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9 - Graph Theory Advanced Topics

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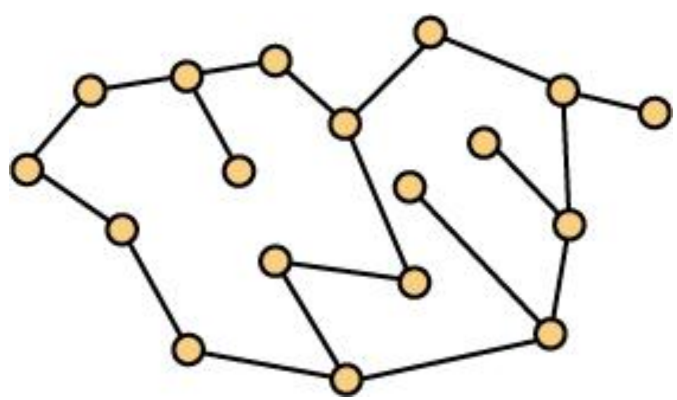
The Girth of a Graph

Definition A graph containing no cycles is called a forest. In a forest, every component is a tree. So a tree is a forest. We say that the girth of a forest is infinite.

Definition When G is not a forest, we define the girth of G as the size of the smallest cycle in G. The graph shown below has girth 8.



Chromatic Number and Girth

Observation The three constructions studied earlier produce triangle-free graphs with large chromatic number produce graphs with small girth. Although the proof is a bit beyond our scope in this course, here is a historically very important result in applications of probability to combinatorics.

Theorem (Erdős, '59) For every pair (g, t) of positive integers with $g, t \geq 3$, there is a graph G with girth g and chromatic number t.

On-Line Coloring – A Two Person Game

Builder constructs a graph one vertex at a time.

Assigner colors the graph in an on-line manner.

Fact Even in the class of forests, Builder can force n colors on a graph with 2^{n-1} vertices.

Explanation Let S_n be the Builder's strategy for forcing n colors. Then S_{n+1} can be viewed as adding one new vertex to the disjoint application of S_1, S_2, S_3, …, S_n and then adding one new vertex.

On-Line Coloring for Interval Graphs

Theorem (Kierstead and Trotter, '82) In the class of interval graphs, there is a strategy for Assigner that will enable her to color an interval graph with $3k - 2$ colors provided Builder keeps the maximum clique size at most k. Builder does not need to know the value of k in advance. Furthermore, this bound is best possible, since there is a strategy for Builder that will force assigner to use at least $3k - 2$ colors, regardless of the strategy used in assigning colors.

Game Coloring for Graphs

Definition The game chromatic number of a graph is the least positive integer t for which there is a strategy for Alice that will enable her, working in "cooperation" with Bob, to color the graph using t colors and alternating turns.

Note The issue as to who goes first can be important.

Theorem (Kierstead and Trotter, '94) The game chromatic number of a planar graph is at most 33.

Two Challenging Exercises

Observation The chromatic number of a tree is two if it has an edge. However, the game chromatic number of a tree is at most 4 and this result is best possible. This is a good exercise for a senior level undergraduate course in graph theory.

Follow-Up Note Kierstead and Zhu have been carrying on a running competition for 20 years, and it is now known that the game chromatic number of a planar graph is at most 17 with Zhu in the winning position for now. From below, a lower bound of 7 is known. If you really want to get an A+++, move either bound.

List Colorings of Graphs

Definition The list chromatic number of a graph is the smallest integer t so that a proper coloring of the graph can always be found using colors from prescribed lists of size t, one list for each vertex. Note that different vertices can have different lists.

Example When $n = C(2t-1, t)$, the complete bipartite graph $K_{n,\,n}$ has list chromatic number $t + 1$.

Theorem (Thomasen, 1994) The list chromatic number of a planar graph is at most 5.