FINAL EXAMAM SOLUTIONS

Student Name and ID Number

MATH 3012 Final Exam, December 15, 2010, WTT

Note. The answers given here are more complete and detailed than students are expected to provide when taking a test. The extra information given here should help current students to understand the questions being asked on the test. At the same time, it is intended to help future students by supplementing explanations given in the text.

1. Consider the 9-element set X consisting of the five letters $\{a, b, c, d, e\}$ and the four digits $\{0, 1, 2, 3\}$.

a. How many strings of length 7 can be formed if repetition of symbols is permitted?

> This is just asking for the number of strings of length 7 that can be formed from an alphabet of size 9. Answer: 9^7.

b. How many strings of length 7 can be formed if repetition of symbols is *not* permitted?

> This is the number of permutations of length 7 that can be formed from a set of 9 objects. So the answer is $P(9,7) = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$.

c. How many strings of length 7 can be formed using exactly two 3's, three a's and two c's?

> This is the classic "Mississippi" problem. Answer is the multinomial coefficient:

$$\binom{7}{2,3,2} = \frac{7!}{2!\,3!\,2!}$$

d. How many strings of length 7 can be formed if exactly three characters are digits and exactly two of the remaining characters are c's? Here, repetition is allowed.

> Choose the three positions for the digits. Of the remaining four positions, choose two for the c's. The two positions that are not c's can be any of the remaining four letters. Answer: $\binom{7}{3}4^3\binom{4}{2}4^2$.

e. How many symmetric binary relations are there on X?

> For each $x \in X$, we choose whether (x, x) belongs to the relation or not. Each of the two choices is allowed. This gives a factor of 2^9. For each of the $\binom{9}{2}$ pairs $\{x, y\}$ from X with $x \neq y$, we either put both of (x, y) and (y, x) in the relation, or we put neither. This gives another $2^{\binom{9}{2}}$ choices. Answer: $2^9\, 2^{\binom{9}{2}}$.

f. How many equivalence relations are there on X with class sizes 3, 3, 2 and 1?

> This is another multinomial coefficient problem with the extra complication that we can permute the classes. So the answer is:

$$\frac{7!}{3!\,3!\,2!\,1!\,2!\,1!\,1!}$$

> The problem is so small that the form of the correct answer may not be clear. Perhaps a better problem would be to ask how many equivalence relations there are on a 59 element set if the class sizes are $7, 7, 7, 7, 7, 4, 4, 4, 2, 2, 2, 2, 2, 2$. Now the answer is:

$$\frac{59!}{7!\,7!\,7!\,7!\,7!\,4!\,4!\,4!\,2!\,2!\,2!\,2!\,2!\,2!\,5!\,3!\,6!}$$

2. Bob has a job with the Math department at a university (of sorts) some 60 miles from Atlanta. Bob is responsible for paperclip inventory, i.e., counting the department's paperclips and storing them for safe-keeping. Being thoroughly conscientious in his assignment, Bob determines that they have exactly $2,835$ paperclips on hand. Now Bob will distribute these paperclips among three Storage Rooms. Room 1 is in the math building, Room 2 is in the central administration building, and Room 3 is located underneath the bleachers in the football stadium. In other words, Bob will choose non-negative integers x_1, x_2 and x_3 with $x_1 + x_2 + x_3 = 2,835$, and then store x_i paperclips in Room i, for $i = 1, 2, 3$. Count the number of ways Bob can store the paperclips, subject to the following restrictions:

a. $x_i \geq 0$ for $i = 1, 2, 3$ (i.e., no restrictions).

For each room, we add an artificial paperclip. The new total is 2838. Now choose two gaps from the $2837 = 2838 - 1$ gaps they form when lined up (something that Bob would love to do). This separates the list of paperclips into three group, each containing at least one paperclip. Then subtract one from each group. This results in the non-negative number going to each of the three rooms. Note, for suspect reasons, this question came before the next one, which is really the starting point for this type of problem. Answer: $\binom{2837}{2}$.

b. $x_i > 0$ for $i = 1, 2, 3$.

As mentioned above, this question should really be answered first. Bob lines up the 2835 paperclips. There are 2834 gaps. He chooses two gaps, and the paperclips are partitioned left to right into three groups. Those in the first group (say the leftmost) are put in Room 1. Those in the next group (say the middle group) are placed in Room 2, and those in the third group are placed in Room 3. Answer: $\binom{2834}{2}$.

c. $x_i > 0$ for $i = 1, 2, 3$ and $x_3 > 700$.

Set aside 700 paperclips in advance for Room 3. There are $2135 = 2835 - 700$ remaining. Choose two gaps among the 2134. Answer: $\binom{2134}{2}$. Note that because Room 3 will get at least one more, the total for Room 3 will be strictly larger than 700.

d. $x_i > 0$ for $i = 1, 2, 3$ and $x_3 < 700$.

Following the same reasoning as in part c, there are $\binom{2135}{2}$ ways resulting in $x_3 > 699$, which is the same as requiring $x_3 \geq 700$. So we subtract these from the answer to part b to find the number with $x_3 < 700$. Answer: $\binom{2834}{2} - \binom{2134}{2}$.

3. Use the Euclidean algorithm to find

a. $d = \gcd(7735, 1638)$.

Using long division, we make the following calculations:

$$\begin{aligned}
7735 &= 4 \cdot 1638 + 1183 \\
1638 &= 1183 + 455 \\
1183 &= 2 \cdot 455 + 273 \\
455 &= 273 + 182 \\
273 &= 182 + 91 \\
182 &= 2 \cdot 91 + 0
\end{aligned}$$

This shows that d, the last positive remainder, is $91 = 7 \cdot 13$.

b. Use your work in the first part of this problem to find integers a and b so that $d = 7735a + 1638b$.

We begin by rewriting the top five statements above as:

$$\begin{aligned}
91 &= 273 - 182 \\
182 &= 455 - 273 \\
273 &= 1183 - 2 \cdot 455 \\
455 &= 1638 - 1183 \\
1183 &= 7735 - 4 \cdot 1638
\end{aligned}$$

Substituting, we see that:

$$\begin{aligned}
91 &= 273 - 182 \\
&= 273 - (455 - 273) = -455 + 2 \cdot 273 \\
&= -455 + 2(1183 - 2 \cdot 455) = 2 \cdot 1183) - 5 \cdot 455 \\
&= 2 \cdot 1183 - 5(1638 - 1183) = -5 \cdot 1638 + 7 \cdot 1183 \\
&= -5 \cdot 1638 + 7(7735 - 4 \cdot 1638) = 7 \cdot 7735 - 33 \cdot 1638
\end{aligned}$$

It follows that we make take $a = 7$ and $b = -33$. *Note.* There are infinitely many choices for a and b. The ones given here are just what result from backtracking through the calculations for d.

Note further that as discussed in class, both the greatest common divisor $d = \gcd(m, n)$ as well as integers a and b so that $d = am + bn$ can be computed with a loop. No backtracking is actually required.

c. Using your previous work, factor 1638 completely into a product of primes. You will need this answer later on this test.

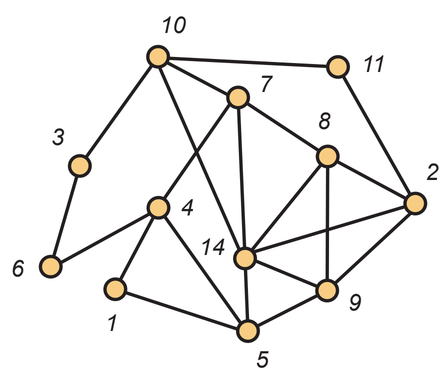
We know 1638 is divisible by 91 and just a little work shows that $1638 = 91 \cdot 18 = 2 \cdot 3^2 \cdot 7 \cdot 13$.

4. For a positive integer n, let t_n count the number of ways to tile a $3 \times n$ array with dominoes of the following three sizes: 1×3, 3×1 and 2×3. Note that dominoes of size 3×2 are not permitted. Then $t_1 = 1$, $t_2 = 1$ and $t_3 = 4$. Develop a recurrence for t_n and use it to find t_6.

We develop a recurrence for t_n when $n \geq 4$. We consider how the last column is covered. It could be done by a single 3×1 domino. That results in t_{n-1} ways to cover the first $n-1$ columns. It could happen that the last column is covered by three 1×3 dominoes. This results in t_{n-3} tilings for the other columns. It could happen that the last column is covered by a 1×3 domino and a 2×3 domino. There are two ways for this to happen—either the 1×3 domino is on the top, or it is on the bottom. In either case, we get t_{n-3} ways to cover the remaining columns. So the recurrence is $t_n = t_{n-1} + 3t_{n-3}$. And it follows that

$$\begin{aligned} t_4 &= t_3 + 3t_1 = 4 + 3 = 7 \\ t_5 &= t_4 + 3t_2 = 7 + 3 = 10 \\ t_6 &= t_5 + 3t_3 = 10 + 12 = 22 \end{aligned}$$

5. Use the algorithm developed in class to find an Euler circuit in the graph G shown below (use node 1 as root):



We start with node 1 and proceed greedily:

$$1, 4, 5, 1$$

Not all edges have been visited, so we proceed along this path to the first vertex incident with an edge we have not already visited. This is vertex 4. From there we proceed greedily to obtain:

$$4, 6, 3, 10, 7, 4$$

This loop is inserted into our original sequence, resulting in

$$1, 4, 6, 3, 10, 7, 4, 5, 1$$

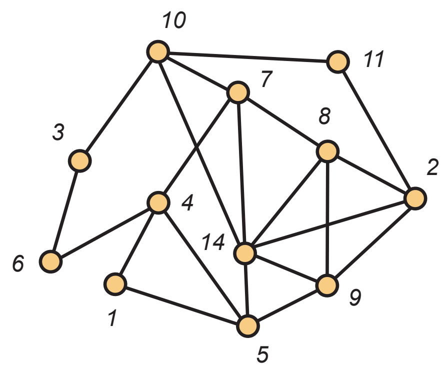
Again we find the first vertex incident with an edge not already visited. This is vertex 10. From there we proceed greedily:

$$10, 11, 2, 8, 7, 14, 2, 9, 5, 14, 9, 8, 14, 10$$

Inserting this subsequence, we have the final answer:

$$1, 4, 6, 3, 10, 11, 2, 8, 7, 14, 2, 9, 5, 14, 9, 8, 14, 10, 7, 4, 5, 1$$

6. We show the same graph G again.



a. Explain why $\{1, 4, 5\}$ is a maximal clique.

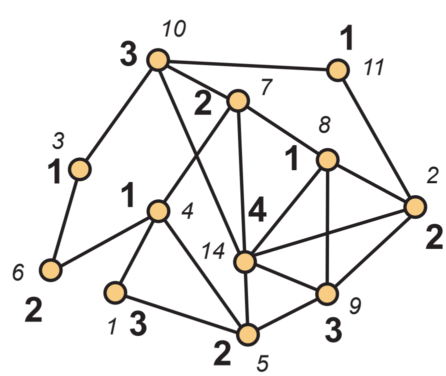
It is a clique because each pair of vertices in the set is adjacent. It is maximal since no other vertex of G is adjacent to all three of these vertices.

b. Find the maximum clique size $\omega(G)$ and find a set of vertices that form a maximum clique.

The maximum clique size (by inspection for now) is 4 and $\{2, 8, 9, 14\}$ is a maximum clique.

c. Show that $\chi(G) = \omega(G)$ by providing a proper coloring of G. You may indicate your coloring by writing directly on the figure.

We provide another drawing of the graph with the bold face numbers providing a coloring with 4 colors. Note that we have now truly established that $\omega(G) = 4$, since we have found both a clique of size 4 and a 4-coloring.



d. Despite the fact that $\chi(G) = \omega(G)$, the graph G is not perfect. Explain why.

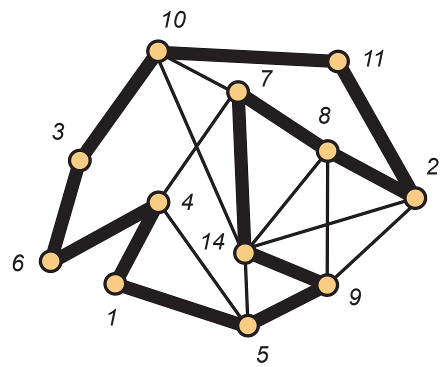
In order for a graph G to be perfect, we need $\chi(H) = \omega(H)$ for every induced subgraph H. In this case, the induced subgraph generated by the vertices $\{2, 7, 8, 10, 11\}$ is a 5-cycle which has maximum clique size 2 and chromatic number 3. So G is not perfect. Note that $\{3, 4, 6, 7, 10\}$ is also a 5-cycle.

7. Show that the graph G from the first two problems is hamiltonian by writing an appropriate listing of the vertices, starting and ending with node 1.

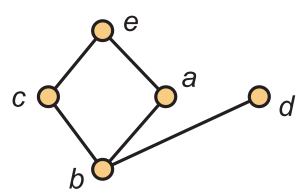
By inspection, we find the hamiltonian cycles:

$$(1, 4, 6, 3, 10, 11, 2, 8, 7, 14, 9, 5, 1) \quad \text{and} \quad (1, 4, 6, 3, 10, 11, 2, 9, 8, 7, 14, 5, 1)$$

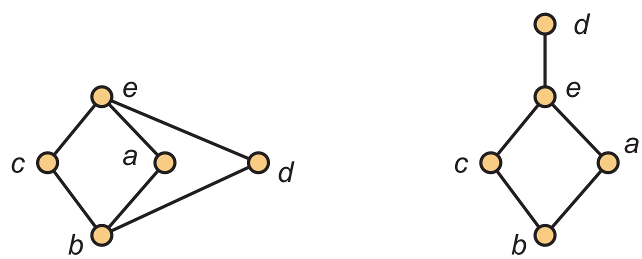
The first of these two cycles is illustrated with the darkened edges in the figure below.



8. Count the number of linear extensions of the following poset:



We consider the incomparable pair ed. Either $e > d$ or $e < d$ in a linear extension of P and both possibilities can occur. The choice $e > d$ leads to the poset P_1 shown below on the left, while $d > e$ leads to the poset P_2 on the right.



It is easy to see that P_1 has 6 linear extensions, while P_2 has 2. Therefore P has $8 = 6 + 2$ linear extensions. We note that while this divide and conquer approach *always* works, it can (and typically does) take exponentially many steps.

9. For the subset lattice $\mathbf{2}^{12}$,

a. The total number of elements is:

This is just the number of subsets of a 12-element set; equivalently, the number of bit strings of length 12. Answer: 2^{12}

b. The total number of maximal chains is:

Start with a string of length 12, all entries set to zero. Choose one of the 12 positions and toggle this entry to a 1. There are 11 remaining 0's. Choose one and toggle it to a 1. Continue in this way until you reach the top element, the string of all 1's. Answer: 12!

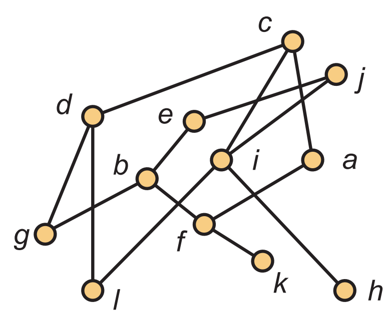
c. The number of maximal chains through $\{1, 3, 6, 7, 9\}$ is:

Now you have 5! ways to proceed from the string of all 0's to the string $(1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0)$, which is associated with this set. Similarly, you have 7! ways to continue on to the top. Answer: 5! 7!

d. The width of $\mathbf{2}^{12}$ is:

This is just a specific instance of Sperner's theorem: The width of the subset lattice $\mathbf{2}^n$ is just the size of a middle rank, i.e, the binomial coefficient $\binom{n}{\lceil n/2 \rceil}$. Answer: $\binom{12}{6}$.

10. For the poset P shown below,



a. List all elements comparable with a.

Clearly, these elements are $\{c, f, k\}$. Note that $c > a$, while $f < a$ and $k < a$.

b. List all elements covered by a.

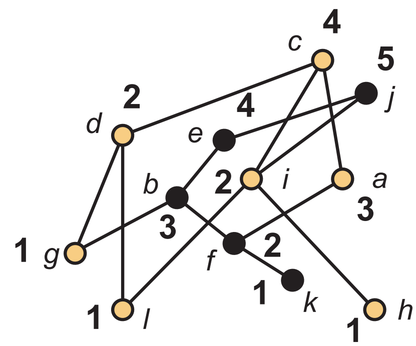
There is only one such element, i.e., the element f.

c. By inspection (not by algorithm), explain why this poset is not an interval order.

We search for a copy of $\mathbf{2} + \mathbf{2}$ contained in P. There are many. Among them are the subposets generated by: $\{e, j, a, c\}$, $\{b, e, c, a\}$, $\{b, e, c, i\}$ and $\{g, d, a, f\}$.

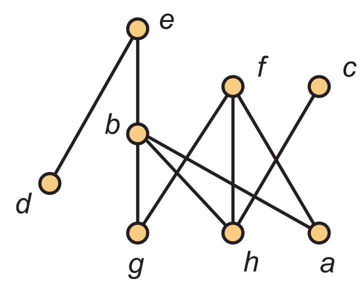
d. Find the height h and a partition into h antichains by recursively stripping off the set of minimal elements. You may display your answer by writing directly on the diagram. Then darken a set of points that form a maximum chain.

Our answer is provided in the following diagram in which the minimal elements are marked with a boldface 1. If these elements are removed, then the minimal elements among those that remain are marked boldface 2, etc.



We note that the height is 5; the darkened points, i.e., $\{b, e, f, j, k\}$ form a 5-element chain.

11. The poset P shown below is an interval order:



a. Find the down sets and the up sets. Then use these answers to find an interval representation of P that uses the least number of end points.

In the table below, we list the up sets and down sets for each of the elements in P, noting that there are four distinct down sets which can be labeled as D_1, D_2, D_3 and D_4 so that $D_1 \subsetneq D_2 \subsetneq D_3 \subsetneq D_4$.

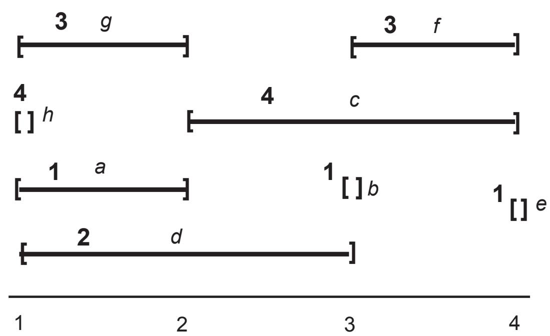
Similarly, there are four distinct up sets and we label them as U_1, U_2, U_3 and U_4 so that $U_4 \subsetneq U_3 \subsetneq U_2 \subsetneq U_1$. The bold face numbers correspond to these labelings.

Finally, in the right most column, we associate with each element of P the interval determined by the assignment of the bold face numbers in the first two columns

$D(a) = \emptyset$ **1**	$U(a) = \{b, e, f\}$ **2**	$I(a) = [1, 2]$
$D(b) = \{a, g, h\}$ **3**	$U(b) = \{e\}$ **3**	$I(b) = [3, 3]$
$D(c) = \{h\}$ **2**	$U(c) = \emptyset$ **4**	$I(c) = [2, 4]$
$D(d) = \emptyset$ **1**	$U(d) = \{e\}$ **3**	$I(d) = [1, 3]$
$D(e) = \{a, b, d, g, h\}$ **4**	$U(e) = \emptyset$ **4**	$I(e) = [4, 4]$
$D(f) = \{a, g, h\}$ **3**	$U(f) = \emptyset$ **4**	$I(f) = [3, 4]$
$D(g) = \emptyset$ **1**	$U(g) = \{b, e, f\}$ **2**	$I(g) = [1, 2]$
$D(h) = \emptyset$ **1**	$U(h) = \{b, c, e, f\}$ **1**	$I(h) = [1, 1]$

b. In the space below, draw the representation you have found. Then use the First Fit Coloring Algorithm for interval graphs to solve the Dilworth Problem for this poset, i.e., find the width w and a partition of P into w chains. You may display your answers by writing the colors directly on the intervals in the diagram.

Our work is captured in the following diagram. In this figure, we are coloring the intervals using First Fit and respecting the ordering by left endpoints. When there is a tie, we proceed alphabetically, so for example interval d was colored before g although both have the same left endpoint.



c. Find a maximum antichain in P:

From the diagram above, we see that the width of P is 4 because the highest color used is 4. There are three maximum antichains: $\{a, d, g, h\}$, $\{a, c, d, g\}$ and $\{b, c, d, f\}$. The coloring algorithm will find the first two of these using the left endpoint of intervals receiving the highest color. The third maximum antichain could be found by a linear scan looking for all the intervals containing integer i where $1 \leq i \leq 4$.

12.

a. Write all the partitions of the integer 9 into odd parts:

We proceed by inspection. There is no general way to do this.

$$\begin{aligned} 9 &= 9 \\ &= 7+1+1 \\ &= 5+3+1 \\ &= 5+1+1+1+1 \\ &= 3+3+3 \\ &= 3+3+1+1+1 \\ &= 3+1+1+1+1+1+1 \\ &= 1+1+1+1+1+1+1+1+1 \end{aligned}$$

We note that there are 8 partitions of this type.

b. Write all the partitions of the integer 9 into distinct parts:

Again, we proceed by inspection.

$$\begin{aligned} 9 &= 9 \\ &= 8+1 \\ &= 7+2 \\ &= 6+3 \\ &= 5+4 \\ &= 6+2+1 \\ &= 5+3+1 \\ &= 4+3+2 \quad \text{We note that there are 8 partitions of this type.} \end{aligned}$$

c. Use generating functions to prove that the number of partitions of an integer into odd parts equals the number of partitions into distinct parts.

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the number of partitions of the integer n into distinct parts, and let $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be the generating function for the number of partitions of n into odd parts. We show that $f(x) = g(x)$.

The starting point in discussions of partition of an integer is the following basic formula for the sum of a geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x5 + x^6 + x^7 + \cdots = \sum_{n=0}^{\infty} x^n$$

Substituting x^m for x when $m \geq 2$, we have

$$\frac{1}{1-x^m} = 1 + x^m + x^{2m} + x^{3m} + x^{4m} + x^{5m} + x^{6m} + x^{7m} + \cdots = \sum_{n=0}^{\infty} x^{nm}$$

Now consider the function $p(x)$ defined as an infinite product:

$$p(x) = \Pi_{m=1}^{\infty} \frac{1}{1-x^m}$$

Now it is easy to see that $p(x)$ is just the generating function for the number of partitions of the integer n. Similarly, the function $f(x)$ defined by

$$f(x) = \Pi_{m=1}^{\infty} \frac{1}{1-x^{2m-1}}$$

is the generating function for the number of partitions of n into odd parts.
Now let $g(x)$ be the generating function for the number of partitions of n into distinct parts. Clearly,

$$g(x) = \Pi_{m=1}^{\infty}(1+x^m)$$

Since $1 + x^m = (1 - x^{2m})/(1 - x^m)$, it follows that

$$\begin{aligned} g(x) &= \Pi_{m=1}^{\infty} \frac{1 - x^{2m}}{1 - x^m} \\ &= \frac{\Pi_{m=1}^{\infty} 1 - x^{2m}}{\Pi_{m=1}^{\infty} 1 - x^m} \\ &= \frac{\Pi_{m=1}^{\infty} 1 - x^{2m}}{\Pi_{m=1}^{\infty} 1 - x^{2m-1} \Pi_{m=1}^{\infty} 1 - x^{2m}} \\ &= \Pi_{m=1}^{\infty} \frac{1}{1 - x^{2m-1}} \\ &= f(x) \end{aligned}$$

This shows that the number of partitions of an integer n into odd parts equals the number of partitions of n into distinct parts.

13. Find the general solution to the advancement operator equation:
$(A - 2)^4 (A + 7)^2 (A - 9) f = 0$

The general solution is given by

$$f(n) = c_1 2^n + c_2 n 2^n + c_3 n^2 2^n + c_4 n^3 2^n + c_5 (-7)^n + c_6 n (-7)^n + c_7 9^n$$

14. Find the solution to the advancement operator equation:
$(A^2 - 12A + 35) f(n) = 0, \quad f(0) = -2$ and $f(1) = 12$.

We can readily factor $A^2 - 12A + 35$ as $(A - 5)(A - 7)$, so the general solution is $c_1 5^n + c_2 7^n$. The initial conditions require that $c_1 + c_2 = -2$ and $5c_1 + 7c_2 = 12$. This results in $c_1 = -13$ and $c_2 = 11$. So the general solution is $f(n) = -13 \cdot 5^n + 11 \cdot 7^n$.

15.

a. Write the inclusion/exclusion formula for the number of onto functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$.

Let n and m be positive integers with $n \geq m$. Then let X denote the set of all functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$. For an integer $i \in \{1, 2, \ldots, m\}$, we say that a function $f \in X$ satisfies property P_i if i is not in the range of f. A function is onto when it satisfies none of the properties. If S is a set of i properties, then the number of functions satisfying all the properties in S is $(m - i)^n$; and there are $\binom{m}{i}$ sets S of size i. The following inclusion/exclusion formula for the number $S(n, m)$ of onto functions follows easily.

$$S(n, m) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m - i)^n$$

b. Evaluate your formula when $n = 5$ and $m = 3$.

The formula yields

$$\begin{aligned} S(5, 3) &= \binom{3}{0} 3^5 - \binom{3}{1} 2^5 + \binom{3}{2} 1^5 - \binom{3}{3})0^5 \\ &= 243 - 3 \cdot 32 + 3 = 150 \end{aligned}$$

16.

a. Write the inclusion/exclusion formula for the number of derangements on $\{1, 2, \dots, n\}$.

Let X consist of all permutations on the set $\{1, 2, \dots, n\}$. For each $i = 1, 2, \dots, n$, we say that a permutation $\sigma \in X$ satisfies property P_i if $\sigma(i) = i$. A derangement is a permutation satisfying none of the properties. If S is a set of i properties, then there are $(n - i)!$ permutations satisfying all the properties in S. Now there are $\binom{n}{i}$ sets of size i, so the following inclusion/exclusion formula for the number d_n of derangements follows immediately.

$$d_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!$$

b. Evaluate your formula when $n = 5$.

The formula yields

$$\begin{aligned} d_5 &= \binom{5}{0}5! - \binom{5}{1}4! + \binom{5}{2}3! - \binom{5}{3}2! + \binom{5}{4}1! - \binom{5}{5}0! \\ &= 5! - 5 \cdot 4! + 10 \cdot 6 - 10 \cdot 2 + 5 \cdot 1 - 1 \cdot 1 \\ &= 120 - 120 + 60 - 20 + 5 - 1 \\ &= 44 \end{aligned}$$

c. Verify the correctness of your answer by writing all derangements when $n = 5$.

We elected to proceed lexicographically. Here are the 44 derangements when $n = 5$.

21453 21534 23154 23451 23514 24153 24513 24531 25134 25413 25431
31254 31452 31524 34152 34251 34512 34521 35124 35214 35412 35421
41253 41523 41532 43152 43251 43512 43521 45123 45132 45213 45231
51234 51423 51431 53124 53214 53412 53421 54123 54132 54213 54231

17. Previously, you factored 1638 into a product of primes. Using this factorization, evaluate the euler ϕ-function $\phi(1638)$.

Let n be an integer with $n \geq 2$ and let $p_1, p_2, \dots, p_k$ be the distinct prime factors of n. Then the inclusion/exclusion formula for $\phi(n)$ is:

$$\phi(n) = n \, \Pi_{i=1}^{k} (1 - \frac{1}{p_i})$$

Earlier in the test, we found that $1638 = 91 \cdot 18 = 2 \cdot 3^2 \cdot 7 \cdot 13$. It follows that

$$\begin{aligned} \phi(1638) &= 1638 \, (1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{7})(1 - \frac{1}{13}) \\ &= 1638 \, \frac{1}{2} \, \frac{2}{3} \, \frac{6}{7} \, \frac{12}{13} \\ &= 432 \end{aligned}$$

18. Let G be a graph on 23 vertices in which every vertex has 19 neighbors. Explain why G is hamiltonian but not planar.

A terrible typo here. Your professor gets a thousand whacks with an ∞ sign. There is no such graph!! One of the basic facts about graphs is that the number of vertices of odd degree is even! This comes from the following identity. Let $\deg_G(x)$ denote the degree of the vertex x in a graph G, let V denote the vertex set of G, and let q be the number of edges in G. Then

$$2q = \sum_{x \in V} \deg_G(x)$$

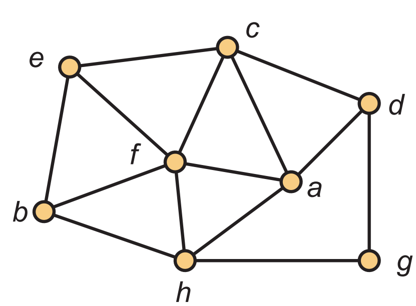
This equation follows from the fact that each edge in the graph is counted twice in the sum, once from each endpoint.

To see what I had in mind in posing the problem, change the 19 to an 18. We show that a graph G with 23 vertices in which every vertex has degree 18 is hamiltonian but not planar.

The fact that G is hamiltonian can be deduced from Dirac's Theorem which provides a sufficient condition for a graph to be hamiltonian: A graph on n vertices is hamiltonian if every vertex has degree at least $\lceil n/2 \rceil$. Here, this requires that all vertices have degree at least $\lceil 23/2 \rceil = 12$, which they do.

On the issue of planarity, we recall one of the corollaries to Euler's formula: The maximum number of edges in a planar graph on n vertices is $3n - 6$. Should our graph be planar, it could have at most $3 \cdot 23 - 6 = 63$ edges. But since every vertex has degree 18, the number of edges in G is $23 \cdot 18/2 = 207$, which means that G is *very* nonplanar!

19. Verify Euler's formula for the planar graph shown below.

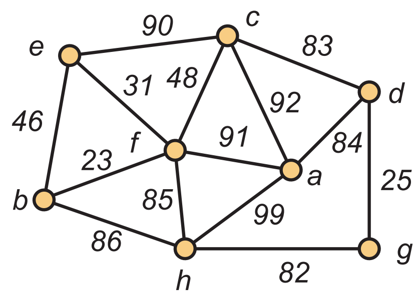


Given a drawing of a planar connected graph, let V denote the number of vertices, E the number of edges and F the number of faces (including the unbounded exterior face). Then Euler's formula is:

$$V - E + F = 2$$

For the graph shown, $V = 8$, $F = 8$ and $E = 14$, so that $8 - 14 + 8 = 2$, as required.

20. Consider the following weighted graph:



In the space below, list *in order* the edges which make up a minimum weight spanning tree using, respectively Kruskal's Algorithm (avoid cycles) and Prim's Algorithm (build tree). For Prim, use vertex a as the root.

To implement Kruskal's algorithm, we consider the edges sorted from least to greatest weight and choose edges greedily as long as we do not form a cycle with edges previously chosen. To implement Prim, we start with a one point tree T consisting of the root node a. We then expand T by choosing the cheapest edge with one endpoint in T and the other one not in T. The results are reflected in the following tables.

Kruskal's Algorithm

bf weight 23
dg weight 25
ef weight 31
cf weight 48
gh weight 82
cd weight 83
ad weight 84

Total weight: 376

Prim's Algorithm

ad weight 84
dg weight 25
gh weight 82
cd weight 83
cf weight 48
bf weight 23
cf weight 31

Total weight: 376

21. A data file `digraph_data.txt` has been read for a digraph whose vertex set is [7]. The weights on the directed edges are shown in the matrix below. The entry $w(i, j)$ denotes the length of the edge from i to j. If there is no entry, then the edge is not present in the graph. Apply Dijkstra's algorithm to find the distance from vertex 1 to all other vertices in the graph. Also, for each x, find a shortest path from 1 to x.

W	1	2	3	4	5	6	7
1	0	32	24	28	68	80	
2		0			30	44	10
3			0		41	56	
4	27	5	8	0		51	
5			12		0	28	3
6	82	5	3	2	2	0	
7		3	2	4	8	12	0

We start with the following candidate paths from the root node 1 to all other nodes—using the notation P_i for the candidate path from the root to node i:

$$\begin{array}{lll} \text{Path } P_2: & (1,2) & \text{Length: } 32 \\ \text{Path } P_3: & (1,3) & \text{Length: } 24 \\ \text{Path } P_4: & (1,4) & \text{Length: } 28 \\ \text{Path } P_5: & (1,5) & \text{Length: } 68 \\ \text{Path } P_6: & (1,6) & \text{Length: } 80 \\ \text{Path } P_7: & (1,7) & \text{Length: } \infty \end{array}$$

Note the length of the path P_7 is infinite, which indicates that there is no edge from node 1 to node 7. Now we taken the shortest temporary path, declare it permanent and then scan from that node to see if we can improve other candidate paths. In this case, the shortes temporary path has length 24, so we declare once and for all the shortest path from 1 to 3 has length 24 and the path $P_3 = (1,3)$ has this length.

Now we scan from node 3 to all other nodes. For starters, an alternate path from 1 to 3 would be $(1,3,2)$. This path has length $24 + w(3,2) = 24 + \infty = \infty$, which is certainly worse that what we now have for a path from node 1 to 2. Similarly, we make no update for the path from 1 to 4. However, we currently have a path of length 68 from 1 to 5. However, the alternate path $(1,3,5)$ has length $24 + w(3,5) = 24 + 41 = 65$, which is shorter. So we update this path.

Currently, we have a candidate path of length 80 from node 1 to 6. When we consider the alternate path $(1,3,6)$, we find that it has length $24 + w(3,6) = 24 + 56 = 80$, i.e., it has the same length as the current path. Here, we make no change, as we will only update when we get a real improvement.

Since $w(3,7) = \infty$, there is no update for the path P_7. With this observation, the scan from node 3 is complete. Here are the resulting candidate paths for all nodes except node 3.

$$\begin{array}{lll} \text{Path } P_2: & (1,2) & \text{Length: } 32 \\ \text{Path } P_4: & (1,4) & \text{Length: } 28 \\ \text{Path } P_5: & (1,3,5) & \text{Length: } 65 \\ \text{Path } P_6: & (1,6) & \text{Length: } 80 \\ \text{Path } P_7: & (1,7) & \text{Length: } \infty \end{array}$$

Now the shortest temporary path is P_4 so again, we declare this path and the distance it provides as permanent. As before, we scan from node 4 for improvements. But it is straightforward to verify that none are to be found. So our shortened list of candidate paths becomes:

$$\begin{array}{lll} \text{Path } P_2: & (1,2) & \text{Length: } 32 \\ \text{Path } P_5: & (1,3,5) & \text{Length: } 65 \\ \text{Path } P_6: & (1,6) & \text{Length: } 80 \\ \text{Path } P_7: & (1,7) & \text{Length: } \infty \end{array}$$

So the path P_2 is declared permanent and the distance from 1 to 2 is 32. Now we scan from node 2 for possible improvements. In this case, we get the following paths, all of which are improvements: $P_5 = (1,2,5)$ which has length $32 + 30 = 62 < 65$; $P_6 = (1,2,6)$ which has length $32 + 44 = 76 < 80$ and $P_7 = (1,2,7)$ which has length $42 < \infty$. So the updated set of candidate paths becomes:

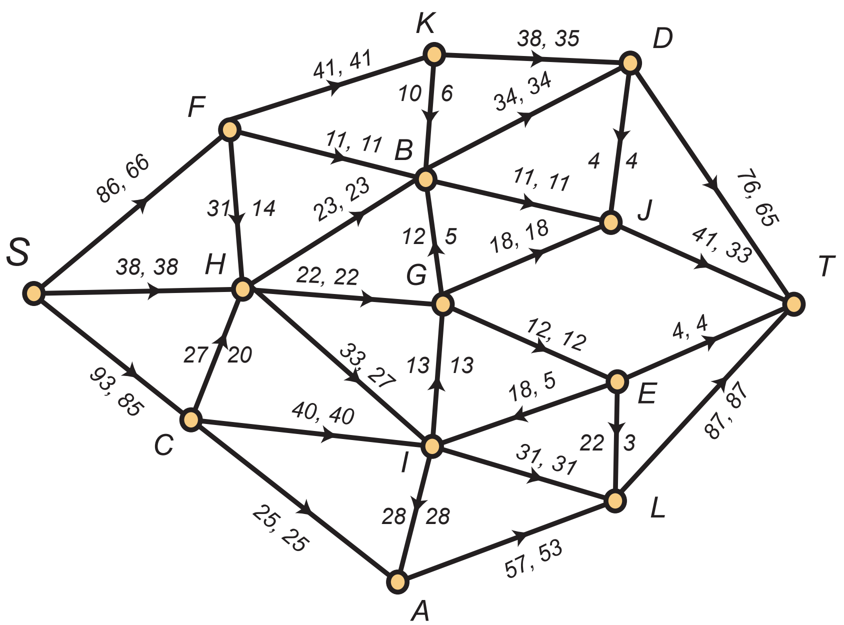
Path P_5 : $(1,2,5)$ Length: 62
Path P_6 : $(1,2,6)$ Length: 76
Path P_7 : $(1,2,7)$ Length: 42

Now P_7 is declared permanent and we scan from node 7. This results in improved paths for 5 and 6. These new paths are $P_5 = (1,2,7,5)$ which has length $42 + 10 = 52 < 62$ and $P_6 = (1,2,7,6)$ which has length $42 + 12 = 54 < 76$.

Path P_5 : $(1,2,7,5)$ Length: 62
Path P_6 : $(1,2,7,6)$ Length: 54

Path P_6 is declared permanent and we make one last scan for an even shorter path to node 5, and we find it since the alternate path $P_5 = (1,2,7,6,5)$ has length $55 + 2 = 57 < 62$. So this becomes the permanent path and true distance from 1 to 5 and the algorithm is complete.

22. Consider the following network flow:



a. What is the current value of the flow?

The value of the flow the amount leaving the source—which is also the amount arriving at the sink. This amount is: $66 + 38 + 85 = 189$.

b. What is the capacity of the cut $V = \{S, G, E, C, I\} \cup \{A, H, B, D, F, J, K, L, T\}$.

We calculate the sum of the capacities of edges from vertices in the first set to vertices in the second. This includes the following edges, listed with their capacities:

(S, F) capacity: 86	(E, T) capacity: 4
(S, H) capacity: 38	(C, A) capacity: 25
(G, B) capacity: 12	(C, H) capacity: 27
(G, J) capacity: 18	(I, A) capacity: 28
(E, L) capacity: 22	(I, L) capacity: 31

Adding these capacities together, we see that the capacity of this cut is:

$$86 + 38 + 12 + 18 + 22 + 4 + 25 + 27 + 28 + 31 = 291$$

c. Carry out the labeling algorithm, using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices.

The following labels will be applied:

S	$(*, +, \infty)$	L	$(E, +, 5)$
C	$(S, +, 8)$	B	$(G, -, 5)$
F	$(S, +, 20)$	A	$(L, -, 5)$
H	$(C, +, 7)$	K	$(B, -, 5)$
I	$(H, +, 6)$	D	$(K, +, 3)$
E	$(I, -, 5)$	T	$(D, +, 3)$
G	$(E, -, 5)$		

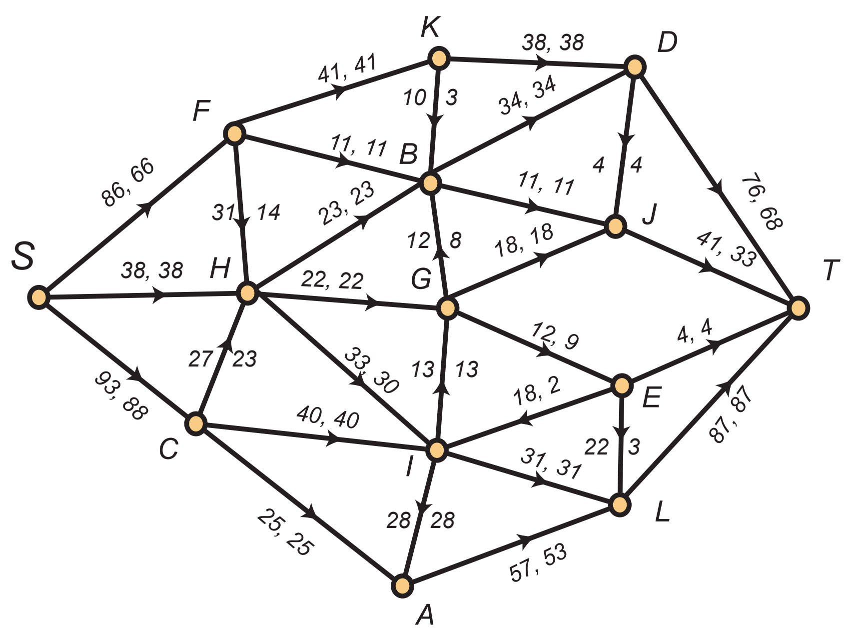
Backtracking from the sink T, we find the path

$$(T, D, K, B, G, E, I, H, C, S)$$

from the sink T back to the source S. Reversing, we have the augmenting path

$$(S, C, H, I, E, G, B, K, D, T)$$

Along this path, all edges are forward *except* (I, E), (E, G) and $(B, K$, which are backwards. We increase the flow on the forward edges by 3 and decrease the flow on the backwards edges by 3. This results in a flow of value $192 = 189 + 3$. We show this updated flow below.



d. Use your work in part c to find an augmenting path and make the appropriate changes directly on the diagram.

e. Carry out the labeling algorithm a second time on the updated flow. It should halt without the sink being labeled. Find a cut whose capacity is equal to the value of the flow.

Here are the results of carrying out the labelling algorithm a second time.

$$\begin{array}{ll} S & (*, +, \infty) \\ C & (S, +, 5) \\ F & (S, +, 20) \\ H & (C, +, 4) \\ I & (H, +, 3) \\ E & (I, -, 2) \end{array} \qquad \begin{array}{ll} G & (E, -, 2) \\ L & (E, +, 2) \\ B & (G, -, 2) \\ A & (L, -, 2) \\ K & (B, -, 2) \end{array}$$

Now the labelling algorithm halts without the sink being labelled. And we have the following partition:

$$\mathcal{L} = \{S, C, F, H, I, E, G, L, B, A, K\} \quad \text{and} \quad \mathcal{U} = \{D, J, T\}.$$

The edges which go from $\mathcal{L}$ to $\mathcal{U}$ are:

$$\begin{array}{ll} (K, D) & \text{capacity: } 38 \\ (B, D) & \text{capacity: } 34 \\ (B, J) & \text{capacity: } 11 \end{array} \qquad \begin{array}{ll} (G, J) & \text{capacity: } 18 \\ (E, T) & \text{capacity: } 4 \\ (L, T) & \text{capacity: } 87 \end{array}$$

It follows that the capacity of this cut is $38 + 34 + 11 + 18 + 4 + 87 = 192$, which is value of the current flow. This shows that the flow is optimal.

23. Consider a poset P whose ground set is $X = \{a, b, c, d, e, f, g, h, i, j\}$. Network flows (and the special case of bipartite matchings) are used to find the width w of P and a minimum chain partition. When the labelling algorithm halts, the following edges are matched:

$$h'd'' \quad a'g'' \quad j'b'' \quad c'i'' \quad e'a'' \quad d'j''$$

a. Find the chain partition of P that is associated with this matching. Also find the value of w.

The rule is that when an edge $x'y''$ is part of the matching, then x is covered by y in one of the chains in the chain partition. Accordingly, the chains are:

$$\begin{aligned} C_1 &= \{h < d < j < b\} \\ C_2 &= \{e < a < g\} \\ C_3 &= \{c < i\} \\ C_4 &= \{f\} \end{aligned}$$

Note that element f makes up a singleton chain, as neither f' nor f'' is used as one of the endpoints in the matching. Since there are 4 chains, the width w of the poset P is 4.

b. We do not have enough information to determine a maximum antichain. Discuss what additional information is needed to do this.

When the labelling algorithm halts, some vertices will be labelled and some vertices will be unlabelled. Of course, the source is always labelled and the sink is always unlabelled. For each $i = 1, 2, 3, 4$, there will be some point $x_i \in C_i$ for which x_i' is labelled and x_i'' is unlabelled. The set $A = \{x_1, x_2, x_3, x_4\}$ will be a maximum antichain.

To see that this claim holds in general, consider a chain $C = u_1 < u_2 < \cdots < u_m$ assembled from a maximum matching of the bipartite graph associated with a poset P. We claim that there is some j with $1 \leq j \leq m$ for which u_j' is labelled and u_j'' is unlabelled. Since u_m is the highest element in the chain C, there is no element v for which u_m' is matched to v''. This means that v_m' is labelled. If u_m'' is unlabelled, we have found the desired point from C, so we may assume without loss of genrality that v_m'' is also labelled.

At the bottom of the chain, we note that u_1'' must be unlabelled, for if u_1'' were labelled, then since there is no edge $u'u_1''$ in the matching, the edge $u_1''T$ in the network flow is empty, and we would be able to label the sink. So there is some least positive integer j for which both u_j' and u_j'' are labelled, with $1 < j \leq m$. The matching contains the edge $u_{j-1}u_j''$ so this edge has flow 1 on it. Since u_j'' is labelled, we could also label u_{j-1}' via a backwards edge with u_j'' at the other end. The minimality of j implies that u_{j-1}'' is unlabelled, so u_{j-1} is the desired point.

c. Explain why element f belongs to every maximum antichain in P.

This is a trivial pigeon-hole argument. Suppose that A is a maximum antichain in P but that $f \notin A$. Since the width of P is 4, we must find the four pigeons in A in one of the three holes: C_1, C_2 and C_3. So two incomparable pigeons must be in the same hole (chain).