1 Answers to Chapter 9, Odd-numbered Exercises

1) **(a)** $(A^2 - A - 2)f = 0$.

   (c) $(A^3 - 5A + 1)f = 3^n$.

   (e) $(A^5 - 4A^4 - A^2 + 3)f = (-1)^n$.

3) $g_n = c_1 2^n + c_2$. We write the recurrence as the advancement operator equation

$$(A^2 - 3A + 2)f = 0.$$

This can be rewritten as

$$(A - 2)(A - 1)f = 0.$$

Hence, our solutions are $c_1 2^n$ and $c_2 1^n = c_2$. Combining these to generate the entire family of solutions gives us the general solution $g_n = c_1 2^n + c_2$.

5)

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

We solve the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 1, f_1 = 1$ (note that, for convenience, we start our recurrence at f_0 instead of f_1). The recurrence can be written as an advancement operator equation

$$(A^2 - A - 1)f = 0.$$

The roots of the above polynomial are

$$\frac{1+\sqrt{5}}{2} \text{ and } \frac{1-\sqrt{5}}{2}.$$

Hence, our family of solutions is given by

$$c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Since $f_0 = 1$ and $f_1 = 1$, we get two linear equations in two variables:

$$c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = 1$$
$$c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

Solving this yields $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$.

7) $(-6/7)(-5)^n + (20/7)2^n$. We factor

$$A^3 + 3A - 10 = (A + 5)(A - 2)$$

to deduce that $c_1(-5)^n$ and $c_2 2^n$ are solutions to the advancement operator equation. Using the initial conditions $f(0) = 2$ and $f(1) = 10$, we get two linear equations in two variables

$$\begin{aligned} c_1 + c_2 &= 2 \\ -5c_1 + 2c_2 &= 10 \end{aligned}$$

Solving this yields $c_1 = -6/7$, $c_2 = 20/7$.

9) **(a)** $c_1 5^n + c_2(-2)^n - \frac{1}{10}3^n$. We try $f_0(n) = d3^n$. By multiplying out the advancement operator equation, we see that $d3^n$ must satisfy

$$a_{n+2} - 3a_{n+1} - 10a_n = 3^n.$$

Plugging it in yields

$$d3^{n+2} - 3d3^{n+1} - 10d3^n = 3^n.$$

So, $d = \frac{-1}{10}$.

(c) $c_1 3^n + c_2 n 3^n + c_3 n^3 3^n + (-3/8)n + (-11/16)$. We try $f_0(n) = an + b$. By multiplying out the advancement operator equation, we see that $an + b$ must satisfy

$$a_{n+3} - 9a_{n+2} + 27a_{n+1} - 27a_n = 3n + 1.$$

Plugging it in yields

$$a(n+3) + b - 9(a(n+2) + b) + 27(a(n+1) + b) - 27(an + b) = 3n + 1.$$

From this, we see that

$$-8an = 3n \text{ and } 12a - 9b = 1.$$

So $a = -3/8$ and $b = -11/16$.

(e) $c_1 2^n + c_2 4^n + \frac{1}{3}n^2 + \frac{-2}{27}n + \frac{-7}{243} + \frac{1}{25}9^n$. We try out $f_0(n) = an^2 + bn + c + d9^n$. By multiplying our the advancement operator equation, we see that $an^2 + bn + c + d9^n$ must satisfy

$$a_{n+2} - 6a_{n+1} + 8a_n = 3n^2 + 9^n.$$

Plugging it in yields

$$a(n+2)^2 + b(n+2) + c + 81d9^n - 6(a(n+1)^2 + b(n+1) + c + 9d9^n + 8(an^2 + bn + c + d9^n) = 3n^2 + 9^n.$$

From this, one can solve to find

$$a = \frac{1}{3}; b = \frac{-2}{27}; c = \frac{-7}{243}; d = \frac{1}{25}.$$

(g) $c_1 3^n + c_2 n 3^n + c_3(-1)^n + \frac{1}{36}n^2 3^n$. Since 3 is a repeated root to the advancement operator equation, $c_1 3^n$ and $c_2 n 3^n$ are distinct solutions to the recurrence relation. Hence, we try $dn^3 3^n$ to get a nonhomogeneous solution to the recurrence. By multiplying/expanding our advancement operator equation, we see that $dn^2 3^n$ must satisfy

$$a_{n+3} - 5a_{n+2} + 3a_{n+1} + 9a_n = 2 \cdot 3^n.$$

Plugging it in yields

$$27d(n+3)^2 - 45d(n+2)^2 + 9d(n+1)^2 + 9dn^2 = 2.$$

This reduces to

$$72d = 2$$

and so $d = 1/36$.

(i) $d_1 2^n + d_2 n 2^n + d_3 - \frac{1}{8}n^2 \cdot 2^n$. Since the nonhomogenous part of the equation is $(3n^2+1)2^n$, we want to try $(an^2 + bn + c)2^n$. However, since 2 is a repeated root, we will not generate distinct solutions unless we multiply by n^2 (if 2 was a single root, we would multiply only by n, and if it was not a root at all, we would not have to multiply by any extra factors). Hence, we try $f_0(n) = (an^4 + bn^3 + cn^2)2^n$. By multiplying/expanding our advancement operator equation, we see that $f_0(n)$ must satisfy

$$a_{n+3} - 5a_{n+2} + 8a_{n+1} - 4a_n = 2n^2 2^n + 2^n.$$

Plugging $f_0(n)$ in for a_n yields

$$\begin{aligned} &8(a(n+3)^4 + b(n+3)^3 + c(n+3)^2) - 20(a(n+2)^4 + b(n+2)^3 + c(n+2)^2) + \\ &+ 16(a(n+1)^2 + b(n+1) + c) - 4(an^4 + bn^3 + cn^2) = 2n^2 + 1. \end{aligned}$$

Solving this yields $a = 0, b = 0, c = -1/8$.

11) $t_n = 2^n - 1$. First, observe that t_n satisfies the following recurrence relation $t_n = 2t_{n-1} + 1$ with $t_1 = 1$. To see this, in order to move the largest disc to the rightmost peg, the only way to do this is by first moving all of the other pegs onto the center peg first. This is done in t_{n-1} steps. Then, we do 1 step by moving the largest disc to the rightmost peg, and now we must again perform t_{n-1} steps to move all the other discs on top of the rightmost peg again.

Now, to determine a formula for t_n, we solve the recurrence relation by observing that it can be modeled by the advancement operator equation

$$(A-2) = 1.$$

So, $t_n = c2^n + d$. We first solve for d (the nonhomogeneous part of the solution), then we use our initial condition to finally solve for c.

We must have that d satisfies the recurrence, and so $d = 2d + 1$ implying $d = -1$. Now, $1 = t_1 = c2^1 + -1$, implying $c = 1$.

13) $t_n = (-1)^n - \frac{1}{\sqrt{3}}(1-\sqrt{3}) + \frac{1}{\sqrt{3}}(1+\sqrt{3})$. If we are to tile a $2 \times n$ rectangle (2 rows, n columns), there is 1 way to do this for $n = 1$ (we must use 1×1 squares), and there are 5 ways to do this for $n = 2$ (the four rotations of the L-tile, along with the configuration to use all 1×1 tiles). For $n = 3$, there are 11 ways to tile it (just draw out all 11 cases). So $t_1 = 1$, $t_2 = 5$, $t_3 = 11$. We now analyze t_n.

Consider the upper rightmost square of the grid. *Case 1:* If it is filled by a 1×1 square, then there are two possibilities. Either there is a 1×1 square below it, or there is an L-tile around it. There are t_{n-1} ways to tile the former, and t_{n-2} ways to tile the latter. *Case 2:* If it is an L-tile, then there are three orientations the L-tile could be in. If the orientation is such that there is a 1×1 square in the bottom left square, then there are t_{n-2} ways to tile the rest of the grid. Otherwise, there are $t_{n-2} + t_{n-3}$ ways to tile the grid. Summing these distinct cases up results in

$$t_n = t_{n-1} + t_{n-2} + t_{n-2} + 2(t_{n-2} + t_{n-3}) = t_{n-1} + 4t_{n-2} + 2t_{n-3}.$$

The advancement operator polynomial is

$$A^3 - A^2 - 4A - 2 = 0$$

which can be factored as

$$(A+1)(A-(1-\sqrt{3}))(A-(1+\sqrt{3})) = 0.$$

Hence,

$$t_n = a(-1)^n + b(1-\sqrt{3})^n + c(1+\sqrt{3})^n.$$

Using the initial conditions $t_1 = 1, t_2 = 5, t_3 = 11$, we get the system of equations

$$\begin{aligned} -a + b(1-\sqrt{3}) + c(1+\sqrt{3}) &= 1 \\ a + b(1-\sqrt{3})^2 + c(1+\sqrt{3})^2 &= 5 \\ -a + b(1-\sqrt{3})^3 + c(1+\sqrt{3})^3 &= 11 \end{aligned}$$

Solving this (preferably using a computer algebra system or Wolfram Alpha) results in $a = 1, b = -1/\sqrt{3}, c = 1/\sqrt{3}$.

15) $r_n = 3/5(-2)^n + 2/53^n$. Let $r(x)$ be the generating function for the sequence r_n. So,

$$\begin{aligned} r(x) &= \sum_{n=0}^{\infty} r_n x^n \\ -x \cdot r(x) &= \sum_{n=0}^{\infty} -4r_n x^{n+1} = \sum_{n=1}^{\infty} -4r_{n-1}x^n \\ -6x^2 \cdot r(x) &= \sum_{n=0} -6r_n x^{n+2} = \sum_{n=2}^{\infty} -6x^2 r_{n-2} x^n \end{aligned}$$

Summing these together results in

$$r(x)(1 - x - 6x^2) = 1 + 3x - 4x + \sum_{n=2}^{\infty}(r_n - 4r_{n-1} - 6r_{n-2})x^n = 1 - x$$

So,

$$r(x) = \frac{1-x}{1-x-6x^2} = \frac{1-x}{(2x+1)(-3x+1}.$$

A partial fraction expansion gives

$$r(x) = \frac{\frac{3}{5}}{1+2x} + \frac{\frac{2}{5}}{1-3x} = \frac{3}{5}\sum_{n=0}^{\infty}(-2)^n x^n + \frac{2}{5}\sum_{n=0}^{\infty} 3^n x^n.$$

So the coefficent of x^n is $(3/5)(-2)^n + (2/5)3^n$.

17) Let $b(x)$ be the generating function for the sequence b_n. So,

$$
\begin{aligned}
b(x) &= \sum_{n=0}^{\infty} b_n x^n \\
-4xb(x) &= \sum_{n=0}^{\infty} -4b_n x^{n+1} = \sum_{n=1}^{\infty} -4b_{n-1}x^n \\
x^2 b(x) &= \sum_{n=0}^{\infty} b_n x^{n+2} = \sum_{n=2}^{\infty} b_{n-2}x^n \\
6x^3 b(x) &= \sum_{n=0}^{\infty} 6b_n x^{n+3} = \sum_{n=3}^{\infty} 6b_{n-3}x^n
\end{aligned}
$$

Now, observe that if $b_0 = 1, b_2 = 1, b_3 = 4$ and, by the recurrence relation, $b_3 = 4b_2 - b_1 - 6b_0 + 3^0$, we get that $b_1 = -5$. Summing the right hand sides in the above equations as well as the left hand sides results in

$$r(x)(1 - 4x + x^2 + 6x^3) = 1 - 5x + x^2 - 4x + 20x^2 + x^2 + \sum_{n=3}^{\infty} 3^{n-3}x^n = 1 - 9x + 22x^2 + \frac{\frac{1}{27}}{1-3x}.$$

We can now solve for $r(x)$ and do a partial fraction expansion:

$$
\begin{aligned}
r(x) &= \frac{1 - 9x + 22x^2}{1 - 4x + x^2 + 6x^3} + \frac{\frac{1}{27}}{(1-3x)(1 - 4x + x^2 + 6x^3)} \\
&= -\frac{\frac{8}{3}}{1-2x} + \frac{1}{1-3x} + \frac{\frac{8}{3}}{1+x} \\
&+ \frac{\frac{8}{81}}{1-2x} - \frac{\frac{7}{48}}{1-3x} + \frac{\frac{1}{12}}{(1-3x)^2} + \frac{\frac{1}{1296}}{1+x} \\
&= \frac{\frac{-208}{81}}{1-2x} + \frac{\frac{41}{48}}{1-3x} + \frac{\frac{1}{12}}{(1-3x)^2} + \frac{\frac{3457}{1296}}{1+x}
\end{aligned}
$$

Expanding these as infinite sums gets us that r_n is equal to

$$r_n = \frac{-208}{81} \cdot 2^n + \frac{41}{48} \cdot 3^n + \frac{1}{12}(n+1) \cdot 3^n + \frac{3457}{1296} \cdot (-1)^n.$$