1 Answers to Chapter 7, Odd-numbered Exercises

1) 11 students do not like any of the flavors. To see this, draw a Venn diagram with a circle for each flavor, and fill in the numbers in each area starting from the center (18 for all three flavors, then 7 for chocolate and strawberry since it adds to 25, and so on.)

3) 40. There are 50 positive integers less than or equal to 100 that are divisible by 2. They are $\{2 \cdot k : k = 1, \ldots, 50\}$. There are 20 positive integers less than or equal to 100 that are divisible by 5. They are $\{5 \cdot k : k = 1, \ldots, 20\}$. Let A be the first set and B be the second set. If we can determine $A \cup B$, then the number of positive integers less than or equal to 100 that are not divisible by 2 or 5 will be $100 - |A \cup B|$.

   By inclusion-exclusion, $|A \cup B| = |A| + |B| - |A \cap B|$. $A \cap B$ is the set of positive integers less than or equal to 100 that are divisible by 2 and 5 – ie. divisible by 10. Therefore,

$$|A \cup B| = 50 + 20 - 10 = 60.$$

   So the number divisible by neither 2 or 5 is $100 - 60 = 40$.

5) 560. Let A, B, C be the set of positive integers less than or equal to 1000 that are divisible by 3, 8, and 25, respectively. So,

$$\begin{aligned} A &= \{3 \cdot k : k = 1, \ldots, 333\}, \\ B &= \{8 \cdot k : k = 1, \ldots, 125\}, \\ C &= \{25 \cdot k : k = 1, \ldots, 40\}. \end{aligned}$$

   We want to find $1000 - |A \cup B \cup C|$. By inclusion-exclusion,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

   Observe that $A \cap B$, $A \cap C$, $B \cap C$, $A \cap B \cap C$ are the set of positive integers less than or equal to 1000 that are divisible by $\text{lcm}\{3, 8\} = 24$, $\text{lcm}\{3, 25\} = 75$, $\text{lcm}\{8, 25\} = 200$, $\text{lcm}\{3, 8, 25\} = 600$, respectively. They have sizes 41, 13, 5, 1, respectively. Hence,

$$|A \cup B \cup C| = 333 + 125 + 40 - 41 - 13 - 5 + 1 = 440.$$

   So the number divisible by neither 3, 8, or 25 is $1000 - 440 = 560$.

7) The number of solutions to $x_1 + x_2 + x_3 + x_4 = 32$ without restrictions (ie. $x_i \geq 0$) is $\binom{35}{3}$. Let A_i denote the set of solutions to $x_1 + x_2 + x_3 + x_4 = 32$ with $x_i > 10$. The number of solutions with $0 \leq x_i \leq 10$ will be

$$\binom{35}{3} - |A_1 \cup A_2 \cup A_3 \cup A_4|.$$

   Hence, we use inclusion-exclusion to determine the size of the union of the A_i.

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum_{i=1}^{4} |A_i| - \sum_{1 \leq i < j \leq 4} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq 4} |A_i \cap A_j \cap A_k| - |\bigcap_{i=1}^{4} A_i|$$

By the counting methods introduced in Chapter 2 (Section 2.5), one can see that

$$|A_i| = \binom{24}{3}$$
$$|A_i \cap A_j| = \binom{13}{3}$$
$$|A_i \cap A_j \cap A_k| = 0$$
$$|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$$

Hence,

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 4 \cdot \binom{24}{3} - 6\binom{13}{3}$$

and so the number of solutions that we are looking for is

$$\binom{35}{3} - \left[4 \cdot \binom{24}{3} - 6\binom{13}{3}\right].$$

9) Yes, he ate alone 1 time. Label his friends $1, 2, 3, 4, 5, 6$, and let A_i be the set of weeks that he ate lunch with friend i. Since there are 15 weeks in the semester, the number of times he ate alone is

$$15 - \left|\bigcup_{i=1}^{6} A_i\right|.$$

By inclusion-exclusion, the size of the union is

$$|\bigcup_{i=1}^{6} A_i| = \sum_{i=1}^{6} |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \ldots - |A_1 \cap A_2 \cap \ldots \cap A_6|.$$

The problem statement gives us the size of each of these intersections, so

$$|\bigcup_{i=1}^{6} A_i| = 6 \cdot 11 - \binom{6}{2} \cdot 9 + \binom{6}{3} \cdot 6 - \binom{6}{4} \cdot 4 + \binom{6}{5} \cdot 4 - 1 = 14.$$

So, he ate alone $15 - 14 = 1$ time.

11) **(a)**. No, because $f(2) = 2$ (and also $f(6) = 2$). Yes, it satisfies P_3. It also satisfies P_5 and P_7.

(b). Yes. Let $g(x) = x$ for $1 \leq x \leq 7$ and $g(8) = 1$. Then g satisfies no P_i for $i \leq 7$.

(c). No, it is not possible because since $9 > 8$, there will always be an element in $\{1, 2, \ldots, 9\}$ that does not get mapped to by h.

13) **(a)**. Yes, because $12 \in [15]$ and 3 divides 12. It does not satisfy P_5 since 5 does not divide 12. It also satisfies P_1, P_2, P_4, P_6, P_12.

(b). 2 satisfies only P_1 and P_2. Any other prime will also work.

(c). 6 satisfies P_1, P_2, P_3, P_6.

(d). 4 satisfies P_1, P_2, P_4.

15) $S(10,4)$. This is equal to the number of surjections from a set of 10 elements (the books) to a set of 4 elements (John, Paul, Ringo, and George). Since we are counting surjections, this means that each of them will get at least one book.

17) $S(11,5)+S(11,6)$. Label the students with elements from [6], and label the topics with labels from [12]. Let Katie be labeled 6, and let the hardest topic be labeled 12. The number of ways that the professor can assign topics is the number of mappings from [12] to [6] such that the map is a surjection and 12 gets mapped to 6. If Katie does not receive any extra topics, then this is equal to the number of surjections from [11] to [5] which is $S(11,5)$. If Katie receives some extra topics, then this is equal to the number of surjections from [11] to [6] which is $S(11,6)$. Since these cases are disjoint, their sum gives us the total number of mappings.

19) The number of derangements of a 9 element set are, by Theorem 7.10, equal to

$$d_9 = \sum_{k=0}^{9} (-1)^k \binom{9}{k} (9-k)!.$$

We calculate that this is equal to 133496.

21) $d_4 \cdot \binom{7}{3}$. Label each employee with an element in [7]. There are $\binom{7}{3}$ ways to choose which employees receive the correct checks. There are d_4 ways to arrange the remaining 4 checks so that no other employee receives their correct check.

23) $\phi(18) = 6$. The elements relatively prime to 18 are $\{1,5,7,11,13,17\}$. By Theorem 7.13,

$$\phi(18) = 18 \cdot \left(\frac{2-1}{2} \cdot \frac{3-1}{3}\right) = 18 \cdot \frac{1}{3} = 6.$$

25) By Theorem 7.13,

$$\begin{aligned}\phi(1625190883965792) &= 1625190883965792 \cdot \left(\frac{2-1}{2} \cdot \frac{3-1}{3} \cdot \frac{11-1}{11} \cdot \frac{13-1}{13} \cdot \frac{23-1}{23} \cdot \frac{181-1}{181}\right) \\ &= 432431285299200\end{aligned}$$

27) .