loop, then the graph has two non-trivial components.

Data Structure Issues

Remark When we read the data for the graph, we must build for each vertex x a structure that keeps track of the neighbors of x. As the algorithm progresses, we must keep track of the neighbors of x for which we have already walked on the edge xy. So either we have to "flag" edges already visited or have a convenient way to delete them from the neighborhood.

Remark The Greedy Algorithm taught in class tries to capture the spirit of these complexities, but in fact an actual implementation might follow a quite different track.

Hamiltonian Paths and Cycles

Definition When G is a graph on $n \geq 3$ vertices, a cycle $C = (x_1, x_2, \ldots, x_n)$ in G is called a Hamiltonian cycle, i.e, the cycle C visits each vertex in G exactly one time and returns to where it started.

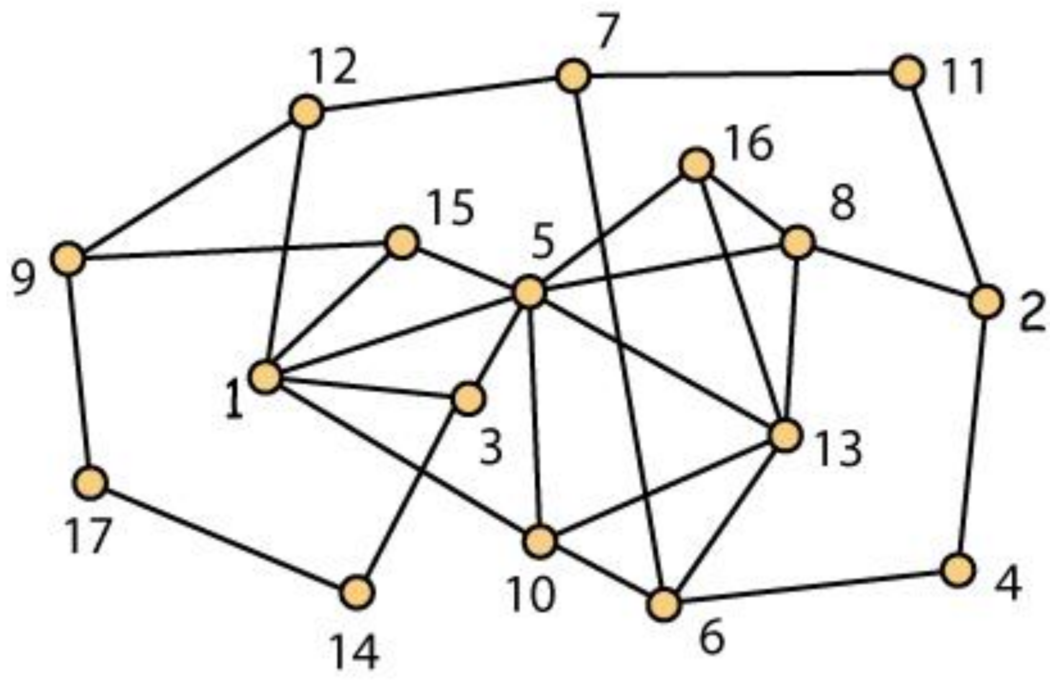
Definition When G is a graph on $n \geq 3$ vertices, a path $P = (x_1, x_2, \ldots, x_n)$ in G is called a Hamiltonian path, i.e, the path P visits each vertex in G exactly one time. In contrast to the first definition, we no longer require that the last vertex on the path be adjacent to the first.

Hamiltonian Paths and Cycles (2)

Remark In contrast to the situation with Euler circuits and Euler trails, there does not appear to be an efficient algorithm to determine whether a graph has a Hamiltonian cycle (or a Hamiltonian path). For the moment, take my word on that but as the course progresses, this will make more and more sense to you.

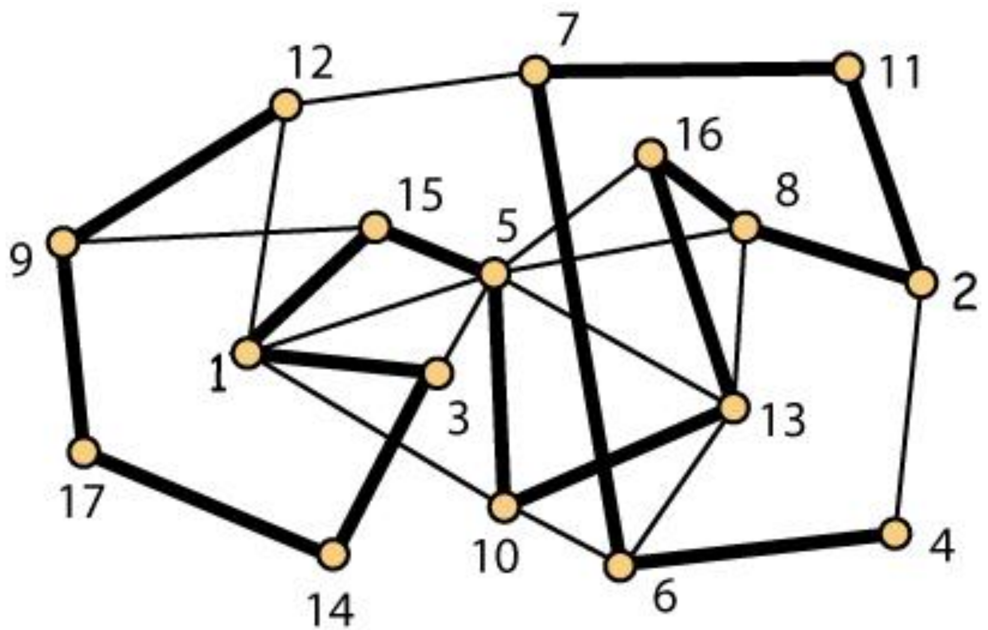
Hamiltonian Paths

Question Does the graph shown below have a Hamiltonian path?



Hamiltonian Paths (2)

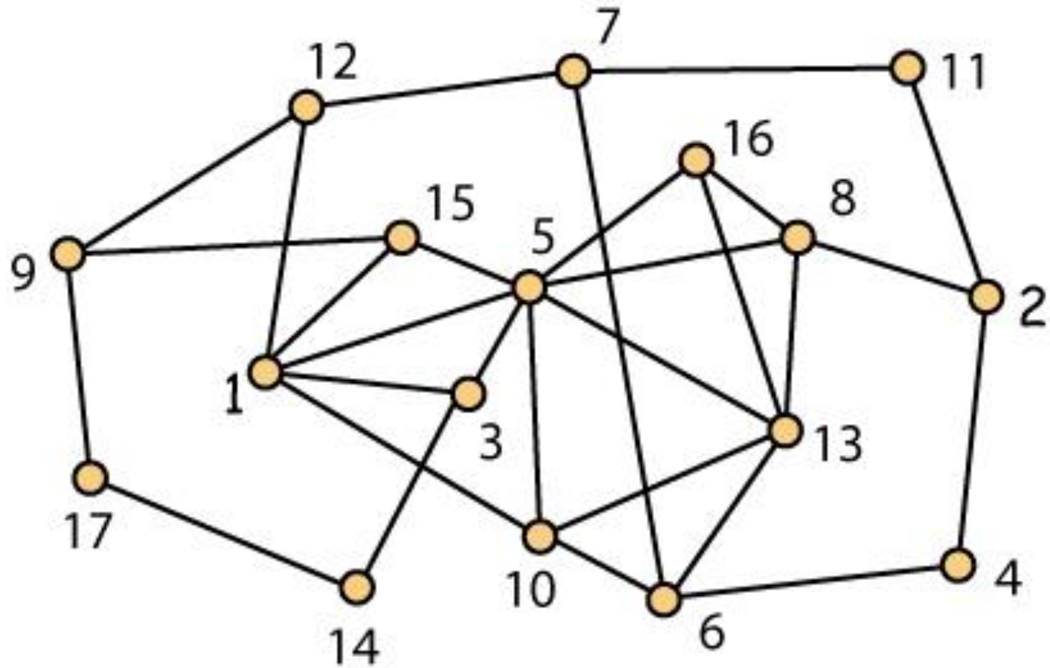
Answer Yes!!



(12, 9, 17, 14, 3, 1, 15, 5, 10, 13, 16, 8, 2, 11, 7, 6, 4)

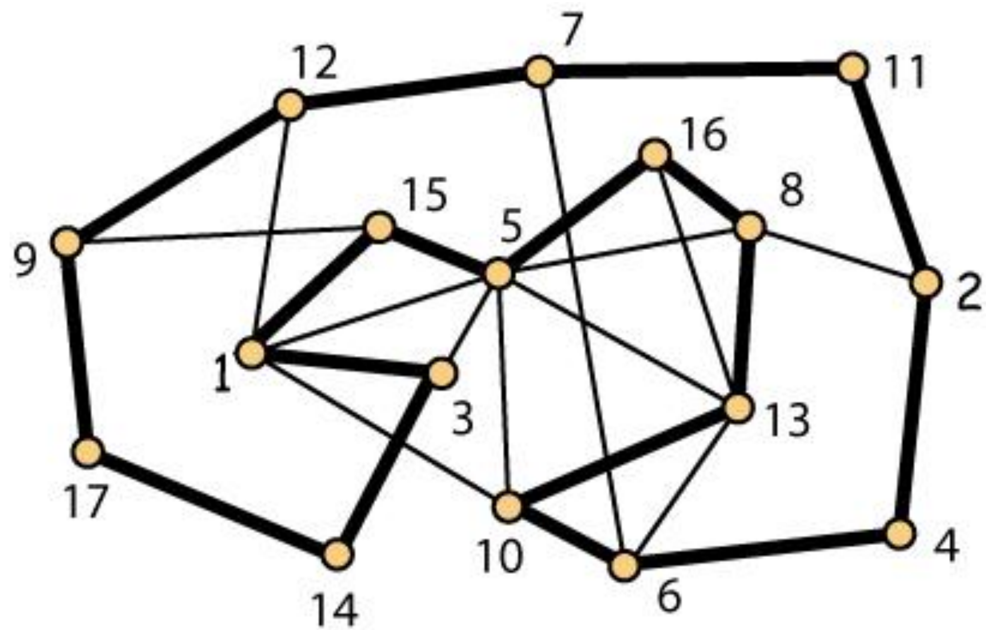
Hamiltonian Cycles

Question Does the graph shown below have a Hamiltonian cycle?



Hamiltonian Cycles (2)

Answer Yes!!



(1, 3, 14, 17, 9, 12, 7, 11, 2, 4, 6, 10, 13, 8, 16, 5, 15)

Certificates for "Yes" Answer

Remark Given a graph G, a "yes" answer to the question: Does G have a Hamiltonian path?" can be validated by providing a certificate in the form of a permutation of the vertex set of G. An impartial referee (computer) can quickly check the essential details. Is every vertex listed exactly once? Are consecutive vertices adjacent in the graph?

Remark An analogous statement applies for Hamiltonion cycles.

Hamiltonian Paths (3)

Question Does this graph have a Hamiltonian path?

```
Graph_data.txt
6
5  2
1  4
6  5
3  6
1  3
4  5
4  6
6  1
```

Answer Yes!!

Certificate (6, 3, 1, 4, 5, 2)

Note The correctness of the answer can be verified quickly by an impartial referee (computer).

Hamiltonian Cycles (3)

Question Does this graph have a Hamiltonian cycle?

```
Graph_data.txt
6
5  2
1  4
6  5
3  6
1  3
4  5
4  6
6  2
```

Answer Yes!!

Certificate (6, 3, 1, 4, 5, 2)

Note The correctness of the answer can be verified quickly by an impartial referee (computer).

Certificates for "No" Answer

Remark Given a graph G, there does not seem to be a way to provide a certificate to validate a "no" answer to the question: Does G have a Hamiltonian cycle?" To be more precise, there does not seem to be a way to provide an impartial referee (computer) with information which can be effectively checked and will satisfy the referee that your answer is correct.

There Are Exceptions

Question Does this graph have a Hamiltonian cycle?

```
Graph_data.txt
6
5   2
1   4
6   5
3   6
1   3
4   5
4   6
6   1
```

Answer No!!

Certificate Vertex 2 has degree 1. If a graph has a Hamiltonian cycle, every vertex has degree at least 2.

Note The correctness of the answer can be verified quickly by an impartial referee (computer).

Certificates for "No" Answer

Remark Given a graph G, there does not seem to be a way to provide a certificate to validate a "no" answer to the question: Does G have a Hamiltonian cycle?" To be more precise, there does not seem to be a way to provide an impartial referee (computer) with information which can be effectively checked and will satisfy the referee that your answer is correct, at least not in general. This does not preclude there being a justification for a "no" answer in *some* cases.

Computational Complexity

A Very Informal Perspective The class **P** consists of all "yes-no" questions for which the answer can be determined using an algorithm which is provably correct and has a running time which is polynomial in the input size.

Examples

1. Given a list of n numbers, is 2388643 in the list?
2. Given a list of n numbers, can you find three distinct numbers a, b and c in the list so that $a + b = c$?
3. Given a graph G, does it have an Euler circuit?

Computational Complexity (2)

A Very Informal Perspective The class **NP** consists of all "yes-no" questions for which there is a certificate for a "yes" answer whose correctness can be verified with an algorithm whose running time is polynomial in the input size. *Any question in* **P** is also in **NP**.

Examples

1. Given a list of n numbers, is there a fair division?
2. Given a graph G, is there a clique whose size is at least $n/2$?
3. Given a graph G, does it have a Hamiltonian cycle?

Computational Complexity (3)

Observation As we have already noted, any problem which is in **P** is also in **NP**, but no one knows whether the converse statement is true or not. The current reward for settling this question:

$$\mathbf{P} = \mathbf{NP}\,?$$

Stands at $1, 000, 000 USD.

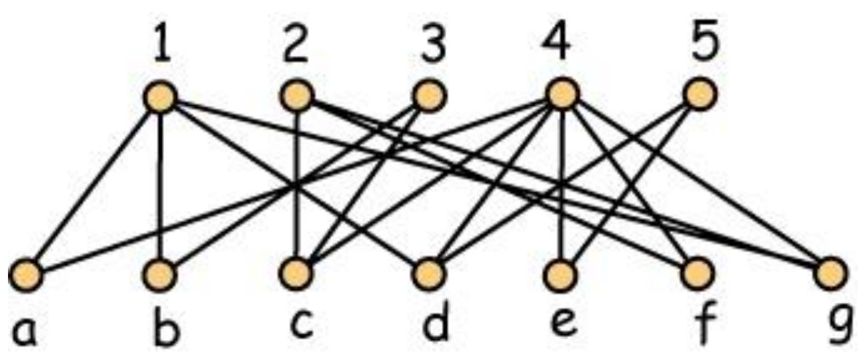
http://www.claymath.org/millennium-problems

Revisiting Euler Circuits

Remark Given a graph G, a "no" answer to the question: Does G have an Euler circuit?" can be validated by providing a certificate. Now this certificate is one of the following. Either the graph is not connected, so the referee is told of two specific vertices for which the graph does not contain a path between them. On the other hand, if the graph is connected, then the referee is told that there is vertex of odd degree.

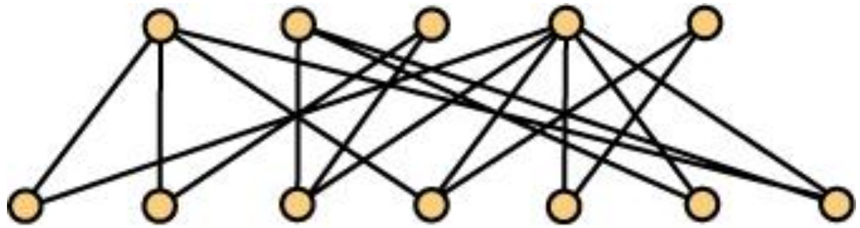
Bipartite Graphs

Definition A bipartite graph is a triple (A, B, E) where A and B are disjoint finite sets and E is a collection of 2-element sets, each of which contains one element of A and one element of B. In the bipartite graph shown below, A = {a, b, c, d, e, f, g} and B = {1, 2, 3, 4, 5}



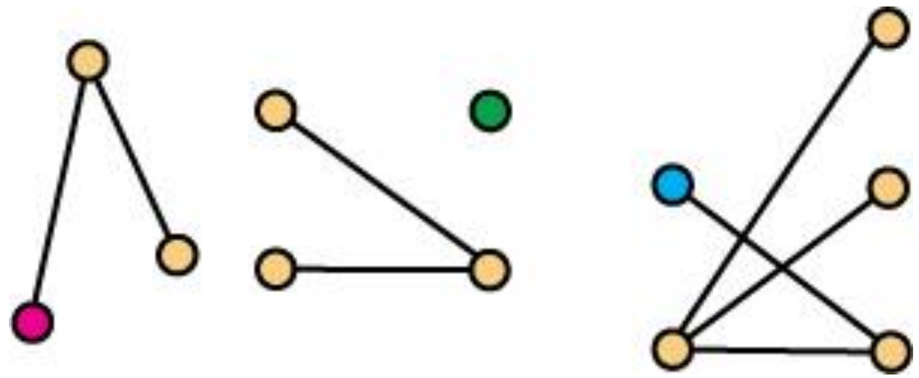
Unlabelled Bipartite Graphs

Caution In a discussion of unlabelled bipartite graphs, care has to be exercised regarding which elements belong to A and which belong to B. The potential for confusion is minor when the graph is connected.



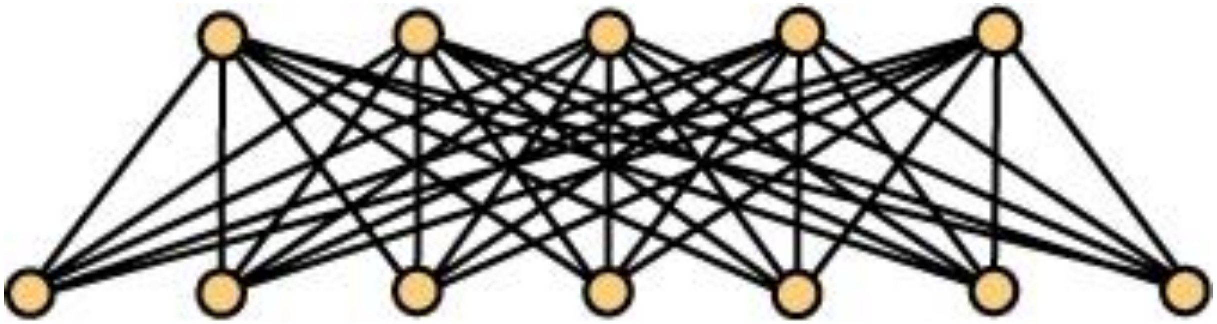
Unlabelled Bipartite Graphs

Caution But there are real problems when the graph is disconnected. For example, consider the red, blue and green points in the graph shown below. Which side are they on?



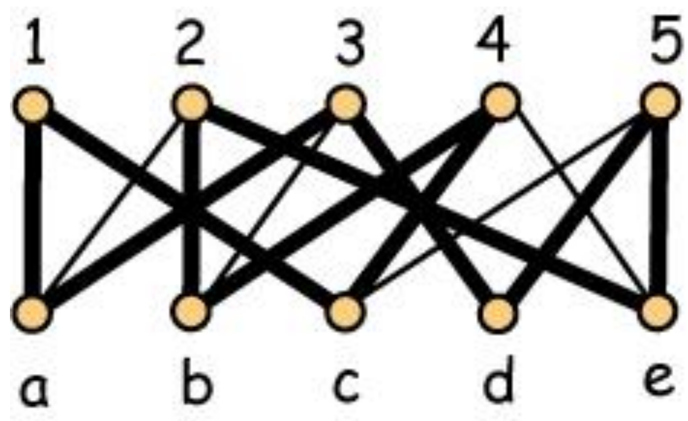
Complete Bipartite Graphs

Definition For $m, n \geq 1$, the complete bipartite graph $K_{m,n}$ has $m + n$ vertices, with m on one side and n on the other. There are mn edges in $K_{m,n}$, i.e., each vertex on one side is adjacent to every vertex on the other. Here is a drawing of $K_{7,5}$.



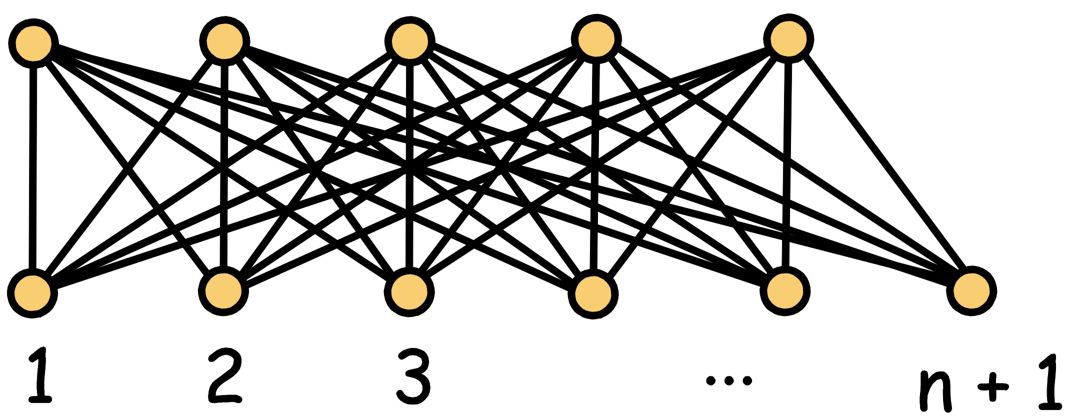
Hamiltonian Cycles in Bipartite Graphs

Observation If a bipartite graph $G = (A, B, E)$ has a Hamiltonian cycle, then it is connected and $|A| = |B|$.



Hamiltonian Cycles in Bipartite Graphs (2)

Observation In particular, the complete bipartite graph $K_{n,\,n+1}$ does not have a Hamiltonian cycle, even though every vertex is adjacent to (nearly) half the other vertices.



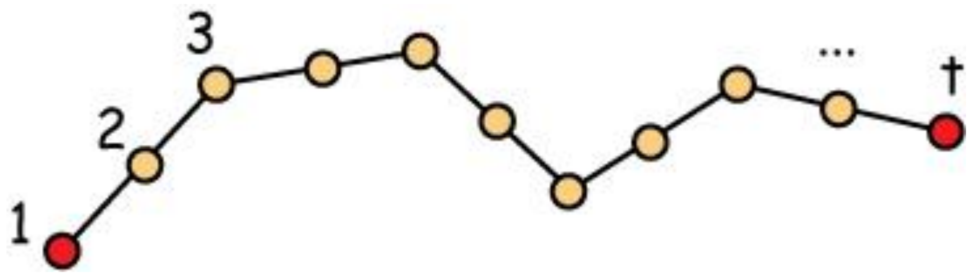
Dirac's Theorem

Theorem If G is a graph on n vertices and every vertex in G has at least $n/2$ neighbors, then G has a Hamiltonian cycle.

Note The complete bipartite graph $K_{n,\, n+1}$ has $2n + 1$ vertices but the vertices in the larger part have only n neighbors and $n < (2n + 1)/2$.

An Algorithm to Find a Hamiltonian Cycle

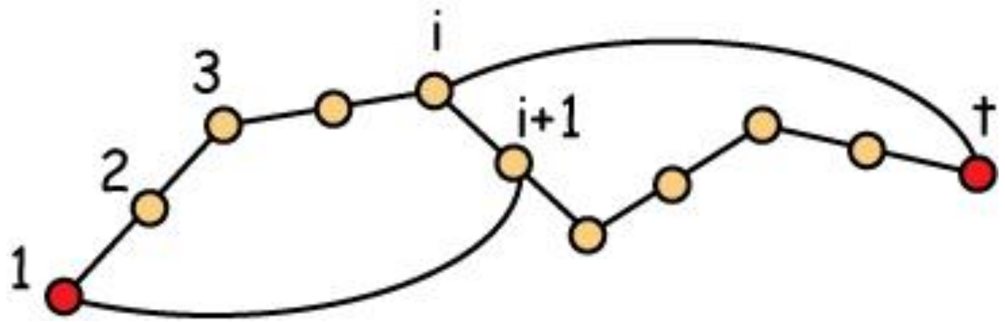
Initialization: Build Long Path



Note We may assume that all the neighbors of the end (red) vertices are on the path; otherwise we get a longer path. This implies $t > 1 + n/2$.

A Two-Phase Algorithm

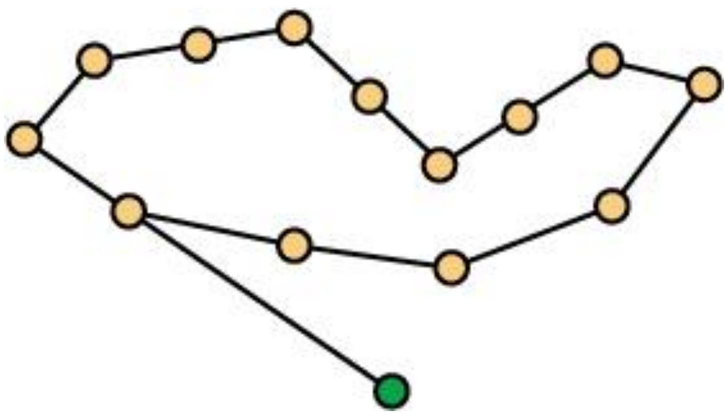
Phase 1 – Path of size t produces cycle of size t



Note Using the pigeon-hole principle, there are consecutive vertices i and $i+1$ on the path with $\{1, i+1\}$ and $\{i, t\}$ as edges in G.

A Two-Phase Algorithm (2)

Phase 2 – Cycle of size t produces path of size t+1



Note Since the size of the cycle is at least n/2, any vertex not on the cycle has a neighbor on the cycle. Therefore, if the cycle has size t, we get a path of size t + 1.