1 Answers to Chapter 5, Odd-numbered Exercises

1) **(a)** 2.

   (b) 4.

   (c) There are five vertices of degree two. They are $1, 4, 6, 8, 9$.

   (d) One cycle of length 8 is $1, 5, 6, 2, 3, 10, 4, 7, 1$. There are others.

   (e) The length of the shortest path is two: $3, 10, 4$. It cannot be length one since $\{3, 4\}$ is not an edge.

   (f) The length of the shortest path is three. One such path is $8, 3, 2, 7$. It cannot be length one since 8 is not adjacent to 7. It cannot be length two since no vertex in the neighborhood of 8 is adjacent to 7.

   (g) Such a path is given by $4, 7, 1, 5, 2, 6$.

3) Such a graph cannot exist since by Corollary 5.2, it must have an even number of odd degree vertices. This graph would have three odd degree vertices which is a contradiction.

5) **(a)** Yes.

   (b) No. $\{c, j\}$ is not an edge in G.

   (c) No. It must have edge $\{h, d\}$ to be an induced subgraph of G.

7) G_1 and G_3 are isomorphic. One isomorphism is given by the mapping $\{(v_3, w_4), (v_2, w_3), (v_4, w_2), (v_1, w_1), ($ G_1 and G_2 are not isomorphic since G_1 contains a cycle of length 3 and G_2 does not. G_4 is not isomorphic to any of the other graphs because it contains a vertex of degree 1.

9) It is Eulerian. The following walk is an Eulerian cycle:

$$a, b, l, d, h, m, g, n, m, i, d, j, m, c, i, f, e, a, k, f, c, j, l, a.$$

   It is not Hamiltonian. This is because if it were, it would have to include edges $\{m, g\}$, $\{g, n\}$, and $\{n, m\}$. This would create a subcycle of length 3 and Hamilton cycles cannot have subcycles.

11) Suppose G has an Eulerian trail. If G also has an Eulerian circuit, then all the vertices in G are of even degree, and so at most two are odd. If it does not also have an Eulerian circuit, we do the following. Take the Eulerian trail and add an edge (possibly parallel) between the endpoints. This new graph now has an Eulerian circuit and hence every vertex in the new graph has even degree. Removing the added edge, we see that we only have two odd degree vertices in G.

    If a graph is connected and has at most two vertices of odd degree, we consider two cases. If there are no odd degree vertices, then clearly it has an Eulerian circuit, and hence an Eulerian trail. It cannot have only one odd degree vertex because of Corollary 5.2. If it has two odd degree vertices, again, add an edge between them (possibly parallel), and the graph has an Eulerian circuit. Removing this added edge creates an Eulerian trail with endpoints which are the two odd degree vertices.

13) $\chi(G) = 2$. Greedily color vertices red and blue to obtain a two coloring.

15) Let G be the graph with vertex set labeled $1, 2, \dots, 10$ by the chemicals and edge set E where $\{i, j\} \in E$ if and only if the (i, j)th entry in the matrix is 1. The smallest number of rooms to store the chemicals is the chromatic number of this graph. Since 1, 2, 4, and 5 are all adjacent to each other, they must all receive different colors. Hence, the chromatic number of this graph is at least 4. It is easy to find a 4-coloring of this graph, and hence we need at least 4 rooms in the warehouse.

17) If T is a tree on at least 2 vertices, then its chromatic number is two. It is at least 2 since it has two vertices and it is connected. To find a 2-coloring of any tree, root it at an arbitrary vertex and color each level of the tree by alternating colors. Alternatively, one can argue that since trees do not contain cycles, they cannot contain odd cycles. Hence, trees are 2-colorable.

19) The graph G_4 has

$$13 + 5 \cdot \binom{13}{5}$$

vertices. This is because we have an independent I set of size 13 and for every 5-element subset of I – of which there are $\binom{13}{5}$ – we create a copy of $G_3 = C_5$.

21) Let n_t be the number of vertices in the graph G_t from the Kelly and Kelly proof. We have that $n_3 = 5$. In general,

$$n_{t+1} = t(n_t - 1) + 1 + \binom{t(n_t - 1) + 1}{n_t}.$$

23) The girth of G_t is at least 4 since, as seen in the proof of Proposition 5.9, G_t does not contain a triangle. G_t is at most 5 since $G_3 = C_5$, and $G_3 \subseteq G_t$ for all t. We claim that G_t does not contain a 4-cycle. G_3 does not contain a 4-cycle. Suppose that G_t does not contain a 4-cycle for all $t < k$ for some $k > 3$. For a contradiction, assume that G_k contains a 4-cycle. By the induction hypothesis, it cannot be contained in G_{k-1}. Hence, it must use at least one vertex, say v, from the independent set I. Since v is only connected to one vertex in each copy of G_{k-1}, the 4-cycle must have one edge going from v to a vertex u_1 in one copy of G_{k-1} and another edge going from v to a vertex u_2 in a different copy of G_{k-1}. Now, the only neighbors of u_1 are v and other vertices in that particular copy of G_{k-1}. Hence, it cannot be connected to a neighbor of u_2. Therefore, there is no 4-cycle in G_k, and the girth is 5.

25) .

27) **(a)** Order the vertices of G from left to right as $v_1, \dots, v_n$. In step 1, color v_1 with color 1. In step i, color vertex v_i with color t where t is the smallest color not used to color an adjacent vertex to the left of v_i. By induction, we prove that we have used at most $D + 1$ colors at any step in the algorithm. At step 1, this is of course true. Suppose that it is true for all steps $t < k$ for some $k > 1$. At step k, we color v_k. Since v_k has degree at most D, there are only D colors that it cannot be. Hence, there is a color available to color v_k, proving the claim.

(b) Let G be a graph with 1000 vertices on the left and 1000 vertices on the right. Connect each vertex on the left to every vertex on the right. Every vertex has degree 1000, but the graph is clearly bipartite.

29) .

31) .

33)
- c) cannot be a graph since it has an odd number of odd degree vertices, contradicting Corollary 5.2.
- a) and e) could be planar graphs. The other graphs have too many edges to satisfy Theorem 5.12.
- a) could be the degree sequence of a tree. It could not be any others because trees have at least two leaves (degree 1 vertices).
- b) is the degree sequence of an eulerian graph because all vertices have even degree and it is connected since a graph on 9 vertices with a degree 8 vertex must be connected.
- d) is the degree sequence of a graph that must be hamiltonian since it is a graph on 10 vertices with minimum degree 5 (Theorem 5.5).

35) a) could be Hamiltonian. Below are two graphs with the degree sequence shown in a) but one is Hamiltonian and the other is not. a) could be planar. Below are two graphs with the degree sequence shown in a) but one is planar and the other is not.

37) $prufer(T) = 1, 4, 6, 9, 4, 9, 1, 4$

39) $prufer(T) = 9, 3, 9, 5, 9, 4, 5, 14, 1, 6, 5, 1$