Solutions

Student Name and ID Number

MATH 3012 Final Exam, May 3, 2018, WTT

12

3, 3, 3, 3

1. Consider the 36-element set consisting of all 26 lower case letters of the English alphabet and the 10 digits in $\{0, 1, 2, \ldots, 9\}$. From this alphabet, identification strings of length 12 will be constructed using three letters, followed by a dash, and then 9 digits. For example $xbx - 009357882$ is one possible identification string.

a. What is the total number of identification strings (repetition of characters is allowed)?

$26^3\, 10^9$

b. How many identification strings are possible if repetition of characters is *not* permitted?

$P(26,3)\, P(10,9)$ OR $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$

c. How many identification strings can be formed using exactly two m's, one b, three 0's, five 2's and one 7?

$C\binom{3}{2}\binom{9}{3,5,1}$ OR $\frac{3!}{2!\,1!}\;\frac{9!}{3!\,5!\,1!}$

d. How many identification strings can be formed using exactly two m's, one b, three 0's, five 2's and one 7 if all characters of each type must occur consecutively?

$P(2,2) \cdot P(3,3)$ OR $2!\,3!$ OR $2 \cdot 6$ OR 12

3

2. How many lattice paths from $(0,0)$ to $(24,31)$ pass through the point $(9,25)$?

$\binom{34}{9} \cdot \binom{21}{6}$ other correct answers via $\binom{n}{k} = \binom{n}{n-k}$

12

4, 4, 4

3. A wealthy donor decides to make a generous donation to Georgia Tech to be divided (perhaps not evenly) among the following three schools: Mathematics, Chemistry and Physics. Each school will receive an amount which is a multiple of \$50,000 and the total donation will be exactly \$1,000,000.

$1{,}000{,}000 = 20 \times 50{,}000$ 20 objects

How many different ways can the donation be made with the following restrictions imposed?

a. Each of the three schools receives at least \$200,000.

Set aside 150K = 3 · 50K for each, Total of 9.
11 "objects" remain $\boxed{\binom{12}{2}}$

b. The award to Mathematics is at least \$400,000?

350K = 7 · 50K 13 object remain + 2 artificial. = 15

$\boxed{\binom{14}{2}}$

c. The award to Mathematics is at least \$400,000? and Physics and Chemistry each receive at least \$200,000,

Set aside 7 = 350K for Math 3 each = 150K for Physics and Chem. 13 set aside; 7 remain

$\boxed{\binom{6}{2}}$

10
5/5

4a. Find two integers m and n, each at least 100 and at most 200 such that $23 = \gcd(m, n)$, $m/23$ is a prime; and $n/23$ is a prime. Even though you already know that $23 = \gcd(m, n)$, carry out the Euclidean algorithm to verify this fact.

$23 \cdot 3 = 69$ ✗ $23 \cdot 5 = 115$ ✓ m $23 \cdot 7 = 161$ ✓ n $23 \cdot 11 = 253$ ✗

$161 = 1 \cdot 115 + 46$ $115 = 2 \cdot 46 + 23$ $46 = 2 \cdot 23 + 0$

23

b. Use your work in the preceding problem to find integers a and b so that $d = am + bn$.

$23 = 115 - 2 \cdot 46$

$46 = 161 - 1 \cdot 115$

$23 = 115 - 2[161 - 1 \cdot 115]$

$\boxed{23 = 3 \cdot 115 - 2 \cdot 161}$

$345 - 322 = 23$ ✓

10
M, 7

5a. For a positive integer n, let q_n count the number of quaternary strings (alphabet $= \{0, 1, 2, 3\}$) that do not contain either 11 or 002 as substrings of consecutive characters. Find initial conditions by specifying q_1, q_2 and q_3.

$n=1$
$q_1 = 4$ all good

$n=2$ total $= 16$
1 bad
$q_2 = 15$

$n=3$ ~~total~~ 64
002 bad 1
*11 4 bad
1 4 bad } double count 111
7

$\boxed{q_3 = 64 - 8 = 56}$

b. Develop a recurrence for q_n when $n \geq 4$ and use it to compute q_4 and q_5.

Note: You want to work this left-to-right

| 0 | all-bad $= q_{n-1} - $ | 0 | 0 | 2 | $= q_{n-1} - q_{n-3}$

1 for empty string when n=4

✓ not 1
| 1 | * | 2nd not 1 $3q_{n-2}$ - bad | 1 | 0 | 0 | 2 | $=$ ~~q_{n-2}~~ $= q_{n-4}$

| 2 | q_{n-1}

| 3 | q_{n-1}

$q_n = 3q_{n-1} + 3q_{n-2} - q_{n-3} - q_{n-4}$

$q_4 = 3 \cdot 56 + 3 \cdot 15 - 4 - 1$
$= 168 + 45 - 5 = 208$

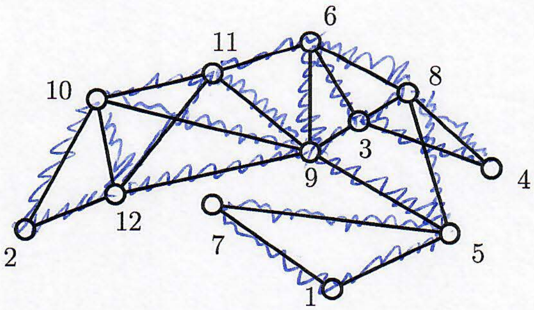
must count $q_0 = 1$ for empty string

$q_5 = 3 \cdot 208 + 3 \cdot 56 - 15 - 4$
$= 624 + 168 = 792$

$792 - 19 = 773$

6. Use the greedy algorithm developed in class (always proceed to the lowest legal vertex) to find an Euler circuit in the graph G shown below (use node 1 as root):

10



(1, 5, 7, 1)

5, 8, 3, 4, 8, 6, 3, 9, 5

1, 5, 8, 3, 4, 8, 6, 3, 9, 5, 7, 1

(6, 9, 10, 2, 12, 9, 11, 6)

(1, 5, 8, 3, 4, 8, 6, 9, 10, 2, 12, 9, 11, 6, 3, 9, 5, 7, 1)

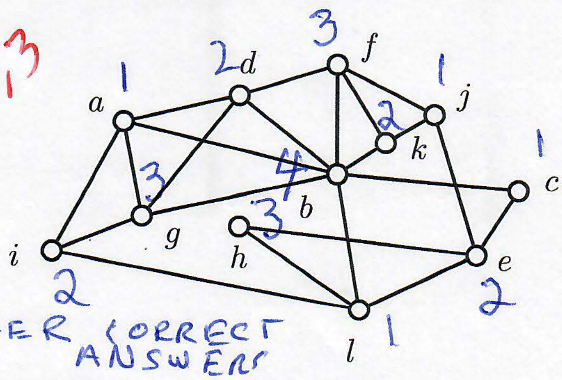
10, 11, 12, 10

(1, 5, 8, 3, 4, 8, 6, 9, 10, 11, 12, 10, 2, 12, 9, 11, 6, 3, 9, 5, 7, 1)

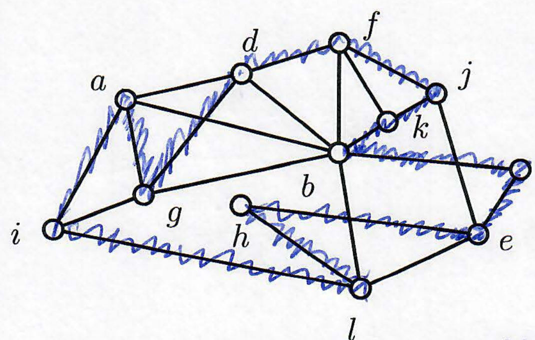
15

3, 3, 3, 3, 3
2, 1

7. Two copies of a graph G are shown below:



OTHER CORRECT ANSWERS



OTHER CORRECT ANSWERS

a. Find a clique of size 4 in G.

{a, b, d, g}

b. Find an induced cycle of size 5 in G.

{e, c, b, f, j} or {b, f, j, e, l}

c. Show that $\chi(G) \leq 4$ by producing a proper coloring using the elements of $\{1, 2, 3, 4\}$ as colors. Write directly on the first copy of G to give your answer.

DONE

d. Explain why this graph does not have an Eulerian circuit.

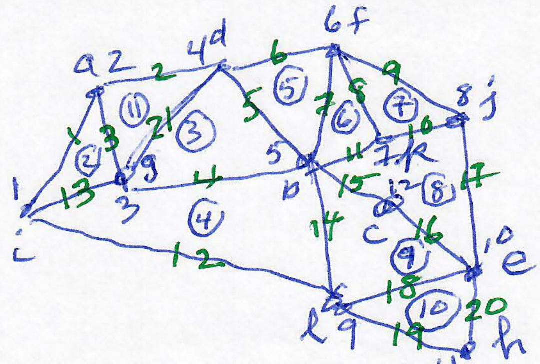
It has vertices of odd degree : {i, k, j, b}

e. Show that the graph is Hamiltonian by either (1) listing an appropriate sequence of vertices below, or (2) darkening an appropriate set of edges on the second copy.

DONE

8. The graph G from the preceding problem is non-planar. In the space below, show that the graph G' obtained from G by deleting the edge ab is planar by drawing G' in the plane without edge crossings. Then verify Euler's formula for your drawing.

8



$V = 12$

$E = \cancel{20}\ 21$

$F = \cancel{10}\ 11$

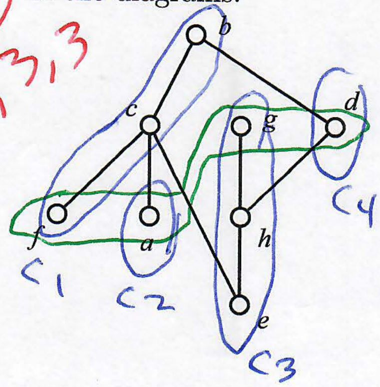
$V - E + F = 2$

$23 - 21 = 2$

9. Find by inspection the width w of the following poset, and find a partition of the poset into w chains. Also find a maximum antichain. You may indicate the partition by writing directly on one of the diagrams.

9

3, 3, 3



chain partition shown

a. The width w is 4 and $\{a, f, g, d\}$ is a maximum antichain.

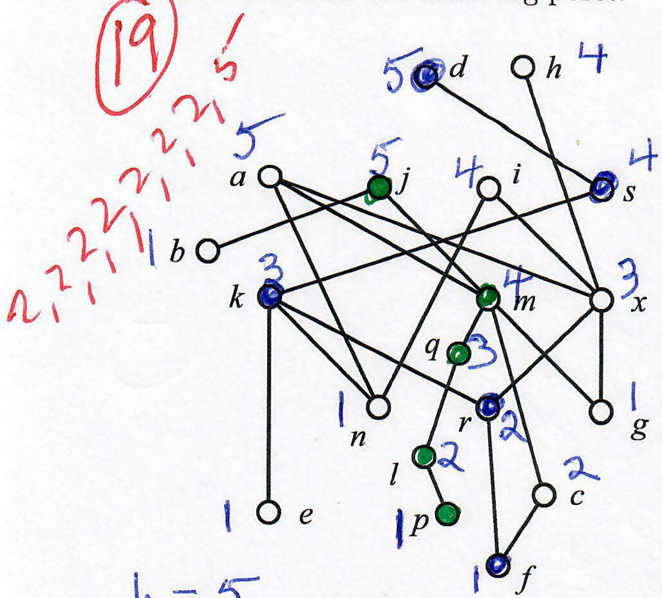
b. This poset is not an interval order. Find by inspection four points which form a copy of $\mathbf{2} + \mathbf{2}$. $\{c, f, g, h\}$

OTHER CORRECT ANSWERS

OTHER CORRECT ANSWERS

10. Consider the following poset.

19



h = 5

a. Find all points comparable to x.

$\{i, h, r, g, f\}$

b. Find all points which cover x.

$\{a, i, h\}$

c. Find all points which are covered by x.

$\{r, g\}$

d. Find a maximal chain of size 2.

$\{j, b\}$ or $\{n, i\}$ or $\{n, a\}$

e. Find a maximal chain of size 3.

$\{h, x, g\}$ or $\{i, x, g\}$ or $\{j, m, g\}$

f. Find the set of all maximal elements.

$\{a, j, i, d, h\}$

g. Find the set of all minimal elements.

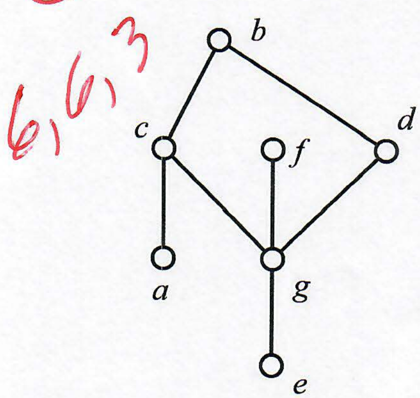
$\{b, e, n, p, f, g\}$

There is one more.

h. Using the algorithm taught in class (recursively removing the set of minimal elements), find the height h of the poset and a partition of P into h antichains. Also find a maximum chain. You should indicate the partition writing directly on the diagram, i.e., each element should be labeled with an integer from $\{1, 2, \ldots, h\}$. You may indicate a chain of maximum size by darkening an appropriate set of points on the figure. h = 5 two maximum chains are shown. One blue, one green.

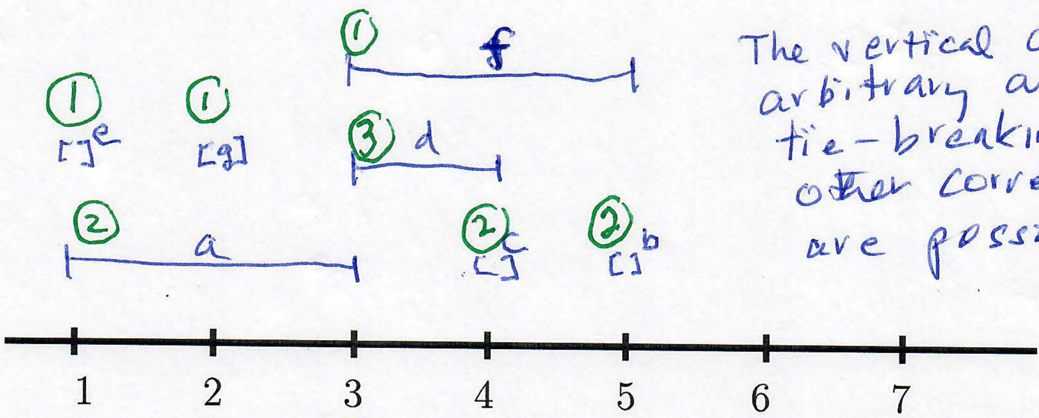
15

11a. Shown below is the diagram of a poset P which is an interval order P. Using the taught in class to find an interval representation for P, determine the down-sets and up-sets in the space provided.



$D(a) =$ ∅	1	3	$U(a) =$ bc
$D(b) =$ acdefg	5	5	$U(b) =$ ∅
$D(c) =$ age	4	4	$U(c) =$ b
$D(d) =$ ge	3	4	$U(d) =$ b
$D(e) =$ ∅	1	1	$U(e) =$ bcdfg
$D(f) =$ ge	3	5	$U(f) =$ ∅
$D(g) =$ e	2	2	$U(g) =$ bcdf

b. Use the information obtained in completing the first part of this problem to find an interval representation of P and display the resulting intervals in the space below. Then use the First Fit coloring algorithm to find the width w and a partition of the poset into w chains. Also, find a maximum antichain.



The vertical displacement is arbitrary and so is the tie-breaking rule. So other correct "colorings" are possible.

The width w of P is 3 and $\{a, d, f\}$ is a maximum antichain.

8

4,4

12a. Write in product form the generating function for the number of partitions of an integer n into parts, all of which have size at most 4. There is no limit on the number of parts of any given size.

$$f(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^4}$$

b. Write in product form the generating function for the number of partitions of an integer n into parts, all of which have sizes which are a multiple of 3, if no two parts can have the same size.

$$f(x) = (1+x^3)(1+x^6)(1+x^9)(1+x^{12})(1+x^{15})\cdots$$

6

13. Write the inclusion-exclusion formula for the number d_n of derangements of $\{1, 2, \ldots, n\}$. Then use your formula to find d_4. Please carry out all necessary operations to evaluate d_4 as an integer.

$$d_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!$$

$$d_4 = \sum_{i=0}^{4} (-1)^i \binom{4}{i} (4-i)!$$

$$= \binom{4}{0} 4! - \binom{4}{1} 3! + \binom{4}{2} 2! - \binom{4}{3} 1! + \binom{4}{4} 0!$$

$$= 1 \cdot 24 - 4 \cdot 6 + 6 \cdot 2 - 4 \cdot 1 + 1 \cdot 1$$

$$= 24 - 24 + 12 - 4 + 1 = 9$$

6

14. Write the formula for the number $S(n, m)$ for the number of surjections from $[n] = \{1, 2, \ldots, n\}$ to $[m] = \{1, 2, \ldots, m\}$. Then use your formula to find $S(5, 3)$. Again, find $S(5, 3)$ explicitly.

$$S(n, m) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m-i)^n$$

$$S(5, 3) = \sum_{i=0}^{3} (-1)^i \binom{3}{i} (3-i)^5$$

$$= \binom{3}{0} 3^5 - \binom{3}{1} 2^5 + \binom{3}{2} 1^5 - \cancel{\binom{3}{3} 0^5}$$

$$= 1 \cdot 243 - 3 \cdot 32 + 3 \cdot 1 = 150$$

$$\begin{array}{r} 246 \\ 96 \\ \hline 150 \end{array}$$

4

15. This problem is concerned with the Euler ϕ-function. You are trapped on a desert island with no electricity, calculators, computers, etc. Fortunately, you have plenty of paper and pens for writing. The only way to escape is to solve one of the following two problems by hand:

(1) The number $n = 56652933174483971459$ is the product of two primes. Find $\phi(n)$. Note that n has only 20 digits.

(2) The integers $m = 307625422503012706920514605395861669272917327549 61$ and $p = 313539589974026666385010319707341761012894704055733952484113$ are primes. Note that the product mp has 110 digits. Find $\phi(mp)$.

Explain which of the two problems you would tackle and explain whether your escape would take minutes, hours, days, weeks, months, years or centuries.

$$\phi(mp) = mp\left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{p}\right)$$

$$= \cancel{mp}\left(\frac{m-1}{\cancel{m}}\right)\left(\frac{p-1}{\cancel{p}}\right)$$

$$= (m-1)(p-1)$$

We would take (2). I could do this multiplication in an hour or two, maybe less.

8

2, 2, 2, 1 ✓

16a. The height of the subset lattice $\mathbf{2}^{13}$ is:

14

b. The width of the subset lattice $\mathbf{2}^{13}$ is:

$\binom{13}{6}$ or $\binom{13}{7}$

c. The number of maximal chains in the subset lattice $\mathbf{2}^{13}$ is:

$13!$

d. The number of maximal chains in the subset lattice $\mathbf{2}^{13}$ passing through 0101100110000 is:

$5!\,8!$

14

4, 4, 6, 1

17. This question concerns advancement operator equations.

a. Find the general solution to the advancement operator equation:
$(A+4-2i)^3(A-7)^4 f(n) = 0$

$$f(n) = c_1(-4+2i)^n + c_2 n(-4+2i)^n + c_3 n^2(-4+2i)^n + c_4 7^n + c_5 n 7^n + c_6 n^2 7^n + c_7 n^3 7^n$$

b. Consider the following non-homogenous advancement operator equation:

$$(A-5)f(n) = 2n \cdot 5^n.$$

If you were forced to find a particular solution, what is the *form* for such a solution? *Note.* Please do not attempt to find an actual particular solution. You are only supposed to give its form.

$$f(n) = d_1 \cancel{n \cdot 5^n} \cancel{+ d_2 n^2}$$
$$= d_1 n 5^n + d_2 n^2 5^n$$

c. Find the solution to the advancement operator equation:
$(A^2 - 10A + 16)f(n) = 0, \quad f(0) = -8$ and $f(1) = 2$.

$$\boxed{f(n) = (-11)2^n + 3 \cdot 8^n}$$

$(A-2)(A-8)f(n) = 0$

$f(n) = c_1 2^n + c_2 8^n$

$f(0) = -8 = c_1 + c_2$

$f(1) = 2 = 2c_1 + 8c_2$

$-16 = 2c_1 + 2c_2$

$18 = 6c_2$

$c_2 = 3$

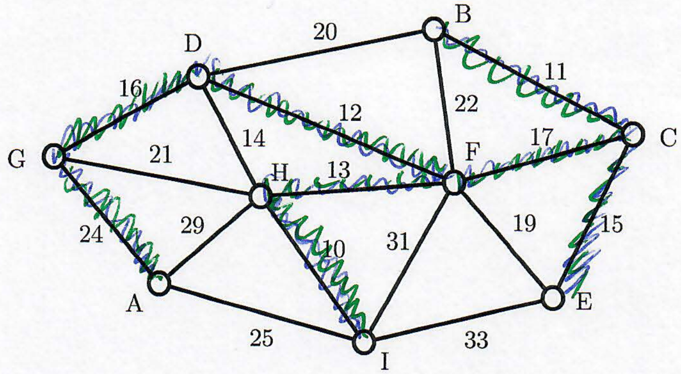
$-8 = c_1 + 3$

$c_1 = -11$

12

6, 6

18. A graph with weights on edges is shown below. In the space to the right of the figure, list *in order* the edges which make up a minimum weight spanning tree using, respectively, Kruskal's Algorithm (avoid cycles) and Prim's Algorithm (build tree). For Prim, use vertex A as the root.



Kruskal	Prim
HI = 10	AG = 24
BC = 11	GD = 16
DF = 12	DF = 12
FH = 13	HF = 13
CE = 15	HI = 10
GD = 16	FC = 17
FC = 17	BC = 11
AG = 24	CE = 15

13

3, 5, 5

19a. What is the probability that a hand of five cards from a standard deck of 52 cards would be classified as "full house"? This means the five cards are of the form $\{x, x, x, y, y\}$.

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} \quad \text{OR} \quad \frac{13 \cdot 4 \cdot 12 \cdot 6}{\binom{52}{5}}$$

b. In a Bernoulli trial experiment, the probability of success is 1/8. If 20 trials are conducted, what is the probability that the number of successes is less than 3?

$$\binom{20}{0}\left(\frac{1}{8}\right)^0\left(\frac{7}{8}\right)^{20} + \binom{20}{1}\left(\frac{1}{8}\right)^1\left(\frac{7}{8}\right)^{19} + \binom{20}{2}\left(\frac{1}{8}\right)^2\left(\frac{7}{8}\right)^{18}$$

c. Yolanda rolls a fair die. She wins three matchsticks if she rolls a six. With any other result, she then rolls repeatedly until one of the following statements holds: (a) she rolls the same result as her initial roll, and in this case she wins four matchsticks; (b) she rolls a six, and in this case, she loses five matchsticks. What is the expected value of this game? *Note.* The answer can be positive, negative or zero.

$$\frac{1}{6} \cdot 3 + \frac{5}{6} \cdot \frac{1}{2} \cdot 4 + \frac{5}{6} \cdot \frac{1}{2}(-5)$$

$$= \frac{3}{6} + \frac{10}{6} - \frac{25}{12}$$

$$= \frac{6}{12} + \frac{20}{12} - \frac{25}{12} = \boxed{\frac{1}{12}}$$

15

20. A data file `digraph_data.txt` has been read for a digraph whose vertex set is $\{1, 2, \ldots, 6\}$. The weights on the directed edges are shown in the matrix below. The entry $w(i, j)$ denotes the length of the edge from i to j. If there is no entry, then the edge is not present in the graph. Apply Dijkstra's algorithm to find the distance from vertex 1 to all other vertices in the graph. Also, for each x, find a shortest path from 1 to x. Please show your work.

W	1	2	3	4	5	6
1	0	40	10		20	23
2		0	44	8	30	23
3		18	0	60	9	
4	27	30	28	0	10	45
5		7	2		0	21
6	82	55	3	20	2	0

(1, 2)
40

(1, 3)
10 Perm

(1, 4)
∞

(1, 5)
20

(1, 6)
23

(1, 3, 2)
10 + 18 = 28

(1, 3, 4)
10 + 60 = 70

(1, 3, 5)
10 + 9 = 19
Perm

(1, 6)
23

(1, 3, 5, 2)
19 + 7 = 26

(1, 3, 4)
70

1, 6
23 PERM

(1, 6, 4)
23 + 20 = 43

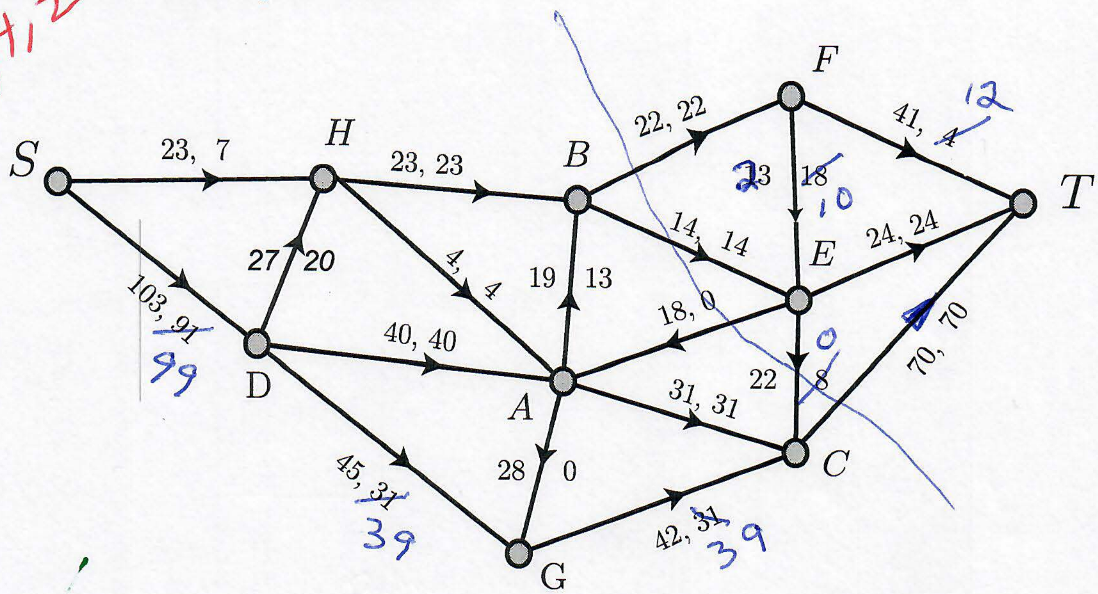
(1, 3, 5, 2)
26 perm

1, 3, 5, 2, 4
26 + 8 = 34 perm

21

2,2,5,2,2,2,4,2

21. Consider the following network flow:



a. What is the current value of the flow?

91 + 7 = 98

23
19
31
42
115

b. What is the capacity of the cut $V = \{S, A, D, G, H\} \cup \{T, B, C, E, F\}$.

23 + 19 + 31 + 42 = 115

c. Carry out the Ford-Fulkerson labeling algorithm, using the pseudo-alphabetic order on the vertices, and list below the labels which will be given to the vertices. Use two columns if cramped for space.

S(*, +, ∞)
D(S, +, 12)
H(S, +, 16)
G(D, +, 12)
C(G, +, 9)
A(C, −, 9)
E(C, −, 8)

B(A, +, 6)
F(E, −, 8)
T(F, +, 8)

d. What is the augmenting path identified by the labeling algorithm?

S, D, G, C, E, F, T

e. The augmenting path informs us how the flow should be increased. Make these changes by marking directly on the diagram for the network.

DONE

f. What is the value of the new flow?

98 + 8 = 106

g. Carry out the labeling algorithm on the updated network flow, again using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices. Hint. The algorithm should halt without the sink being labeled. Again, use two columns if cramped for space.

S $(*, +, \infty)$ A $(C, -, 3)$
D $(S, +, 4)$ B $(A, +, 6)$
H $(S, +, 16)$
G $(D, +, 4)$
C $(G, +, 3)$

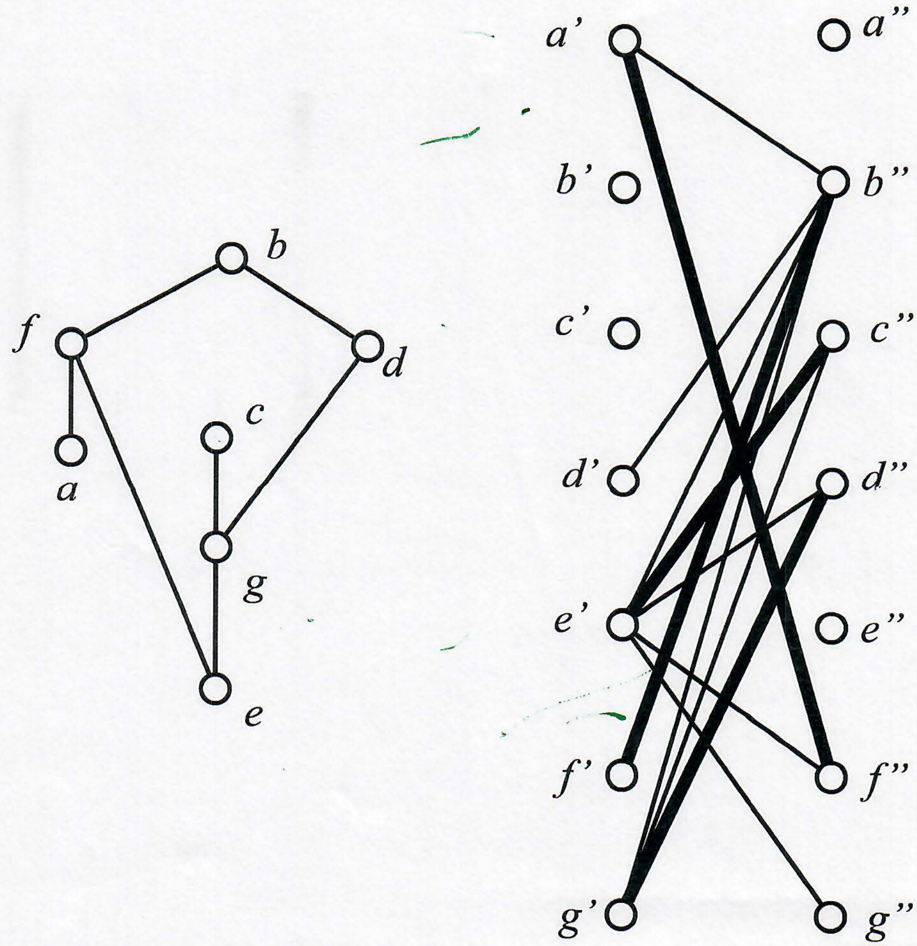
h. Find a cut whose capacity is equal to the value of the current flow.

$\{S, A, B, C, D, G, H\} \cup \{T, E, F\}$

22.

8

4, 4



A poset P is shown above on the left while the bipartite graph G associated with P is shown on the right. Some of the edges in the graph have been darkened.

a. What matching in G is associated with the chain partition $\{b, e, f\} \cup \{a\} \cup \{d\} \cup \{c, g\}$ of the poset P?

$e' f''$ $f' b''$ $g' c''$

b. What chain partition of P is associated with the maximum matching given by the darkened edges?

$\begin{vmatrix} b \\ f \\ a \end{vmatrix}$ $\begin{vmatrix} c \\ e \end{vmatrix}$ $\begin{vmatrix} d \\ g \end{vmatrix}$ $\{a, b, f\} \cup \{c, e\} \cup \{g, d\}$

17

$\frac{1}{2}$ each

23. True–False. Mark in the left margin.

T 1. $C(10, 3) = 120$. $\frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = \frac{720}{6}$ ✓

F 2. $P(10, 3) = 120$. $10 \cdot 9 \cdot 8 = 720$

F 3. Any graph in which all vertices have degree 4 has an Euler circuit. Could be disconnected

T 4. There is a connected graph with 500 vertices and 5000 edges which does not have a Hamiltonian cycle.

T 5. The number of lattice paths from $(0, 0)$ to $(12, 12)$ which do not go above the diagonal is the Catalan number $C(24, 12)/13$.

T 6. If G is a graph and $\chi(G) = 8$, then $\omega(G) \leq 8$.

T 7. The running time of an optimal implementation of the Euler Circuit algorithm is linear in the input size.

T 8. The decision problems: Is $\chi(G) \leq 3$? and Is G Hamiltonian? belong to the class NP.

T 9. The decision problems: Is $\chi(G) \leq 2$? and Is G Eulerian? belong to the class P.

T 10. There is a graph G with $\omega(G) = 2$ and $\chi(G) = 9$.

F 11. There is a graph G with $\omega(G) = 9$ and $\chi(G) = 2$.

T 12. There is a graph G with $\omega(G) = 3$ and $\chi(G) = 9$.

F 13. There is a planar graph G with $\omega(G) = 2$ and $\chi(G) = 9$.

F 14. There is a perfect graph G with $\omega(G) = 2$ and $\chi(G) = 9$.

T 15. If $\chi(G) = 2$, then G is perfect.

F 16. If $\chi(G) = 3$, then G is perfect. $120 \cdot 120 = 14{,}400$

T 17. There is a graph G with 240 vertices and 10,000 edges such that $\chi(G) = \omega(G) = 2$.

T 18. There is a graph with 240 vertices and 24,000 edges. $\frac{240 \cdot 239}{2} > 240 \cdot 100$

F 19. There is a planar graph with 240 vertices and 10,000 edges. $6 \cdot 240 - 6 << 10{,}000$

T 20. There is a poset with 585 points having width 31 and height 23. $31 \cdot 23 > 600$

F 21. There is a poset with 855 points having width 31 and height 23. $31 \cdot 23 < 855$

F 22. When $n \geq 4$, the shift graph S_n contains the triangle $\{\{1,2\},\{2,3\},\{3,4\}\}$..

T 23. When $n \geq 3$, the shift graph S_n has $\binom{n}{3}$ edges.

T 24. When $n \geq 2$, the shift graph S_n has $\binom{n}{2}$ vertices.

T 25. When $n \geq 2$, the chromatic number of the shift graph S_n is $\lceil \lg n \rceil$.

F 26. To test whether a graph G is an interval graph, we use a 2-phase algorithm. In the first phase, we test whether G is a cover graph of a poset P. In the second phase, we test whether P is an interval order.

F 27. To implement Kruskal's algorithm, it is not necessary to sort the edges by weight. One can simply take the edges in any order and take the first one avoiding a cycle when added to those edges already chosen.

F 28. The key idea behind the Ford-Fulkerson algorithm for network flows is to find at each step an augmenting path which uses the maximum number of edges.

T 29. All network flow problems are also linear programming problems.

F 30. All linear programming problems posed with integral constraints have integral solutions.

T 31. All network flow problems posed with integer valued capacities have an optimum solution in which all flow values are integers.

F 32. Let X be a finite set. Then a function P mapping the subsets of X to $[0,1]$ is a probability measure on X provided $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \emptyset$.

F 33. The expected value of a random variable is always non-negative.