Student Name and ID Number

MATH 3012 Quiz 2, October 30, 2003, WTT

1. Note that $3960 = 2^3 \times 3^2 \times 5 \times 11$. Compute $\phi(3960)$.

$$\begin{aligned}\phi(3960) &= 3960\,(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1-\frac{1}{11}) \\ &= 3960\,\frac{1}{2}\,\frac{2}{3}\,\frac{4}{5}\,\frac{10}{11} \\ &= 1320\end{aligned}$$

2. Consider the partitions of an integer n. What is the conclusion to be drawn from the following computation? Verify your answer when $n = 7$.

$$\begin{aligned}&(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\cdots \\ &= \frac{1-x^2}{1-x}\,\frac{1-x^4}{1-x^2}\,\frac{1-x^6}{1-x^3}\,\frac{1-x^8}{1-x^4}\,\frac{1-x^{10}}{1-x^5}\,\frac{1-x^{12}}{1-x^6}\cdots \\ &= \frac{1}{1-x}\,\frac{1}{1-x^3}\,\frac{1}{1-x^5}\,\frac{1}{1-x^7}\,\frac{1}{1-x^9}\,\frac{1}{1-x^{11}}\cdots\end{aligned}$$

The computation shows that the number of partitions of an integer into distinct parts is equal to the number of partitions into odd parts. For $n = 7$, there are 5 of each type:

Distinct parts: $7 \quad 6+1 \quad 5+2 \quad 4+3 \quad 4+2+1$

Odd parts: $7 \quad 5+1+1 \quad 3+3+1 \quad 3+1+1+1+1 \quad 1+1+1+1+1+1+1$

3. Let A denote the advancement operator, i.e., $A\,f(n) = f(n+1)$. Find the general solution of the following equation:

$$(A^2 + 4A - 12)f(n) = 0$$

The polynomial factors as $(A+6)(A-2)$, and the roots are -6 and $+2$. So the general solution is $f(n) = c_1(-6)^n + c_2 2^n$.

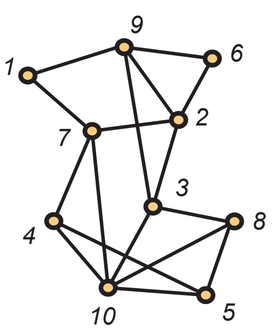
4. For the equation in the preceding problem, find the particular solution given $f(0) = 8$ and $f(1) = -8$.

We solve the following two equations:

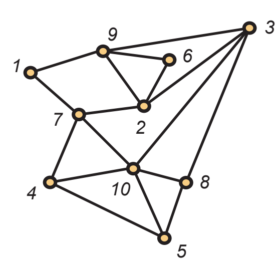
$$\begin{aligned}c_1 + c_2 &= 8 \\ -6c_1 + 2c_2 &= -8\end{aligned}$$

to obtain $c_1 = 3$ and $c_2 = 5$. So the particular solution is $f(n) = 3(-6)^n + 5(2)^n$.

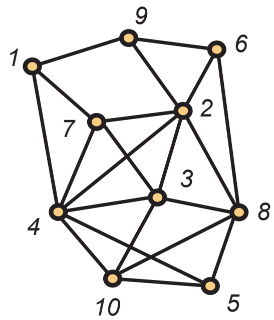
5. Show that G is planar.



Here's a redrawing of the graph without edge crossings, which shows that G is planar.



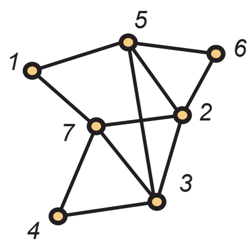
6. Show that G is hamiltonian by listing the vertices in an appropriate order, starting with vertex 1.



The following sequence forms a hamiltonian cycle in G:

$$1, 9, 6, 2, 3, 8, 5, 10, 7, 1$$

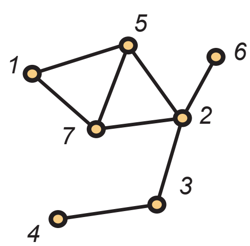
7. Show that G has an euler circuit by listing the vertices (with repetition allowed) in an appropriate order, starting with vertex 1.



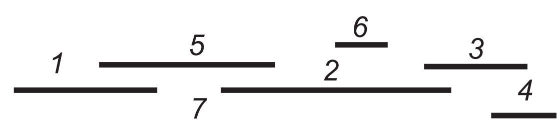
The following sequence forms an euler circuit in G:

$$1, 5, 6, 2, 5, 3, 2, 7, 3, 4, 7, 1$$

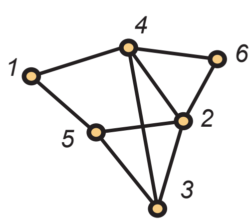
8. Show that G is an interval graph.



The following intervals form a representation of the given graph.

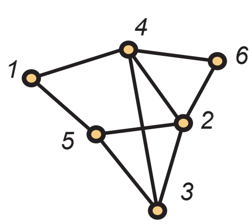


9. Find $\omega(G)$, and list a set of vertices which forms a clique of maximum size.

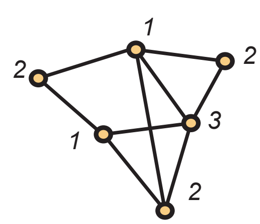


The maximum clique size is 3. The vertices $\{2, 4, 6\}$ form a clique of size 3. There are several other such sets.

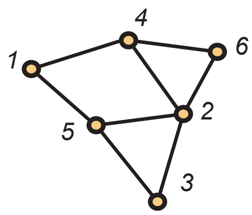
10. Find a proper coloring of G using $\chi(G)$ colors.



Here is a proper coloring of the graph using the colors $\{1, 2, 3\}$.



11. Verify Euler's formula for the following planar graph.



For this drawing, $V = 6$, $E = 8$ and $F = 4$. Thus

$$V - E + F = 6 - 8 + 4 = 2.$$

12. Suppose G is a graph with 120 vertices and 4037 edges. Explain why G is non-planar. Also explain why G contains a triangle.

By Euler's formula, we know that a planar graph on n vertices contains at most $3n - 6$ edges, so a planar graph with 120 vertices has at most 357 edges. By Turán's theorem, a triangle-free graph on n vertices has at most $\lceil \frac{n^2}{4} \rceil$ edges, so a triangle free graph with 120 vertices has at most 3600 edges.